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THOMAS CRAIG, Associate Editor.

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Memoir on Seminvariants.

BY CAPT. P. A. MACMAHON, R. A.

I continue the discussion of the asyzygetic seminvariants which was commenced in the *American Journal of Mathematics*, Vol. VI, No. 2; reference is made also to two papers on the same subject by Professor Cayley and to one by myself in Vol. VII, No. 1 of the same Journal.

I will, for the present, consider the seminvariants of the quantic with derived coefficients

$$ax^n - nbx^{n-1}y + n(n-1)cx^{n-2}y^2 - n(n-1)(n-2)dx^{n-3}y^3 + \ldots$$

which are identical with the non-unitary symmetric functions of the equation

$$x^n - bx^{n-1} + cx^{n-2} - dx^{n-3} + \ldots = 0,$$

n being infinite, and which therefore satisfy the partial differential equation

$$d_0 = \frac{d}{db} + b\frac{d}{dc} + c\frac{d}{dd} + d\frac{d}{de} + \ldots = 0.$$

SEC. 1. *The Three Cardinal Laws.*

The first may be called the derivation law and may be enunciated as follows: "The derivative of a form with respect to any letter is obtained with changed sign by performing the operation d_0 upon the derivative with respect to the next preceding letter."

For using suffixed letters for convenience and putting

$$d_\lambda = a_0\frac{d}{da_\lambda} + a_1\frac{d}{da_{\lambda+1}} + a_2\frac{d}{da_{\lambda+2}} + \ldots$$

we have $$\frac{d}{da_\lambda} = d_\lambda - H_1 d_{\lambda+1} + H_2 d_{\lambda+2} - H_3 d_{\lambda+3} + \ldots$$

wherein H_w represents the total symmetric function of weight w according to the usual notation; whence

$$d_1 \frac{d}{da_\lambda} = d_1 d_\lambda - H_1 d_1 d_{\lambda+1} + H_2 d_1 d_{\lambda+2} - H_3 d_1 d_{\lambda+3} + \dots$$
$$- d_{\lambda+1} + H_1 d_{\lambda+2} - H_2 d_{\lambda+3} + \dots$$

since
$$d_\lambda H_w = (-)^{\lambda+1} H_{w-\lambda};$$

also d_λ, $d_{\lambda+1}, \dots$ by their operation produce seminvariants (*vide* Hammond, Proc. Lon. Math. Soc., Vol. XIV, pp. 119-129), therefore

$$d_1 \frac{d}{da_\lambda} = -\frac{d}{da_{\lambda+1}},$$

which establishes the theorem.

This includes the theorem that the derivative with respect to the highest letter is itself a seminvariant, first proved I believe by Professor Sylvester (*cf.* *American Journal of Mathematics*, Vol. V, No. 1, p. 82). What appears to be the second fundamental property is in its essence Professor Sylvester's; it may be stated as follows: "If any seminvariant be operated upon by substituting for each letter, the letter of weight higher by unity, the resulting terms are those of highest degree in some seminvariant of higher weight"; this, as well as the converse proposition, is absolutely true in the case of the quantic with derived coefficients, but only true as regards forms when the binomial quantic is under consideration; it follows, of course, from a mere inspection of the operator d_λ, and is in fact identical with the converse processes of 'capitation' and decapitation (*cf. Cayley*, *American Journal of Mathematics*, Vol. VII, No. 1).

I have remarked elsewhere (Quart. Jour. of Math., Vol. XX, No. 80) that 'diminishing' and 'decapitation' are alike the performance of the operator D_θ on a form of degree θ (*cf.* Hammond, *ante*).

The third property, which appears to be absolutely fundamental, may be termed 'The Conjugate Law,' which was foreshadowed in the symmetrical seminvariant tables in Vol. VI, No. 2, *American Journal of Mathematics*. It is in its essence an extension and refinement of Hermite's Law of Reciprocity, showing clearly the source whence that great classical theorem springs; the theorem is: "To every seminvariant corresponds another, the partitions of whose terms are the Ferrers-conjugates of the partitions of its own terms."

This appears from the following consideration: suppose P and Q to be conjugate terms, having therefore each the same number of different parts in their partitions; by operating upon P with the component $a_\lambda \dfrac{d}{da_{\lambda+1}}$ of the operator d_1 a certain term will be produced, which, so far as its literal part is concerned, is merely P with the index of $a_{\lambda+1}$ diminished by unity; call this term p; now operating on Q with the element $a_\mu \dfrac{d}{da_{\mu+1}}$, $\mu+1$ being the $(\lambda+1)^{\text{th}}$ part of the partition of Q in descending order, a term q will be obtained, which is the conjugate of p; consequently the performance of d_1 on Q produces terms which are the conjugates of those obtained by performing d_1 on P, and there is an exact one-to-one correspondence; *ex. gr.* take the conjugate terms $a_1 a_2 a_3 a_5 a_7$ and $a_1^2 a_2^2 a_3 a_4 a_5$; $\dfrac{d}{da_1}$ and $a_4 \dfrac{d}{da_5}$ operating on these respectively produce conjugate terms; so also do $a_1 \dfrac{d}{da_2}$ and $a_3 \dfrac{d}{da_4}$, $a_2 \dfrac{d}{da_3}$ and $a_2 \dfrac{d}{da_3}$, $a_4 \dfrac{d}{da_5}$ and $a_1 \dfrac{d}{da_2}$, $a_6 \dfrac{d}{da_7}$ and $\dfrac{d}{da_1}$; it follows then that in seeking a form $\Sigma A.P$ by means of the differential equation, if we obtained an equation

$$\alpha A + \beta B + \gamma C + \ldots = 0,$$

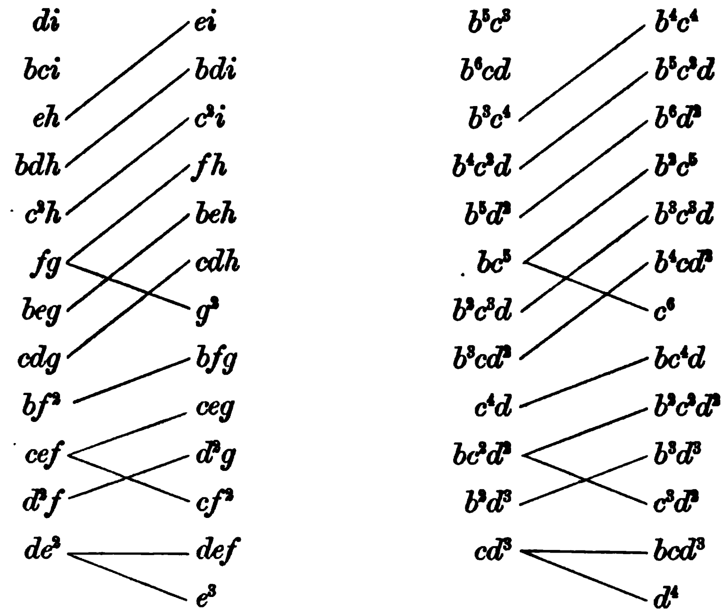
we must, in seeking a form $\Sigma A_1.Q$, arrive at a similar equation

$$\alpha_1 A_1 + \beta_1 B_1 + \gamma_1 C_1 + \ldots = 0,$$

and that therefore if the form $\Sigma A.P$ exist, so also must the conjugate form $\Sigma A_1.Q$.

Another proof presents itself as a result of considering together Arbogast's Method of Derivations, and a new rule of Professor Cayley's, which, in some cases, is simpler in its performance; his rule for forming the combinations of a given degree and weight in the letters $(a, b, c, d, \ldots)$ from those of the same degree and of a weight lower by unity is (1) to multiply the latter throughout by b, with the exception of those terms already of the proper degree, and (2) further, to raise the weight of the first letter by unity, whenever such letter occurs to the first power only; now this rule and Arbogast's rule of the last, and the last but one letter are conjugate to one another, and by means of a scheme initiated by Professor Cayley establishes the conjugate law above enunciated. An example having reference to the form whose leading and ending terms are ei and e^3 respectively will show how this comes about.

Form $ei + e^3$ — ARBOGAST'S RULE. Conjugate Form $b^4c^4 + d^4$ — CAYLEY'S RULE.



It is clear from the above example that the last-and-last-but-one terms fg, cef, de^2 which by Arbogast's rule are doubly operated upon, give rise, in the conjugate form, to terms in which the first letter has an index unity, and which therefore are doubly operated upon by Cayley's rule; in fact the two rules are completely conjugate and rigorously establish the conjugate law. We might of course apply Cayley's rule to the form $ei + e^3$, and Arbogast's to the conjugate with equally satisfactory results.

It is to be observed in regard to the definition of the conjugate law above given, that in any form every literal term is supposed to be present that is at once posterior to the leading term in counter order* and anterior to the ending term in alphabetical order; it may happen that certain of these terms, not being any of them a leading or an ending term, are affected with zero coefficients; but it does not thence follow that the conjugate terms are absent in the conjugate form.

* The alphabetical order of terms is that of Prof. Cayley's Seminvariant Tables in Vol. VII, No. 1 of this Journal. The counter order is got by taking the conjugates of these in reverse order. The former proceeds according to degree only, and the latter simply by weight of highest letter. These names are Professor Cayley's, and he fully explains them in a memoir shortly to appear, I believe, in this Journal.

SEC. 2. *The formation of an Asyzygetic Series of Seminvariants of a given weight.*

It will be convenient, although not necessary, to consider a particular weight; suppose the weight 11. If we wish to discover the simplest source having an ending term b^2d^3, we have to combine the form (53^2) with those whose symbols are posterior in counter order to it, in such a manner that a maximum number of the terms of highest extent in the letters may vanish, and thus it is manifest that an ending term, as having a partition the conjugate of a non-unitary partition, must of necessity be power-ending; that is, it must contain the highest letter involved in it to a power greater than unity; hence also it follows that a leading term must be a non-unitary one, for by the conjugate law it must be conjugate to a power-ending term.

It becomes a matter of importance to determine the best arrangement of the terms for the purpose of presenting to view an asyzygetic series, bearing in mind that it is desirable that the leading and ending terms shall indicate respectively the highest letter and the degree of the form. In the *American Journal of Mathematics*, Vol. VII, No. 1, Professor Cayley has arranged them in alphabetical order, which is one essentially of degree only, and possesses the advantage that the ending term of necessity indicates the degree correctly; the leading term, however, does not with certainty involve the most advanced letter in the form. The author's tables I to XII in the same journal do as a fact so far indicate correctly both of these, but in neither case necessarily; such an arrangement (after Durfee) is well adapted to showing the symmetry consequent on the conjugate law, but is objectionable, for the reasons above given, in any theoretical discussion of the forms; table XI differs from the remainder, and shows a modified arrangement; the first half of the terms were ranked according to weight of highest letter (the self-conjugate terms excluded), no attention being paid to degree; the self-conjugate terms then come, followed by the conjugates of the first half of the terms in reverse order. The last half therefore proceeded essentially as regards degree only, and it would appear *à priori* that with no other continuous arrangement of terms would there exist so high a probability of the leading and ending terms showing respectively the highest letter and the degree of the source. '*In medio tutissimus ibis.*' However, such a sequence of terms would in fact only be less imperfect than others of a continuous nature, for it is manifestly impossible for the terms to proceed both as regards weight of highest letter and degree.

It appears then to be imperative to abandon any continuous order, and as a consequence the following method has been adopted: after Professor Cayley I call the order according to weight of highest letter the counter order. Consider the weight n, and starting with the letter of weight n proceed with terms which follow both the alphabetical and counter-orders, the alphabetical order being dominant; that is to say, the order is an alphabetical one, with those terms omitted which do not as well obey the counter-order; the remaining terms are then taken, and starting with the first one which was omitted from the first series, a second series is formed which is in both orders; of the remaining terms a third series is formed, and so forth, until all the terms are exhausted.

As an example the arrangement of weight 11 in four series is exhibited.

Series 1.	Series 2.	Series 3.	Series 4.
l	b^2j	b^3i	b^4h
bk	bci	b^2ch	b^5g
cj	bdh	bc^2g	b^6f
di	c^2h	b^3cg	
eh	cdg	b^3df	
fg	b^2dg	b^2c^2f	
beg	b^2ef	b^4cf	
bf^2	$bcdf$	b^4de	
cef	c^3f	b^5ce	
d^2f	b^3e^2	b^7e	
de^2	b^2cde		
bce^2	bc^3e		
bd^2e	b^3c^2e		
c^2de	b^3cd^2		
cd^3	b^5d^2		
b^2d^3	b^4c^2d		
bc^2d^2	b^6cd		
c^4d	b^8d		
b^2c^3d			
bc^5			
b^3c^4			
b^5c^3			
b^7c^2			
b^9c			
b^{11}			

Each of these series is, taken by itself, perfect in arrangement, for each is in both alphabetical and counter-order; further, each is self-conjugate; that is to say, if a series contain m terms, the r^{th} and the $(m-r+1)^{\text{th}}$ terms from the top are conjugate to one another; this must of necessity be so, for the process is a conjugate one.

Being given the leading and ending terms of a seminvariant of weight 11, an inspection of these series indicates the only possible terms that can occur in it; for instance, consider the form, written for brevity, $fg + b^2d^3$; we have to take in each series those terms which at once coincide with or are posterior to fg in counter-order, and coincide with or are anterior to b^2d^3 in alphabetical order; in series 1, 2, 3, we take the terms from fg to b^2d^3, from cdg to bc^3e, from bc^2g to b^2c^2f respectively; or if we merely required this form by itself, we might start with fg and write down in succession the portions of the series above indicated.

The asyzygetic table for any weight formed on this principle consists of as many separate blocks or parts as there are series of terms; each block of necessity possesses the reversible symmetry of my original tables, and the leading and ending terms must be respectively at the head and at the foot of the corresponding column of some one of the blocks; these two indicator terms do not necessarily appear in the same block.

I annex the new tabulation of weight 8, in two portions, for the binomial quantic.

ASYZYGETIC SEMINVARIANTS

Weight 8.

		Col. $i+e^2$	Col. $cg+cd^2$	Col. $df+b^2d^2$	Col. e^2+c^4	Col. $ce^2+b^2c^3$	Col. $cd^2+b^4c^2$	Col. c^4+b^8
	i	+1						
	bh	−8						
	cg	+28	+1					
	df	−56	−3	+3				
	e^2	+35	+2	−2	+1			
	bde		−1	+1	−8			
	c^2e		−3	+18	+6	+1		
	cd^2		+2	−12		−1	+1	
Part 1.	b^2d^2			+10	+16	+1	−1	
	bc^2d				−24	+2	−6	
	c^4				+9	−1	+4	+1
	b^2c^3					+1	−7	+4
	b^4c^2						+8	+6
	b^6c							−4
	b^8							+1
	b^2g		−1					
	bcf		+3	−9				
	b^3f			+6				
Part 2.	b^2ce			−15		−2		
	b^4e					+1		
	b^3cd					−3	+10	
	b^5d						−4	

This table compares, I think, favorably with the preceding one in many ways; the only blank spaces that can occur must be consequent upon accidental zeros, so that the numbers are brought as close as possible together; should it be found necessary to tabulate weights higher than 12, the splitting up into blocks may be found of considerable advantage.

The minimum forms. By a minimum form I mean the form of lowest degree (*i. e.* the one possessing the earliest ending term in alphabetical order) that has a given leading term; or conversely, given a particular ending term, the minimum form is that one whose leading term is the latest possible in counter-order. In this investigation, for a given weight, we can by simple observation of the known results in the case of lower weights at once narrow the inquiry by imposing limits as regards the degree and weight of highest letter of the ending and leading terms that can be associated with given leading and ending terms respectively; weight 13 will be taken as an example of this, the limits of the forms being determined by means of those in my tables I to XII, which are indubitably correct.

I arrange the leading or non-unitary partition terms in counter-order, that is, according to weight of highest letter only; and by reason of the conjugate law, as well as from considerations which will appear in the sequel, it is natural to place the ending or power-ending partition terms in alphabetical order. Placing these in two vertical columns, columns of limits are ranged to the left and right of these on the following principle.

The limits. The form $(n + *)$ is obviously of degree $\not< 3$; pass on to form $(cl + *)$; by the first fundamental law '*ante,*' the derivative with respect to the highest letter l must be a seminvariant and cannot therefore be other than $(c + b^2)$; thus $(cl + *)$ degree $\not< 3$.

Similarly $(dk + *)$ degree $\not< 4$

$(ej + *)$ degree $\not< 3$

since there exists the minimum form $(e + c^2)$ in Table IV.

And so forth a limit of degree is assigned for the ending term associated with each leading term. For the column of ending terms we can assign a limit to the weight of the highest letter in the associated leading term; for by the second law '*ante,*' when each letter of an ending term is withdrawn one place in alphabetical order, it must be the ending term of a form of lower weight, and the whole of the literal terms of highest degree must be derived from some form of lower weight by advancing each letter therein one place in alphabetical sequence; for example, from the ending term bcf^2, we obtain abe^2, and since Table IX displays a minimum form $(j + be^2)$, the minimum character of the highest-degree terms in the form ending bcf^2 must be $(b^3k + bcf^2)$; that is,

$$(* + bcf^2) \text{ letter } \not< k.$$

These two conjugate processes may be shortly expressed as follows

Leading	Ending
Decapitate	Diminish
Complete	Complete
Deg. $\not<$ (deg. ending + 1).	Letter $\not<$ (letter of leading + 1)

thus:

cg^2	$b^3c^2d^2$
cg	b^3c^2
$cg + cd^2$	$de + b^3c^2$
Deg. $\not< 4$	Letter $\not< f$.

We thus have as under:

THE ALGORITHM $w = 13$.

Shown by	Degree	Lead term	End term	Letter	Shown by
	$\not< 3$	n	bg^2	$\not< n$	
$c + b^2$	$\not< 3$	cl	df^2	$\not< j$	$ci + ce^2$
$d + b^3$	$\not< 4$	dk	bcf^2	$\not< k$	$j + be^2$
$e + c^2$	$\not< 3$	ej	be^3	$\not< i$	$ch + d^3$
$c^2 + b^4$	$\not< 5$	c^2j	cde^2	$\not< h$	$dg + bcd^2$
$f + bc^2$	$\not< 4$	fi	b^3f^2	$\not< j$	$i + e^2$
$cd + b^5$	$\not< 6$	cdi	b^2de^2	$\not< h$	$cg + cd^2$
$\rho\theta - 2w \not< 0$	$\not< 4$	gh	bc^2e^2	$\not< g$	$df + b^2d^2$
$ce + c^3$	$\not< 4$	ceh	bd^4	$\not< g$	$\rho\theta - 2w \not< 0$
$d^2 + b^2c^2$	$\not< 5$	d^2h	c^2d^3	$\not< g$	$\rho\theta - 2w \not< 0$
$c^3 + b^6$	$\not< 7$	c^3h	b^3ce^2	$\not< i$	$h + bd^2$
$cf + bc^3$	$\not< 5$	cfg	b^2cd^3	$\not< g$	$cf + bc^3$
$de + b^3c^2$	$\not< 6$	deg	bc^3d^2	$\not< f$	$de + b^3c^2$
$c^2d + b^7$	$\not< 8$	c^2dg	b^5e^2	$\not< h$	$g + d^2$
$\rho\theta - 2w \not< 0$	$\not< 6$	df^2	b^4d^3	$\not< f$	$ce + c^3$
$\rho\theta - 2w \not< 0$	$\not< 6$	e^2f	$b^3c^2d^2$	$\not< e$	$d^2 + b^2c^2$
$c^2e + b^2c^3$	$\not< 6$	c^2ef	bc^6	$\not< e$	$\rho\theta - 2w \not< 0$
$cd^2 + b^4c^2$	$\not< 7$	cd^2f	b^5cd^2	$\not< g$	$f + bc^2$
$c^4 + b^8$	$\not< 9$	c^4f	b^3c^5	$\not< e$	$cd + b^5$
$cde + b^3c^3$	$\not< 7$	cde^2	b^7d^2	$\not< f$	$e + c^2$
$d^3 + b^5c^2$	$\not< 8$	d^3e	b^5c^4	$\not< d$	$c^2 + b^4$
$c^3d + b^9$	$\not< 10$	c^3de	b^7c^3	$\not< e$	$d + b^3$
$c^2d^2 + b^6c^2$	$\not< 9$	c^2d^3	b^9c^2	$\not< d$	$c + b^2$
	$\not< 13$	c^5d	b^{13}	$\not< d$	

It will be noted that in some cases closer limits are secured by the condition $\rho\theta - 2w \nless 0$ wherein $\rho =$ extent, $\theta =$ degree, $w =$ weight, and it does not appear whether such closer limits are absolutely necessary to the present scheme.

Now I suggest that, beginning from the top, the lead terms are to be paired off with the earliest end terms that satisfy the required conditions, as shown by the limits to the left and right of the columns, and that such an algorithm will give an existent, and furthermore, a minimum system of seminvariants.

I observe that the weights of the letters in the right-hand column of limits are the same as the numbers in the degree column in reverse order, and this is true universally from the conjugate nature of the process. For a like reason the joining lines present the same appearance when the table is inverted, and it is in the symmetry of the algorithm that the *à posteriori* probability of its truth lies. I proceed to its examination. Consider a weight w, supposed odd for convenience, and the cubic form

$$\phi_3 \equiv A_3 3^p 2^{\frac{1}{2}(w-3p)} + B_3 3^{p-2} 2^{\frac{1}{2}(w-3p+6)} + C_3 3^{p-4} 2^{\frac{1}{2}(w-3p+12)} + \ldots$$

in which the form $3^p 2^{\frac{1}{2}(w-3p)}$ is to be combined with forms of lower degree so that a maximum number of terms of higher extent shall vanish; we shall thus arrive at a minimum form whose ending term possesses a partition the conjugate of the partition $3^p 2^{\frac{1}{2}(w-3p)}$, we have to determine the coefficients A_3, B_3, $C_3 \ldots$ (suppose q_3 in number) that such may be the case, and we can only do so by finding $q_3 - 1$ independent linear relations between them.

We need only to consider the non-unitary terms in ϕ_3, since a form is completely given by its non-unitary portion, and we must express in succession the conditions that a_w, $a_2 a_{w-2}$, $a_3 a_{w-3} \ldots$ may be absent from ϕ_3; hence we must have $\frac{d\varphi}{da_w} = 0$, *i. e.* $d_w \phi = 0$ or $[w]\,\phi = 0$ where the portions in [ ] refer to the symmetric functions of the equation

$$1 - y^{-1}D_1 + y^{-2}D_2 - y^{-3}D_3 + \ldots = 0,$$

D_λ being as usual Mr. Hammond's operator.

In operating with $[w]$ on ϕ we are obviously only concerned with the cubic equation

$$1 + y^{-2}D_2 - y^{-3}D_3 = 0,$$

and we thus get one relation between A_3, B_3, C_3, ...; similarly operating with $[w - 2.2]$, $[w - 3.3]$, $[w - 4.4]$, $[w - 4.2^2]$, ... we obtain the relations that must be satisfied by the coefficients if the terms $a_2 a_{w-2}$, $a_3 a_{w-3}$, $a_4 a_{w-4}$, $a_2^2 a_{w-4}$, ... be not present.

The question that now arises is: How many of these operations $[w]$, $[w-2.2]\ldots$ are independent.

The Syzygies.

Consider now not the cubic forms but those of any degree θ; we have the equation of degree θ

$$x^\theta + a_2 x^{\theta-2} - a_3 x^{\theta-3} + \ldots \pm a_\theta = 0;$$

and its non-unitary symmetric functions of weight w $(w > \theta)$ whose partitions contain not more than θ parts; the linear relations connecting them are found by eliminating between them all the non-unitary terms which contain no part $> \theta$; the number of these syzygies will obviously be the excess of the number of non-unitary partitions of w having not more than θ parts over the number of non-unitary partitions having no part greater than θ; that is it will be

$$\text{Co. } a^\theta x^w \frac{1}{(1-a)(1-ax)\ldots(1-ax^w)} - \text{Co. } a^{\theta-1}x^{w-1}\frac{1}{(1-a)(1-ax)\ldots(1-ax^w)}$$
$$- \text{Co. } x^w \frac{1}{(1-x^2)(1-x^3)\ldots(1-x^\theta)}$$
$$= \text{Co. } x^w \frac{1}{(1-x)(1-x^2)\ldots(1-x^\theta)} - \text{Co. } x^{w-1}\frac{1}{(1-x)(1-x^2)\ldots(1-x^{\theta-1})}$$
$$- \text{Co. } x^w \frac{1}{(1-x^2)(1-x^3)\ldots(1-x^\theta)}$$
$$= \text{Co. } x^{w-\theta-1}\frac{1}{1-x.1-x^2.1-x^3\ldots 1-x^\theta},$$

that is to say, their number is equal to the number of partitions of $w-\theta-1$ having no part $> \theta$.

Turning again to the cubic forms and the cubic equation, we can easily form the actual syzygies, for by the Newtonian theorem of the sums of powers

$$(w) + c\,(w-2) - d\,(w-3) = 0,$$

and

$$(3) - 3d = 0$$
$$(2) + 2c = 0,$$

whence a first syzygy

$$(w) - 3\,(w-2.2) - 2\,(w-3.3) = 0, \qquad (1)$$

which, in the case of cubic forms, indicates a linear relation between the equations derived by means of the operators $[w]$, $[w-2.2]$, $[w-3.3]$; hence if a_w and $a_2 a_{w-2}$ be absent, so also must be $a_3 a_{w-3}$, which is obviously consistent with what has gone before, since we know that $a_3 a_{w-3}$ cannot be the lead term of a cubic form.

Again, clearly $\qquad (w-4)\{(4) + c\,(2)\} = 0,$

or

$$(w) - 2\,(w-2.2) + (w-4.4) - 2\,(w-4.2^2) = 0; \qquad (2)$$

the next syzygy, showing, what we know otherwise, that $a_2^2 a_{w-4}$ cannot be the lead term of a cubic form.

This second syzygy is derived from the first by putting $w = 4$ and then multiplying by $(w - 4)$.

The next two are obtained by putting $w = 5$ aud $= w - 2$ and multiplying by $(w - 5)$ and (2) respectively; thus

$$(w) - 5(w - 2.2) - 5(w - 3.3) + (w - 5.5) - 5(w - 5.3.2) = 0 \quad (3)$$

$$(w) - 2(w - 2.2) - 2(w - 3.3) - 3(w - 4.4) - 6(w - 4.2^2)$$
$$- 2(w - 5.5) - 2(w - 5.3.2) = 0 \quad (4)$$

which may be written

$$(w) - 5(w - 2.2) - 5(w - 3.3) + (w - 5.5) - 5(w - 5.3.2) = 0$$
$$(w) - 5(w - 4.4) - 10(w - 4.2^2) - 4(w - 5.5) = 0,$$

indicating that $a_2a_3a_{w-5}$ and a_5a_{w-5} are impossible lead terms for cubic forms.

Again in (1) putting $w = 6$ and $w - 3$ and multiplying respectively by $(w - 6)$ and (3) we obtain the syzygies showing that $a_2a_4a_{w-6}$ and $a_3^2a_{w-6}$ cannot be lead terms. Next

put in (1) $w = 7$ multiply by $(w - 7)$
$w = 5$ . . . . . . $(w - 7)(2)$
$w = 4$ . . . . . . $(w - 7)(3)$
$w = w - 4$ . . . (4)

(N. B.—We do not also multiply the latter by (2^2) by reason of the relation $(4) - 2(2^2) = 0$); between these four syzygies we eliminate the term $(w - 7.3.2^2)$ thus obtaining three significant syzygies showing the nullity of the terms a_7a_{w-7}, $a_2a_5a_{w-7}$, $a_3a_4a_{w-7}$.

This method is easily continued and forces on us the conclusion that cubic forms exist for each lead term a_0a_w, a_2a_{w-2}, a_4a_{w-4}, a_6a_{w-6}, . . . , taken in succession until all the cubic forms are exhausted; those terms of this type which remain over must belong to higher forms. The algorithm is therefore proved so far as regards cubic forms

Passing on now to quartic forms I discuss the linear relations connecting the non-unitary symmetric functions of the equation

$$x^4 + cx^2 - dx + e = 0$$

for a weight $w (> 4)$.

These, as before, will indicate the impossible lead terms of quartic seminvariants.

We have
$$(w) + c(w - 2) - d(w - 3) + e(w - 4) = 0 \quad (a)$$
$$(4) + c(2) + 4e = 0$$
$$(3) - 3d \quad = 0$$
$$(2) + 2c \quad = 0$$

whence a first syzygy

$$(w) - 18(w - 2.2) - 8(w - 3.3) - 9(w - 4.4) - 3(w - 4.2^2) = 0$$

or $a_2^2 a_{w-4}$ is not a lead term.

Put now in (α) $w = 5$ and multiply by $(w - 5)$, whence the next syzygy

$$(w) - 5(w - 2.2) - 5(w - 3.3) + (w - 5.5) - 5(w - 5.3.2) = 0$$

or $a_2 a_3 a_{w-5}$ is not a lead term; $a_5 a_{w-5}$ may be and will be if there be sufficient quartic forms left after the other prior possible lead terms have been utilized.

As before we next have a batch of two syzygies obtained by putting in (α)

$w = 6$ and multiply by $(w - 6)$
$w = w - 2$ and " (2);

without working these out it is easy to see that they will indicate syzygies corresponding to the terms $a_3^2 a_{w-6}$, $a_2^3 a_{w-6}$.

Again in (α) put $w = 7$ and multiply by $(w - 7)$
$w = 5$ " " $(w - 7)(2)$
$w = w - 3$ " " (3)

thus obtaining syzygies which negative $a_2 a_5 a_{w-7}$, $a_3 a_4 a_{w-7}$, $a_2^2 a_3 a_{w-7}$. Also put

$w = 8$ and multiply by $(w - 8)$
$w = 6$ " " $(w - 8)(2)$
$w = 5$ " " $(w - 8)(3)$
$w = w - 4$ " " (4)

and syzygies will result which negative $a_3 a_5 a_{w-8}$, $a_4^2 a_{w-8}$, $a_2^2 a_4 a_{w-8}$, $a_2 a_3^2 a_{w-8}$ as lead terms of quartic forms.

The syzygies are readily continued and material thus obtained for determining '*seriatim*' the lead terms, in counter order, which are associated with the end terms in alphabetical order.

No difficulty occurs in the treatment of the quintic and higher minimum seminvariant forms by the same method; the first syzygy in the case of forms of degree θ is obtained by the elimination of $c, d, e, \ldots, a_\theta$ between the equations

$$\begin{aligned}
(w) + c(w-2) - d(w-3) + e(w-4) - \ldots + (-)^\theta a_\theta (w-\theta) &= 0\\
(\theta) + c(\theta-2) - d(\theta-3) + e(\theta-4) - \ldots + (-)^\theta \theta a_\theta &= 0\\
(\theta-1) + c(\theta-3) - d(\theta-4) + e(\theta-5) - \ldots + (-)^{\theta-1} \theta - 1 . a_{\theta-1} &= 0\\
\ldots\ldots\ldots\ldots\ldots\ldots\ldots\ldots\ldots\ldots\ldots\ldots\ldots\ldots\ldots &\\
(3) - 3d &= 0\\
(2) + 2c &= 0
\end{aligned}$$

and the remaining syzygies in successive batches as before.

Cases occur in which one or more of the coefficients are arbitrary, and the form therefore is, strictly speaking, indeterminate; for example, take the form of weight 17,

$$\phi_4 \equiv A_4 (43 2^5) + B_4 (3^5 2) + C_4 (3^3 2^4) + D_4 (3 2^7);$$

the conditions that r and cp may be absent, give two relations between the coefficients A_4, B_4, C_4, D_4, and since the term '*en*' can also be made to vanish with '*do*' if $A_4 = 0$, it follows that the form $do + bch^2$ properly includes the form $en + fg^2$, and is therefore indeterminate; we can, however, agree to fix the form $do + bch^2$ by making r, cp and en vanish, thus forming three independent equations between the four coefficients; in general a form is fixed by setting apart the lead term and making a maximum number of the remaining non-unitary terms, in counter order, vanish.

The foregoing investigation so far agrees with the algorithm previously set forth, but is, however, insufficient to completely establish it; I do not at present see the way to effect this.

I have verified it as far as weight 17 inclusive.

Sec. 3. *The Calculation of Seminvariants.*

I indicate a method of calculating the covariant sources of the quantic

$$(1, 0, a_2, a_3, a_4, \ldots \between x, y)^n.$$

It is well known that if $\phi(a_2, a_3, a_4, \ldots)$ be any such source, $\phi(A_2, A_3, A_4, \ldots)$ will be the corresponding form for the quantic

$$(1, a_1, a_2, a_3, a_4, \ldots \between x, y)^n,$$

where

$$A_r = a_r - r a_1 a_{r-1} + \tfrac{1}{2} r(r-1) a_1^2 a_{r-2} - \ldots + (-)^r \tfrac{1}{2} r(r-1) a_1^{r-2} a_2 + (-)^{r+1} (r-1) a_1^r.$$

A_r being itself a source, ϕ may be any function whatever, or what is the same thing, ϕ being arbitrary, we have $\phi(a_2, a_3, a_4, \ldots)$ as the non-unitary part of a seminvariant: we can, by the proper determination of the arbitrary constants, reduce the degree of this seminvariant, viz.,

$$A_2 = a_2 - a_1^2$$

$$A_4 = a_4 - 4a_1 a_3 + 6a_1^2 a_2 - 3a_1^4,$$

$a_4 + ka_2^2$ is the non-unitary part of $a_4 - 4a_1a_3 + 6a_1^2a_2 - 3a_1^4 + k(a_2^2 - 2a_1^2a_2 + a_1^4)$ a seminvariant in general of degree 4, but which for $k = 3$, becomes $a_4 - 4a_1a_3 + 3a_2^2$ of degree 2.

Suppose $\phi = \Sigma L a_{\lambda_1} a_{\lambda_2} a_{\lambda_3} \ldots a_{\lambda_\kappa}$, a_{λ_1}, $a_{\lambda_2} \ldots$ being of course non-unitary, I proceed to find the relations that must exist between the L's, in order that the successive possible ending terms may vanish; these are power-ending terms, and

I will consider them arranged so that their conjugates are in counter-order. Thus

End term.	Conjugate.
a_1^w	a_w
$a_1^{w-4}a_2^2$	$a_{w-2}a_2$
$a_1^{w-6}a_2^3$	$a_{w-3}a_3$
$a_1^{w-8}a_2^4$	$a_{w-4}a_4$
$a_1^{w-6}a_3^2$	$a_{w-4}a_2^2$
$\vdots$	$\vdots$

What is done is in reality to establish a rule for the product of any number of terms of the series $A_2, A_3, \ldots$ and then to apply it to a seminvariant considered as a sum of any number of such products.

In order that a_1^w may vanish we must have

$$\Sigma L(-)^{\lambda_1+1}(\lambda_1-1)(-)^{\lambda_2+1}(\lambda_2-1)\ldots(-)^{\lambda_\kappa+1}(\lambda_\kappa-1)=0,$$

or say $$\Sigma(-)^\kappa \Lambda L = 0, \qquad (1)$$

wherein $$\Lambda = (\lambda_1-1)(\lambda_2-1)\ldots(\lambda_\kappa-1);$$

a relation that must be satisfied by the non-unitary portion of every seminvariant of degree $<$ weight.

The condition for the vanishing of the non-power-ending term $a_1^{w-2}a_2$ is similarly found to be $$\Sigma(-)^\kappa\Lambda\Sigma\lambda_1 L = 0,$$

which, since $\Sigma\lambda_1 = w$, is implied in the former, and so in every case it will be found, what we know otherwise, that we need only to consider the power-ending terms.

Let $[1^j]$ denote the sum of the products j together of the integers $\lambda_1, \lambda_2, \ldots \lambda_\kappa$, then for the vanishing of

$$a_1^{w-4}a_2^2 \text{ relation is } \Sigma(-)^\kappa\Lambda[1^2]L = 0 \qquad (2)$$

$$a_1^{w-6}a_2^3 \quad \text{"} \quad \Sigma(-)^\kappa\Lambda[1^3]L = 0 \qquad (3)$$

$$a_1^{w-8}a_2^4 \quad \text{"} \quad \Sigma(-)^\kappa\Lambda[1^4]L = 0 \qquad (4)$$

$$\vdots \qquad\qquad \vdots$$

$$a_1^{w-2j}a_2^j \quad \text{"} \quad \Sigma(-)^\kappa\Lambda[1^j]L = 0.$$

In the case $a_1^{w-6}a_3^2$ we have $\Sigma(-)^\kappa\Lambda\Sigma\lambda_1\lambda_2(\lambda_1-2)(\lambda_2-2)L=0$, which may be reduced to a very simple form if we suppose relations (1), (2), (3), (4) to be already satisfied; for

$$\Sigma\lambda_1\lambda_2(\lambda_1-2)(\lambda_2-2) = [1^2]^2 - 2[1][1^3] + 2[1^4] - 2[1][1^2] + 6[1^3] + 4[1^2],$$

$$= [1^2]^2 - 2w[1^3] + 2[1^4] - 2w[1^2] + 6[1^3] + 4[1^2],$$

or we have $$a_1^{w-6}a_3^2 \text{ relation is } \Sigma(-)^\kappa\Lambda[1^2]^2L = 0, \qquad (5)$$

or this implies that the terms a_1^w, $a_1^{w-4}a_2^2$, $a_1^{w-6}a_2^3$, $a_1^{w-8}a_2^4$, being absent, so also must be $a_1^{w-6}a_3^2$.

So again for $a_1^{w-8}a_2a_3^2$, we have

$$\Sigma(-)^\kappa \Lambda \Sigma\lambda_1\lambda_2\lambda_3(\lambda_1-2)(\lambda_2-2)L=0,$$

and $$\begin{aligned}\Sigma\lambda_1\lambda_2\lambda_3(\lambda_1-2)(\lambda_2-2) &= \Sigma\lambda_1\lambda_2(\lambda_1-2)(\lambda_2-2)(w-\lambda_1-\lambda_2)\\ &\equiv \Sigma\lambda_1\lambda_2(\lambda_1-2)(\lambda_2-2)(\lambda_1+\lambda_2) \text{ from (5)}\\ &\equiv [32]-2[31]-4[2^2]+4[21]\\ &\equiv [1^2][1^3] \text{ by (1), (2), (3), (4) and (5).}\end{aligned}$$

The condition that $a_1^{w-8}a_2a_3^2$ may vanish, as well as those before considered, is consequently $a_1^{w-8}a_2a_3^2$ $\quad \Sigma(-)^\kappa\Lambda[1^2][1^3]L=0.$ (6)

From an examination of the method of obtaining the above relations the following general theorem is seen to be true.

Theorem.—"If $\Sigma La_{\lambda_1}a_{\lambda_2}\ldots a_{\lambda_\kappa}$ be the non-unitary portion of a seminvariant whose developed expression contains no term, the conjugate of which is prior in counter-order to the non-unitary term $a_\alpha a_\beta^b a_\gamma^c a_\delta^d\ldots$; then the condition that the term conjugate to $a_\alpha a_\beta^b a_\gamma^c a_\delta^d\ldots$ may also be absent is

$$\Sigma(-)^\kappa\Lambda[1^\beta]^b[1^\gamma]^c[1^\delta]^d\ldots L=0$$

wherein $\Lambda=(\lambda_1-1)(\lambda_2-1)\ldots(\lambda_\kappa-1)$, and $[1^j]$ is the sum of the products j together of the integers $\lambda_1, \lambda_2, \ldots \lambda_\kappa$."

This theorem obviously enables the calculation of any seminvariant from its non-unitary portion by the formation of a sufficient number of relations between its numerical coefficients, *ex. gr.* there is a quartic invariant containing the non-unitary terms a_2a_4, a_3^2, a_2^3, or say the portion in question is

$$Aa_2a_4+Ba_3^2+Ca_2^3.$$

Hence for

a_1^6 $\quad (2-1)(4-1)A+(3-1)(3-1)B-(2-1)(2-1)(2-1)C=0$ or $3A+4B-C=0$

$a_1^2a_2^2$ $\quad 8$ " $+9$ " -12 " $=0$ or $24A+36B-12C=0$

or if $A=1$, $B=C=-1$,

and there results the portion $a_2a_4-a_3^2-a_2^3$ of the complete form

$$a_2a_4-a_3^2-a_1^2a_4+2a_1a_2a_3-a_2^3;$$

the meaning is that we so determine A, B, C, that

$$\begin{aligned}&A(a_2-a_1^2)(a_4-4a_1a_3+6a_1^2a_2-3a_1^4)\\ &+B(a_3-3a_1a_2+2a_1^3)^2\\ &+C(a_2-a_1^2)^3,\end{aligned}$$

may be without the terms a_1^6, $a_1^2a_2^2$.

Sec. 4. *General form of a Seminvariant.*

For the quantic of order ρ, a seminvariant is of the form

$$u = \alpha\rho^\varkappa + \beta\rho^{\varkappa-1} + \gamma\rho^{\varkappa-2} + \delta\rho^{\varkappa-3} + \dots + J,$$

(ρ being the letter of weight ρ) wherein $\alpha, \beta, \gamma, \delta, \dots J$ are functions of the literal coefficients which do not involve ρ; forming the successive derivatives with regard to ρ, which are of course seminvariants, we have

$$\frac{du}{d\rho} = \varkappa\alpha\rho^{\varkappa-1} + (\varkappa-1)\beta\rho^{\varkappa-2} + (\varkappa-2)\gamma\rho^{\varkappa-3} + (\varkappa-3)\delta\rho^{\varkappa-4} + \dots$$

$$\frac{d^2u}{d\rho^2} = \varkappa(\varkappa-1)\alpha\rho^{\varkappa-2} + (\varkappa-1)(\varkappa-2)\beta\rho^{\varkappa-3} + (\varkappa-2)(\varkappa-3)\gamma\rho^{\varkappa-4} + (\varkappa-3)(\varkappa-4)\delta\rho^{\varkappa-5} + \dots$$

$$\vdots \qquad\qquad\qquad \vdots$$

$$\frac{d^{\varkappa-1}u}{d\rho^{\varkappa-1}} = \frac{\varkappa!}{1!}\alpha\rho + (\varkappa-1)!\,\beta$$

$$\frac{d^\varkappa u}{d\rho^\varkappa} = \varkappa!\,\alpha.$$

From these $\varkappa$ equations are deduced

$$\varkappa!\,\alpha = \frac{d^\varkappa u}{d\rho^\varkappa}$$

$$(\varkappa-1)!\,\beta = \frac{d^{\varkappa-1}u}{d\rho^{\varkappa-1}} - \frac{d^\varkappa u}{d\rho^\varkappa}\rho$$

$$(\varkappa-2)!\,\gamma = \frac{d^{\varkappa-2}u}{d\rho^{\varkappa-2}} - \frac{d^{\varkappa-1}u}{d\rho^{\varkappa-1}}\rho + \frac{1}{2!}\frac{d^\varkappa u}{d\rho^\varkappa}\rho^2$$

and generally

$$(\varkappa-t+1)!\,\alpha_t = \frac{d^{\varkappa-t+1}u}{d\rho^{\varkappa-t+1}} - \frac{d^{\varkappa-t+2}u}{d\rho^{\varkappa-t+2}}\rho + \frac{1}{2!}\frac{d^{\varkappa-t+3}u}{d\rho^{\varkappa-t+3}}\rho^2 - \dots + (-)^{t-1}\frac{1}{(t-1)!}\frac{d^\varkappa u}{d\rho^\varkappa}\rho^{t-1}.$$

Substituting these values in the expression for u there results on reduction,

$$(-)^{\varkappa+1}\varkappa!\,u = \frac{d^\varkappa u}{d\rho^\varkappa}\rho^\varkappa - \varkappa\frac{d^{\varkappa-1}u}{d\rho^{\varkappa-1}}\rho^{\varkappa-1} + \varkappa(\varkappa-1)\frac{d^{\varkappa-2}u}{d\rho^{\varkappa-2}}\rho^{\varkappa-2} - \dots + (-)^s\frac{\varkappa!}{(\varkappa-s)!}\frac{d^{\varkappa-s}u}{d\rho^{\varkappa-s}}\rho^{\varkappa-s} + \dots + (-)^{\varkappa+1}\varkappa!\,J,$$

in which $s \not> \varkappa - 1$.

In this form every seminvariant to the ρ^{ic} may be expressed.

The result may also be written as follows:

$$\left(\frac{d^\varkappa u}{d\rho^\varkappa}, -1!\frac{d^{\varkappa-1}u}{d\rho^{\varkappa-1}}, +2!\frac{d^{\varkappa-2}u}{d\rho^{\varkappa-2}}, \dots (-)^\varkappa\varkappa!\,u \between \rho, 1\right)^\varkappa + (-)^{\varkappa+1}\varkappa!\,J = 0,$$

J being that part of u not involving ρ.

Royal Military Academy, Woolwich, England, June 16, 1885.

Syzygy Tables for the Binary Quintic.

BY J. HAMMOND.

In his Tenth Memoir on Quantics (Phil. Trans., Part II, 1878, pp. 603-661) Prof. Cayley has given an enumeration of the irreducible syzygies for the Binary Quintic, and an incomplete calculation of their values, not extending beyond the deg. order 14.4. Table I of the present paper is meant to replace his enumeration (from which it differs by the omission of twelve syzygies) or that of Prof. Sylvester (*American Journal of Mathematics*, Vol. IV, p. 58). The letters a, b, $c, \ldots w$ denote the same system of 23 groundforms as in the memoir referred to, the numbers show the number of irreducible syzygies for any deg. order. The calculated values of all the irreducible syzygies will be found in Table II.

I hope on some future occasion to prove that my enumeration is the correct one, and shall content myself, for the present, with proving that the syzygy named b^5 by Prof. Cayley is reducible.

This syzygy is

$$-8b^5 + a^2q + 10ab^2j - 9abdg + 3ahj + 12b^3h + 32bcm - 12cdj - 4cgh = 0$$

say $\Sigma = 0$.

Copying from Table II the syzygies named d^2, dh, h^2, and dm, we have

$$(d^2) = aj - b^3 + 2bh - cg - 9d^3$$
$$(dh) = am + 2b^2d + cj - 3dh$$
$$(h^2) = adg - bcg - 12bd^3 - 4cm + h^2$$
$$(dm) = ag + 2b^2j - bdg - hj + 12dm.$$

Whence it is easily verified that

$$\Sigma = a\,(dm) + 8b^2\,(d^2) - 8b\,(h^2) - 12d\,(dh) + 4h\,(d^2).$$

The remaining eleven syzygies have not been calculated by Prof. Cayley.

It may be noticed that the names of the syzygies in Table II are all of them binary combinations of the groundforms, and it has been found (in every case that I have considered) that every irreducible syzygy contains at least one such binary combination among its terms. But I have no proof of this.

Table I.

Order in the Variables.

Degree in the Coefficients	0	1	2	3	4	5	6	7	8	9	10	11	12	13	14	15	16	18	Degree in the Coefficients
1						*a*													1
2			*b*				*c*												2
3				*d*		*e*				*f*									3
4	*g*				*h*		*i*												4
5		*j*		*k*				*l*				1							5
6			*m*		*n*		1		1		1		1		1			1	6
7		*o*				*p*		1		3		1		1		1			7
8	*q*		*r*				2		2		3		3				1		8
9				*s*		1		5		2		2		2					9
10					1		3		4		4				2				10
11		*t*				4		3		3		3							11
12	*u*				2		4		5				2						12
13		*v*		2		3		3		3									13
14			1		2		6				3								14
15				2		3		4		1									15
16					5		2		2		1								16
17				2		3		2											17
18	*w*		2		2		2		1										18
19				2		3													19
20			2		1		2												20
21				3		1				1									21
22			1		2		1												22
23		1		1				1											23
24			2		1														24
25		1				1													25
26			2																26
27				1															27
29		1																	29
31		1																	31
36	1																		36
	0	1	2	3	4	5	6	7	8	9	10	11	12	13	14	15	16	18	

Order in the Variables.

TABLE II.

THE SYZYGIES OF THE QUINTIC.

Deg. Order.		Name.
5.11	$ai + bf - ce = 0$	ce
6.6	$aj - b^3 + 2bh - cg - 9d^2 = 0$	d^2
6.8	$ak + 2bi - 3de = 0$	de
6.10	$a^2g - 12abd - 4b^2c - e^2 = 0$	e^2
6.12	$al - 2ci + 3df = 0$	df
6.14	$a^2h - 6acd - 4bc^2 - ef = 0$	ef
6.18	$a^3d - a^2bc + 4c^3 + f^2 = 0$	f^2
7.7	$am + 2b^2d + cj - 3dh = 0$	dh
7.9	$bl + ck - 3di = 0$	di
7.9	$an - b^2e + 2bl + eh = 0$	eh
7.9	$4bl + 2ck - eh + fg = 0$	fg
7.11	$abh - acg + 6bcd + ei = 0$	ei
7.13	$ade - bce + 2cl + fh = 0$	fh
7.15	$a^2bd - ab^2c + ach - 6c^2d - fi = 0$	fi
8.6	$bn - ej + gi + 3dk = 0$	dk
8.6	$ao - 2bn - ej = 0$	ej
8.8	$2b^2h + bcg + 12cm - 3h^2 - ek = 0$	ek
8.8	$adg - bcg - 12bd^2 - 4cm + h^2 = 0$	h^2
8.10	$abk + ap + bde - fj - 6dl = 0$	dl
8.10	$ap + 2cn + fj = 0$	fj
8.10	$bde - cn - hi = 0$	hi
8.12	$3adh + 2bch - 2c^2g - 18cd^2 + el = 0$	el
8.12	$el - 2i^2 + fk = 0$	fk
8.12	$ab^2d - acj + 9cd^2 + i^2 = 0$	i^2
8.16	$3a^2d^2 - abcd - 2b^2c^2 + 2c^2h - fl = 0$	fl
9.5	$aq + 2b^2j - bdg - hj + 12dm = 0$	dm
9.7	$ar + ij + 3dn = 0$	dn
9.7	$ar - b^2k - bp - em = 0$	em
9.7	$co + 6dn + 3em + gl = 0$	gl
9.7	$co + em - hk = 0$	hk
9.7	$bp + co - ij = 0$	ij
9.9	$ab^2g + 6abm - agh + 2bcj - en = 0$	en
9.9	$abm + 3bcj - 3cdg + ik = 0$	ik
9.11	$adk + bck + cp + fm = 0$	fm
9.11	$bck + 2cp + 3d^2e - hl = 0$	hl
9.13	$bei - de^2 - 2il + fn = 0$	fn
9.13	$3abd^2 - b^2cd - 2c^2j + 3cdh + il = 0$	il

TABLE II—Continued.

Deg. Order.		Name.
10.4	$2br - jk + 3do = 0$	do
10.6	$agj + 2b^3g + 12b^2m - 12bdj - 2bgh - eo = 0$	eo
10.6	$2b^2m - 3cq - k^2 - 3hm = 0$	hm
10.6	$4b^2m + 12bdj - 9d^2g + k^2 = 0$	k^2
10.8	$as - 3bdk - 2im - 3dp = 0$	dp
10.8	$2bdk + bgi - deg + 4im + hn = 0$	hn
10.8	$bdk + cr - im = 0$	im
10.8	$2cr - 3dp + jl = 0$	jl
10.10	$bek + in - kl + ep = 0$	ep
10.10	$ep + 2in + fo = 0$	fo
10.10	$abdg + 3adm - ahj + 3cdj - in = 0$	in
10.10	$3adm - 4bcm - 3cdj + kl = 0$	kl
10.14	$bfk - bi^2 + dei + l^2 + fp = 0$	fp
10.14	$9ad^3 + 3bcd^2 + 4c^2m + l^2 = 0$	l^2
11.5	$bs + km - 3dr = 0$	dr
11.5	$b^2o - 6dr - gp - 3jn - 2ho = 0$	ho
11.5	$b^2o + 2bgk - 6bs - 3ho - 3jn = 0$	jn
11.5	$bgk + eq + gp + 2jn + 6km = 0$	km
11.7	$io + kn + er = 0$	er
11.7	$agm - 6bdm + 3bhj - 3dgh + io = 0$	io
11.7	$b^2dg + 12bdm + 2bhj - 3dgh - kn = 0$	kn
11.9	$ajk - bem - 4dej + 2dgi - 2lm + fq = 0$	fq
11.9	$ado + bem + bhk - 2lm + hp = 0$	hp
11.9	$cs + 3d^2k - lm = 0$	lm
11.11	$bik + dek + ip + fr = 0$	fr
11.11	$abdj - b^2cj + 2bcdg + 6cdm - chj + ip = 0$	ip
11.11	$fr - ip - ln = 0$	ln
12.4	$2b^2q + 2bgm + 4bj^2 - 3dgj + ko = 0$	ko
12.4	$bgm + 3hq + ko + 12m^2 = 0$	m^2
12.6	$bdo - 3bjk + 2iq - 2jp + 6ds = 0$	ds
12.6	$bdo + mn + hr = 0$	hr
12.6	$at - bjk + 2mn - jp = 0$	jp
12.6	$bjk - dgk - iq + mn = 0$	mn
12.8	$bk^2 + 2ir + kp - es = 0$	es
12.8	$kp - lo - 2ir = 0$	ir
12.8	$ajm - 2bcq - 2cgm - cj^2 + kp = 0$	kp
12.8	$6bhm - 2cgm - 18d^2m + 3dhj - lo = 0$	lo
12.8	$beo - egk + 3es - 3lo - 3n^2 = 0$	n^2
12.12	$bkl + 2dik + lp + fs = 0$	fs
12.12	$3ad^2j + 2b^2cm + 3bcdj + 2chm + lp = 0$	lp

TABLE II—Continued.

Deg. Order.		Name.
13.3	$2bt - gs + kq + 2jr = 0$	jr
13.3	$bt - jr + mo = 0$	mo
13.5	$bdq + 3bjm - 4dgm - dj^2 - 3au = 0$	au
13.5	$3bdq - 2bjm + 3dgm - kr = 0$	kr
13.5	$3b^2gj - 2bdg^2 + 6bdq + 12bjm - ghj - no = 0$	no
13.7	$bkm - 3d^2o + 2mp - hs = 0$	hs
13.7	$3bkm - ct + 3mp + lq = 0$	lq
13.7	$ct + djk - mp = 0$	mp
13.9	$dk^2 - lr + is = 0$	is
13.9	$3b^2dm + 3bd^2j - 2cjm + 3dhm + lr = 0$	lr
13.9	$ber - bnk + deo - 2lr - np = 0$	np
14.2	$4bgq - 36bu - gj^2 + o^2 = 0$	o^2
14.4	$js + 2mr - 3dt = 0$	dt
14.4	$bgr + 2bjo - gjk + 3nq - 12mr = 0$	mr
14.6	$2b^2q + b^2gm + bhq + bko + et - av = 0$	av
14.6	$bm^2 + d^2q + djm + cu = 0$	cu
14.6	$bko - 2nr + op - et = 0$	et
14.6	$4bm^2 + 9d^2q + 6djm - ks = 0$	ks
14.6	$2bko + gk^2 - 3ks + 3po - 3nr = 0$	nr
14.6	$4bhq + 2bko + 12bm^2 + 6djm - hj^2 + op = 0$	op
14.10	$3akr + bes + 6dkn - 5ekm - 6ls - 6ft = 0$	ft
14.10	$15bd^2m + 4cm^2 + 9d^3j + ls = 0$	ls
14.10	$bkp - 2dkn + ft + p^2 = 0$	p^2
15.3	$bjq - 6dgq + gjm + 2j^3 + or + 54du = 0$	du
15.3	$2bjq - bv + gjm - or = 0$	or
15.5	$bmo - dgr - djo - gkm - 3eu = 0$	eu
15.5	$2b^2t - 2dgr - 2djo - pq - ht = 0$	ht
15.5	$bmo + 3djo + 2gkm + 3ht - 6ms = 0$	ms
15.7	$6dhq + 3dko + 18dm^2 + 3hjm + 2cjq - cv = 0$	cv
15.7	$dko - pr + it = 0$	it
15.7	$bkr + it + pr + ns = 0$	ns
15.7	$5b^2jm + 3bdj^2 - jk^2 - 3cv + 3pr = 0$	pr
15.9	$adt + 3amr - 2diq + 6dmn - 3em^2 - 3fu = 0$	fu

TABLE II—Continued.

Deg. Order.		Name.
16.4	$3djq + 2j^2m + 2r^3 - 3dv = 0$	dv
16.4	$6b^2u + 3bmq + djq - 2dv - 3hu = 0$	hu
16.4	$6b^2u + 2bmq - 3dv + kt = 0$	kt
16.4	$kt - 2r^3 - os = 0$	os
16.4	$3b^2u + 2bmq + gm^2 - r^3 = 0$	r^3
16.6	$agt - 4bjs - 20bmr + 2bnq - ev = 0$	ev
16.6	$bdt + bmr + dkq + jkm + 3iu = 0$	iu
16.8	$9adu + 2b^2m^2 - 3bd^2q + 2cmq + 2k^2m - lt = 0$	lt
16.8	$2dkr - lt + ps = 0$	ps
16.10	$aht - 6bdkm - 18dmp + 2fjq - 2hiq - fv = 0$	fv
17.3	$boq + gjr + gmo - 2gkq + 2j^2o + 18ku = 0$	ku
17.3	$3ku - qs - 2mt = 0$	mt
17.5	$2b^2jq - b^2v - 2bdgq + 18bdu + nt + hv = 0$	hv
17.5	$bor - 2gkr - jko + 6rs + 3nt = 0$	nt
17.5	$9bdu + 3dmq + 2jm^2 - rs = 0$	rs
17.7	$2ajt - 18d^2t - 18dmr + 6dnq + ijq - iv = 0$	iv
17.7	$d^2t + dmr - km^2 + lu = 0$	lu
18.2	$4gmq + 2j^2q - 36mu - ot - jv = 0$	jv
18.2	$3bgu - bq^2 - jv + ot = 0$	ot
18.4	$4bjt - 2bqr - 3dgt + kv = 0$	kv
18.4	$dgt + doq + 2gmr + 2jmo - 6nu = 0$	nu
18.6	$2bos + 3br^2 - 3dor + gks - 3s^2 + 3pt = 0$	pt
18.6	$27d^2u - 4m^3 - s^2 = 0$	s^2
18.8	$bdkq + 9deu - 4imq + 6m^2n - lv = 0$	lv
19.3	$3bju - rt + mv = 0$	mv
19.3	$6bju - 9dgu + 3dq^2 + 2jmq + 2rt = 0$	rt
19.5	$2j^2s + 10jmr - jnq - nv + 6aw = 0$	aw
19.5	$3bmt + djt + 2dqr + 3pu = 0$	pu
19.5	$b^2gt + b^2qo + 12bmt - ght - hoq - jnq - nv = 0$	nv
20.2	$6bw + gjt - 2gqr - joq + 18ru = 0$	ru
20.2	$12bw + gjt - ov = 0$	ov
20.4	$6bmu + 9dju + 2m^2q - st = 0$	st
20.6	$3bjmo + 5dj^2o + 2dgjr + 2jpq - pv - 12cw = 0$	cw
20.6	$5bjmo + 3dj^2o - 9giu + 3iq^2 - 2kor + 3pv = 0$	pv

TABLE II—Continued.

Deg. Order.		Name.
21.3	$2j^2t - jqr + rv + 18dw = 0$	dw
21.3	$3bou - 2bqt - gmt + rv = 0$	rv
21.3	$4bou - 12dw - gku - gqs + 2moq + 12su = 0$	su
21.5	$gnt - 2jos - 10mor + noq - 6ew = 0$	ew
21.9	$bioq + 2crt - deoq - 6dkmo - 6dpt - hnt + 6fw = 0$	fw
22.2	$3gmu + 3j^2u - mq^2 - t^2 = 0$	t^2
22.4	$3dgoq - 27dou + g^2mr + 2gjmo - 6jmt - o^2r - 3mqr - 9hw = 0$	hw
22.4	$6bru + 3dqt + 2jmt - sv = 0$	sv
22.6	$3bmo^2 + 5djo^2 + 2dgor - gpt + 2poq - 12iw = 0$	iw
23.1	$3gou - 4gqt - oq^2 + 36tu - 12jw = 0$	jw
23.3	$grt + 2jot - oqr + 6kw = 0$	kw
23.7	$bmor + j^2kr + 2jpt - rht + 6lw = 0$	lw
24.2	$3jou - 2jqt + vt + 12mw = 0$	mw
24.2	$3gru + 3jou - q^2r - tv = 0$	tv
24.4	$g^2r^2 + 2gjor + j^2o^2 - 6jrt - 3qr^2 + 9nw = 0$	nw
25.1	$3g^2ju - gjq^2 - 4gqv + 36uv - 12ow = 0$	ow
25.5	$bgrt + bjot - gmor - jmo^2 - jst + mrt + 3nou - 3pw = 0$	pw
26.2	$gt^2 + 3uo^2 - 2qot + 12rw = 0$	rw
26.2	$3gj^2u - 2jqv + 12rw + v^2 = 0$	v^2
27.3	$2jt^2 - 3rou + qrt + 6sw = 0$	sw
29.1	$6gjqu + 3guv - 108ju^2 + 2jq^3 - q^2v + 12tw = 0$	tw
31.1	$3g^2tu - gq^2t + 6goqu + 2q^3o - 108u^2o + 12vw = 0$	vw
36.0	$g^3u^2 + 2g^2q^2u + gq^4 - 72gqu^2 - 8q^3u + 432u^3 - 16w^2 = 0$	w^2

Prüfung grösserer Zahlen auf ihre Eigenschaft als Primzahlen.

(Fortsetzung von Vol. VII, Nro. 3.)

Von P. Seelhoff.

Als Fortsetzung meiner Ausführungen in Nro. 3 gebe ich zunächst eine tabellarische Zusammenstellung von 192 negativen Determinanten. Die Tabelle besteht aus 3 Kolonnen. Die erste mit der Aufschrift D giebt die Determinanten selbst, die zweite die daraus abgeleiteten Hauptformen (Formae principales) und die dritte den Gesammtcharakter dieser Formen. Als wesentlichste und für den vorliegenden Zweck nothwendige Eigenschaft dieser Hauptformen soll noch einmal hervorgehoben werden, dass neben ihnen keine anderen reducirten Formen mit demselben Gesammtcharakter bestehen. Die Bezeichnung der einzelnen Charaktere $3n+r$, $3n-r$, etc. bedeutet, dass die Zahl N, welche dargestellt werden soll, für 3 etc. quadratischer Rest, resp. Nichtrest ist. Haben mehrere Formen denselben Gesammtcharakter, so ist dieser nur bei der ersten Form ausführlich angegeben.

Eine auch für die Zahltypen $8n+3$, $8n+5$ und $8n+7$ durchgeführte Sonderung der Formen erschien mir nicht zweckmässig, weil hierdurch zu viele Wiederholungen nothwendig würden, und wenn ich nach eigener Erfahrung urtheilen darf, so ist die jetzt gewählte Zusammenstellung für den Gebrauch von hinreichender Uebersichtlichkeit.

Ich möchte dies an einem Beispiele nachweisen, welches mir zugleich Gelegenheit geben wird, das in dem früheren Aufsatze über das Verfahren Beigebrachte in einzelnen Punkten zu ergänzen.

Herr J. W. L. Glaisher in Cambridge äussert sich in einem seiner Aufsätze, wie folgt: "The process of determining without a table the factors of a number is excessively laborious. Thus to determine, for example, whether the number 8559091 is or not a prime would require a long day's work." Sehen wir zu!

Es findet sich sofort, dass diese Zahl N quadratischer Rest für 3, 5, 7, 17, 19, 23, dagegen Nichtrest für 11, 13, 29, 31 ist. Unter den Formen, für welche sich einige dieser Charaktere in Verbindung mit dem besonderen Charakter $8n+3$ vereinigt finden, wähle ich die beiden Hauptformen (15, 0, 364) und (60, 0, 91) der Determinante -5460, für welche mein hat $3n+r$, $5n+r$, $7n+r$, $13n-r$; $8n+3$, 7,

1. $$15a + 364b = N$$

Da $N \equiv 1\,(15)$, $364 \equiv 4\,(15)$ und $4.4 \equiv 1\,(15)$ ist, so subtrahire ich 364.4 von N und dividire durch 15. Dann ist

$$a = 575509 - 364k$$
$$b = 4 + 15k.$$

Ferner hat man $575509 \equiv 4\,(7)$, $\equiv 12\,(13)$; setzt man demnach

$$15x^2 + 364y^2 = N,$$

so ist $x = 4t \pm 1 = 7u \pm 2 = 13v \pm 5$ und weil $\sqrt{575509} = 758$ die obere Grenze bildet $x =$ 5, 47, 135, 177, 187, 229, 317, 359, 369, 411, 499, 541, 551, 593, 681, 723, 733.

Leitet man hieraus mit Hülfe von Quadratzahlentafeln k und in zweiter Linie b ab, so wird letzeres für keinen dieser Werte ein Quadrat.

2. $$60a + 91b = N.$$

Aehnlich wie oben findet man

$$a = 142650 - 91k$$
$$b = 1 + 60k$$

$x =$ 12, 40, 51, 79
103, 131, 142, 170
194, 222, 233, 261
285, 313, 324, 362
376.

Nur für $x = 376$ ist b oder $y^2 =$ einem Quadrate, nämlich $841 = 29^2$. Also ist $8559093 = 60.376^2 + 91.29^2$ eine Primzahl.

Ich muss ferner auf das in dem ersten Artikel gesagte noch in einer anderen Beziehung zurückkommen. An ein oder zwei Stellen, wo es sich um mehrere Hauptformen mit gemeinschaftlichem Gesammtcharakter handelt, war meine Ausdrucksweise vielleicht nicht genau genug, ich gestatte mir deshalb, um jedem Misverständnisse vorzubeugen, noch einige Worte.

Aus der Determinante —1012 lassen sich, um ein Beispiel anzuführen folgende Hauptformen ableiten.

$$\begin{Bmatrix} 1, & 0, & 1012 \\ 4, & 0, & 253 \end{Bmatrix} \text{ und } \begin{Bmatrix} 11, & 0, & 92 \\ 23, & 0, & 44 \end{Bmatrix}.$$

Die beiden Formen in der ersten Klammer haben denselben Gesammtcharakter, ebenso die beiden in der zweiten Klammer. Nennt man nun nach Gauss einen solchen Complex von Formen mit demselben Gesammtcharakter eine Gattung (genus), so ergiebt sich für sämmtliche 192 Determinanten folgendes: Erhält man aus *sämmtlichen* Formen einer Gattung nur eine Darstellung der Zahl N, so ist diese eine Primzahl; erhält man keine Darstellung, so ist N zusammengesetzt. Erhält man mehr als eine Darstellung für eine und dieselbe Form oder auch für mehrere der Gattung angehörige, so lässt sich aus einem Paare solcher Darstellungen ein Paar Faktoren von N ableiten.

Wie dies geschehen kann für den Fall, dass beide Darstellungen durch dieselbe Form gebildet werden, habe ich in dem ersten Artikel nach dem Vorgange Eulers gezeigt; es bleibt also noch der andere Fall übrig, dass zwei verschiedene Formen zur Darstellung dienen. Sei also für die Determinante $-mnp$

$$mnx^2 + py^2 = N \quad (1)$$
$$m\xi^2 + np\eta^2 = N \quad (2)$$

so multiplicire man (2) mit nx^2 und (1) mit ξ^2, dann ist

$$mn(\xi x)^2 + p(\xi y)^2 = \xi^2 N$$
$$mn(\xi x)^2 + p(n\eta x)^2 = nx^2 N$$

und hieraus

$$p\{(\xi y)^2 - (n\eta x)^2\} = (\xi^2 - nx^2)N.$$

Bestimmt man jetzt noch den grössten gemeinschaftlichen Divisor von $\xi y + n\eta x$ oder $\xi y - n\eta x$ und N, so ist dieser einer der gesuchten Divisoren.

BREMEN im Mai 1885.

D	Formen.	Charaktere.	D	Formen.	Charaktere.
— 1	1, 0, 1	8n+1, 5	— 42	6, 0, 7	3n+r, 7n—r; 8n+5, 7
— 2	1, 0, 2	8n+1, 3	— 45	1, 0, 45	3n+r, 5n+r; 8n+1, 5
— 3	1, 0, 3	3n+r; 8n+1, 3, 5, 7		5, 0, 9	3n—r, 5n+r; 8n+1, 5
— 4	1, 0, 4	8n+1, 5	— 46	1, 0, 46	23n+r; 8n+1, 7
— 5	1, 0, 5	5n+r; 8n+1, 5		2, 0, 23	" ; "
— 6	1, 0, 6	3n+r; 8n+1, 7	— 48	1, 0, 48	3n+r; 8n+1
	2, 0, 3	3n—r; 8n+3, 5		3, 0, 16	3n+r; 8n+3
— 7	1, 0, 7	7n+r; 8n+1, 3, 5, 7	— 55	1, 0, 55	5n+r, 11n+r; 8n+1, 3, 5, 7
— 8	1, 0, 8	8n+1,		5, 0, 11	" ; "
9	1, 0, 9	3n+r; 8n+1, 5	— 57	1, 0, 57	3n+r, 19n+r; 8n+1, 5
—10	1, 0, 10	5n+r; 8n+1, 3		3, 0, 19	3n+r, 19n—r; 8n+3, 7
	2, 0, 5	5n—r; 8n+5, 7	— 58	1, 0, 58	29n+r; 8n+1, 3
—12	1, 0, 12	3n+r; 8n+1, 5		2, 0, 29	29n—r; 8n+5, 7
	3, 0, 4	3n+r; 8n+3, 7	— 60	1, 0, 60	3n+r, 5n+r; 8n+1, 5
—13	1, 0, 13	13n+r; 8n+1, 5		3, 0, 20	3n—r, 5n—r; 8n+3, 7
—14	1, 0, 14	7n+r; 8n+1, 7		4, 0, 15	3n+r, 5n+r; 8n+3, 7
	2, 0, 7	" ; "		5, 0, 12	3n—r, 5n—r; 8n+1, 5
—15	1, 0, 15	3n+r, 5n+r; 8n+1, 3, 5, 7	— 63	1, 0, 63	3n+r, 7n+r; 8n+1, 3, 5, 7,
	3, 0, 5	3n—r, 5n—r; 8n+1, 3, 5, 7		7, 0, 9	" ; "
—16	1, 0, 16	8n+1	— 66	1, 0, 66	3n+r, 11n+r; 8n+1, 3
—18	1, 0, 18	3n+r; 8n+1, 3		3, 0, 22	" ; "
	2, 0, 9	3n—r; 8n+1, 3		2, 0, 33	3n—r, 11n—r; 8n+1, 3
—20	1, 0, 20	5n+r; 8n+1, 5		6, 0, 11	" ; "
	4, 0, 5	" ; "	— 70	1, 0, 70	5n+r, 7n+r; 8n+1, 7
—21	1, 0, 21	3n+r, 7n+r; 8n+1, 5		2, 0, 35	5n—r, 7n+r; 8n+3, 5
	3, 0, 7	3n+r, 7n—r; 8n+3, 7		5, 0, 14	5n+r, 7n—r; 8n+3, 5
—22	1, 0, 22	11n+r; 8n+1, 7		7, 0, 10	5n—r, 7n—r; 8n+1, 7
	2, 0, 11	11n—r; 8n+3, 5	— 72	1, 0, 72	3n+r; 8n+1
—24	1, 0, 24	3n+r; 8n+1		8, 0, 9	3n—r; 8n+1
	3, 0, 8	3n—r; 8n+3	— 78	1, 0, 78	3n+r, 13n+r; 8n+1, 7
—25	1, 0, 25	5n+r; 8n+1, 5		2, 0, 39	3n—r, 13n—r; 8n+1, 7
—28	1, 0, 28	7n+r; 8n+1, 5		3, 0, 26	3n—r, 13n+r; 8n+3, 5
	4, 0, 7	7n+r; 8n+3, 7		6, 0, 13	3n+r, 13n—r; 8n+3, 5
—30	1, 0, 30	3n+r, 5n+r; 8n+1, 7	— 82	1, 0, 82	41n+r; 8n+1, 3
	2, 0, 15	3n—r, 5n—r; 8n+1, 7		2, 0, 41	" ; "
	3, 0, 10	3n+r, 5n—r; 8n+3, 5	— 84	1, 0, 84	3n+r, 7n+r; 8n+1, 5
	5, 0, 6	3n—r, 5n+r; 8n+3, 5		4, 0, 21	" ; "
—33	1, 0, 33	3n+r, 11n+r; 8n+1, 5		3, 0, 28	3n+r, 7n—r; 8n+3, 7
	3, 0, 11	3n—r, 11n+r; 8n+3, 7		7, 0, 12	" ; "
—34	1, 0, 34	17n+r; 8n+1, 3	— 85	1, 0, 85	5n+r, 17n+r; 8n+1, 5
	2, 0, 17	" ; "		5, 0, 17	5n—r, 17n—r; 8n+1, 5
—36	1, 0, 36	3n+r; 8n+1, 5	— 88	1, 0, 88	11n+r; 8n+1
	4, 0, 9	" ; "		8, 0, 11	11n—r; 8n+3
—37	1, 0, 37	37n+r; 8n+1, 5	— 90	1, 0, 90	3n+r, 5n+r; 8n+1, 3
—39	1, 0, 39	3n+r, 13n+r; 8n+1, 3, 5, 7		9, 0, 10	" ; "
	3, 0, 13	" ; "		2, 0, 45	3n—r, 5n—r; 8n+5, 7
—40	1, 0, 40	5n+r; 8n+1		5, 0, 18	" ; "
	5, 0, 8	5n—r; 8n+5	— 93	1, 0, 93	3n+r, 31n+r; 8n+1, 5
—42	1, 0, 42	3n+r, 7n+r; 8n+1, 3		3, 0, 31	3n+r, 31n—r; 8n+3, 7
	2, 0, 21	3n—r; 7n+r; 8n+5, 7	—100	1, 0, 100	5n+r; 8n+1, 5
	3, 0, 14	3n—r, 7n—r; 8n+1, 3		4, 0, 25	" ; "

D	Formen.	Charaktere.	D	Formen.	Charaktere.
—102	1, 0, 102	3n+r, 17n+r ; 8n+1, 7	—165	1, 0, 165	3n+r, 5n+r, 11n+r ; 8n+1, 5
	2, 0, 51	3n—r, 17n+r ; 8n+3, 5		3, 0, 55	3n+r, 5n—r, 11n+r ; 8n+3, 7
	3, 0, 34	3n+r, 17n—r ; 8n+3, 5		5, 0, 33	3n—r, 5n—r, 11n+r ; 8n+1, 5
	6, 0, 17	3n—r, 17n—r ; 8n+1, 7		11, 0, 15	3n—r, 5n+r, 11n+r ; 8n+3, 7
—105	1, 0, 105	3n+r, 5n+r, 7n+r ; 8n+1, 5	—168	1, 0, 168	3n+r, 7n+r ; 8n+1
	3, 0, 35	3n—r, 5n—r, 7n—r ; 8n+3, 7		3, 0, 56	3n—r, 7n—r ; 8n+3
	5, 0, 21	3n—r, 5n+r, 7n—r ; 8n+1, 5		7, 0, 24	3n+r, 7n—r ; 8n+7
	7, 0, 15	3n+r, 5n—r, 7n+r ; 8n+3, 7		8, 0, 21	3n—r, 7n+r ; 8n+5
—112	1, 0, 112	7n+r ; 8n+1	—177	1, 0, 177	3n+r, 59n+r ; 8n+1, 5
	7, 0, 16	7n+r ; 8n+7		3, 0, 59	3n—r, 59n+r ; 8n+3, 7
—117	1, 0, 117	3n+r, 13n+r ; 8n+1, 5	—180	1, 0, 180	3n+r, 5n+r ; 8n+1, 5
	9, 0, 13	" ; "		4, 0, 45	" ; "
—120	1, 0, 120	3n+r, 5n+r ; 8n+1		5, 0, 36	3n—r, 5n+r ; 8n+1, 5
	3, 0, 40	3n+r, 5n—r ; 8n+3		9, 0, 20	" ; "
	5, 0, 24	3n—r, 5n+r ; 8n+5	—190	1, 0, 190	5n+r, 19n+r ; 8n+1, 7
	8, 0, 15	3n—r, 5n—r ; 8n+7		2, 0, 95	5n—r, 19n—r ; 8n+1, 7
—130	1, 0, 130	5n+r, 13n+r ; 8n+1, 3		5, 0, 38	5n—r, 19n+r ; 8n+3, 5
	2, 0, 65	5n—r, 13n—r ; 8n+1, 3		10, 0, 19	5n+r, 19n—r ; 8n+3, 5
	5, 0, 26	5n+r, 13n—r ; 8n+5, 7	—198	1, 0, 198	3n+r, 11n+r ; 8n+1, 7
	10, 0, 13	5n—r, 13n+r ; 8n+5, 7		9, 0, 22	" ; "
—132	1, 0, 132	3n+r, 11n+r ; 8n+1, 5		2, 0, 99	3n—r, 11n—r ; 8n+3, 5
	4, 0, 33	" ; "		11, 0, 18	" ; "
	3, 0, 44	3n—r, 11n+r ; 8n+3, 7	—205	1, 0, 205	5n+r, 41n+r ; 8n+1, 5
	11, 0, 12	" ; "		5, 0, 41	" ; "
—133	1, 0, 133	7n+r, 19n+r ; 8n+1, 5	—210	1, 0, 210	3n+r, 5n+r, 7n+r ; 8n+1, 3
	7, 0, 19	7n—r, 19n+r ; 8n+3, 7		2, 0, 105	3n—r, 5n—r, 7n+r ; 8n+1, 3
—136	1, 0, 136	17n+r ; 8n+1		3, 0, 70	3n+r, 5n—r, 7n—r ; 8n+1, 3
	8, 0, 17	" ; "		5, 0, 42	3n—r, 5n—r, 7n—r ; 8n+5, 7
—138	1, 0, 138	3n+r, 23n+r ; 8n+1, 3		6, 0, 35	3n—r, 5n+r, 7n—r ; 8n+1, 3
	3, 0, 46	" ; "		7, 0, 30	3n+r, 5n—r, 7n+r ; 8n+5, 7
	2, 0, 69	3n—r, 23n+r ; 8n+5, 7		10, 0, 21	3n+r, 5n+r, 7n—r ; 8n+5, 7
	6, 0, 23	" ; "		14, 0, 15	3n—r, 5n+r, 7n+r ; 8n+5, 7
—142	1, 0, 142	71n+r ; 8n+1, 7	—220	1, 0, 220	5n+r, 11n+r ; 8n+1, 5
	2, 0, 71	" ; "		5, 0, 44	" ; "
—144	1, 0, 144	3n+r ; 8n+1		4, 0, 55	5n+r, 11n+r ; 8n+3, 7
	9, 0, 16	" ; "		11, 0, 20	" ; "
—145	1, 0, 145	5n+r, 29n+r ; 8n+1, 5	—225	1, 0, 225	3n+r, 5n+r ; 8n+1, 5
	5, 0, 29	" ; "		9, 0, 25	" ; "
—150	1, 0, 150	3n+r, 5n+r ; 8n+1, 7	—228	1, 0, 228	3n+r, 19n+r ; 8n+1, 5
	6, 0, 25	" ; "		4, 0, 57	" ; "
	2, 0, 75	3n—r, 5n—r ; 8n+3, 5		3, 0, 76	3n+r, 19n—r ; 8n+3, 7
	3, 0, 50	" ; "		12, 0, 19	" ; "
—154	1, 0, 154	7n+r, 11n+r ; 8n+1, 3	—232	1, 0, 232	29n+r ; 8n+1
	11, 0, 14	" ; "		8, 0, 29	29n—r ; 8n+5
	2, 0, 77	7n+r, 11n—r ; 8n+5, 7	—240	1, 0, 240	3n+r, 5n+r ; 8n+1
	7, 0, 22	" ; "		3, 0, 80	3n—r, 5n—r ; 8n+3
—156	1, 0, 156	3n+r, 13n+r ; 8n+1, 5		5, 0, 48	3n—r, 5n—r ; 8n+5
	12, 0, 13	" ; "		15, 0, 16	3n+r, 5n+r ; 8n+7
	3, 0, 52	3n+r, 13n+r ; 8n+3, 7	—252	1, 0, 252	3n+r, 7n+r ; 8n+1, 5
	4, 0, 39	" ; "		9, 0, 28	" ; "

D	Formen.	Charaktere.
—252	4, 0, 63	$3n+r$, $7n+r$; $8n+3$, 7
	7, 0, 36	" ; "
—253	1, 0, 253	$11n+r$, $23n+r$; $8n+1$, 5
	11, 0, 23	$11n+r$, $23n-r$; $8n+3$, 7
—258	1, 0, 258	$3n+r$, $43n+r$; $8n+1$, 3
	6, 0, 43	" ; "
	2, 0, 129	$3n-r$, $43n-r$; $8n+1$, 3
	3, 0, 86	" ; "
—273	1, 0, 273	$3n+r$, $7n+r$, $13n+r$; $8n+1$, 5
	3, 0, 91	$3n+r$, $7n-r$, $13n+r$; $8n+3$, 7
	7, 0, 39	$3n+r$, $7n+r$, $13n-r$; $8n+3$, 7
	13, 0, 21	$3n+r$, $7n-r$, $13n-r$; $8n+1$, 5
—280	1, 0, 280	$5n+r$, $7n+r$; $8n+1$
	5, 0, 56	$5n+r$, $7n-r$; $8n+5$
	7, 0, 40	$5n-r$, $7n-r$; $8n+7$
	8, 0, 35	$5n-r$, $7n+r$; $8n+3$
—282	1, 0, 282	$3n+r$, $47n+r$; $8n+1$, 3
	3, 0, 94	" ; "
	2, 0, 141	$3n-r$, $47n+r$; $8n+5$, 7
	6, 0, 47	" ; "
—310	1, 0, 310	$5n+r$, $31n+r$; $8n+1$, 7
	10, 0, 31	" ; "
	2, 0, 155	$5n-r$, $31n+r$; $8n+3$, 5
	5, 0, 62	" ; "
—312	1, 0, 312	$3n+r$, $13n+r$; $8n+1$
	3, 0, 104	$3n-r$, $13n+r$; $8n+3$
	8, 0, 39	$3n-r$, $13n-r$; $8n+7$
	13, 0, 24	$3n+r$, $13n-r$; $8n+5$
—328	1, 0, 328	$41n+r$; $8n+1$
	8, 0, 41	" ; "
—330	1, 0, 330	$3n+r$, $5n+r$, $11n+r$; $8n+1$, 3
	2, 0, 165	$3n-r$, $5n-r$, $11n-r$; $8n+5$, 7
	3, 0, 110	$3n-r$, $5n-r$, $11n+r$; $8n+1$, 3
	5, 0, 66	$3n-r$, $5n+r$, $11n+r$; $8n+5$, 7
	6, 0, 55	$3n+r$, $5n+r$, $11n-r$; $8n+5$, 7
	10, 0, 33	$3n+r$, $5n-r$, $11n-r$; $8n+1$, 3
	11, 0, 30	$3n-r$, $5n+r$, $11n-r$; $8n+1$, 3
	15, 0, 22	$3n+r$, $5n-r$, $11n+r$; $8n+5$, 7
—333	1, 0, 333	$3n+r$, $37n+r$; $8n+1$, 5
	9, 0, 37	" ; "
—340	1, 0, 340	$5n+r$, $17n+r$; $8n+1$, 5
	4, 8, 85	" ; "
	5, 0, 68	$5n-r$, $17n-r$; $8n+1$, 5
	17, 0, 20	" ; "
—345	1, 0, 345	$3n+r$, $5n+r$, $23n+r$; $8n+1$, 5
	3, 0, 115	$3n+r$, $5n-r$, $23n+r$; $8n+3$, 7
	5, 0, 69	$3n-r$, $5n+r$, $23n-r$; $8n+1$, 5
	15, 0, 23	$3n-r$, $5n-r$, $23n-r$; $8n+3$, 7
—357	1, 0, 357	$3n+r$, $7n+r$, $17n+r$; $8n+1$, 5
	3, 0, 119	$3n-r$, $7n-r$, $17n-r$; $8n+3$, 7
	7, 0, 51	$3n+r$, $7n+r$, $17n-r$; $8n+3$, 7
—357	17, 0, 21	$3n-r$, $7n-r$, $17n+r$; $8n+1$, 5
—360	1, 0, 360	$3n+r$, $5n+r$; $8n+1$
	9, 0, 40	" ; "
	5, 0, 72	$3n-r$, $5n-r$; $8n+5$
	8, 0, 45	" ; "
—372	1, 0, 372	$3n+r$, $31n+r$; $8n+1$, 5
	4, 0, 93	" ; "
	3, 0, 124	$3n+r$, $31n-r$; $8n+3$, 7
	12, 0, 31	" ; "
—385	1, 0, 385	$5n+r$, $7n+r$, $11n+r$; $8n+1$, 5
	5, 0, 77	$5n-r$, $7n-r$, $11n+r$; $8n+1$, 5
	7, 0, 55	$5n-r$, $7n-r$, $11n-r$; $8n+3$, 7
	11, 0, 35	$5n+r$, $7n+r$, $11n-r$; $8n+3$, 7
—390	1, 0, 390	$3n+r$, $5n+r$, $13n+r$; $8n+1$, 7
	10, 0, 39	" ; "
	3, 0, 130	$3n+r$, $5n-r$, $13n+r$; $8n+3$, 5
	13, 0, 30	" ; "
	2, 0, 195	$3n-r$, $5n-r$, $13n-r$; $8n+3$, 5
	5, 0, 78	" ; "
	6, 0, 65	$3n-r$, $5n+r$, $13n-r$; $8n+1$, 7
	15, 0, 26	" ; "
—408	1, 0, 408	$3n+r$, $17n+r$; $8n+1$
	3, 0, 136	$3n+r$, $17n-r$; $8n+3$
	8, 0, 51	$3n-r$, $17n+r$; $8n+3$
	17, 0, 24	$3n-r$, $17n-r$; $8n+1$
—420	1, 0, 420	$3n+r$, $5n+r$, $7n+r$; $8n+1$, 5
	4, 0, 105	" ; "
	3, 0, 140	$3n-r$, $5n-r$, $7n-r$; $8n+3$, 7
	12, 0, 35	" ; "
	5, 0, 84	$3n-r$, $5n+r$, $7n-r$; $8n+1$, 5
	20, 0, 21	" ; "
	7, 0, 60	$3n+r$, $5n-r$, $7n+r$; $8n+3$, 7
	15, 0, 28	" ; "
—438	1, 0, 438	$3n+r$, $73n+r$; $8n+1$, 7
	6, 0, 73	" ; "
	2, 0, 219	$3n-r$, $73n+r$; $8n+3$, 5
	3, 0, 146	" ; "
—442	1, 0, 442	$13n+r$, $17n+r$; $8n+1$, 3
	17, 0, 26	" ; "
	2, 0, 221	$13n-r$, $17n+r$; $8n+5$, 7
	13, 0, 34	" ; "
—445	1, 0, 445	$5n+r$, $89n+r$; $8n+1$, 5
	5, 0, 89	" ; "
—462	1, 0, 462	$3n+r$, $7n+r$, $11n+r$; $8n+1$, 7
	2, 0, 231	$3n-r$, $7n+r$, $11n-r$; $8n+1$, 7
	3, 0, 154	$3n+r$, $7n-r$, $11n+r$; $8n+3$, 5
	6, 0, 77	$3n-r$, $7n-r$, $11n-r$; $8n+3$, 5
	7, 0, 66	$3n+r$, $7n-r$, $11n-r$; $8n+1$, 7
	11, 0, 42	$3n-r$, $7n+r$, $11n+r$; $8n+3$, 5
	14, 0, 33	$3n-r$, $7n-r$, $11n+r$; $8n+1$, 7
	21, 0, 22	$3n+r$, $7n+r$, $11n-r$; $8n+3$, 5

D	Formen.	Charaktere.	D	Formen.	Charaktere.
−498	1, 0, 498	$3n+r, 83n+r;\ 8n+1, 3$	−600	1, 0, 600	$3n+r, 5n+r;\ 8n+1$
	3, 0, 166	“ ; “		24, 0, 25	“ ; “
	2, 0, 249	$3n-r, 83n-r;\ 8n+1, 3$		3, 0, 200	$3n-r, 5n-r;\ 8n+3$
	6, 0, 83	“ ; “		8, 0, 75	“ ; “
−505	1, 0, 505	$5n+r, 101n+r;\ 8n+1, 5$	−630	1, 0, 630	$3n+r, 5n+r, 7n+r;\ 8n+1, 7$
	5, 0, 101	“ ; “		9, 0, 70	“ ; “
−510	1, 0, 510	$3n+r, 5n+r, 17n+r;\ 8n+1, 7$		2, 0, 315	$3n-r, 5n-r, 7n+r;\ 8n+3, 5$
	15, 0, 34	“ ; “		18, 0, 35	“ ; “
	2, 0, 255	$3n-r, 5n-r, 17n+r;\ 8n+1, 7$		5, 0, 126	$3n-r, 5n+r, 7n-r;\ 8n+3, 5$
	17, 0, 30	“ ; “		14, 0, 45	“ ; “
	3, 0, 170	$3n-r, 5n-r, 17n-r;\ 8n+3, 5$		7, 0, 90	$3n+r, 5n-r, 7n-r;\ 8n+1, 7$
	5, 0, 102	“ ; “		10, 0, 63	“ ; “
	6, 0, 85	$3n+r, 5n+r, 17n-r;\ 8n+3, 5$	−658	1, 0, 658	$7n+r, 47n+r;\ 8n+1, 3$
	10, 0, 51	“ ; “		2, 0, 329	“ ; “
−520	1, 0, 520	$5n+r, 13n+r;\ 8n+1$		7, 0, 94	$7n-r, 47n+r;\ 8n+5, 7$
	5, 0, 104	$5n+r, 13n-r;\ 8n+5$		14, 0, 47	“ ; “
	8, 0, 65	$5n-r, 13n-r;\ 8n+1$	−660	1, 0, 660	$3n+r, 5n+r, 11n+r;\ 8n+1, 5$
	13, 0, 40	$5n-r, 13n+r;\ 8n+5$		4, 0, 165	“ ; “
−522	1, 0, 522	$3n+r, 29n+r;\ 8n+1, 3$		3, 0, 220	$3n+r, 5n-r, 11n+r;\ 8n+3, 7$
	9, 0, 58	“ ; “		12, 0, 55	“ ; “
	2, 0, 261	$3n-r, 29n-r;\ 8n+5, 7$		5, 0, 132	$3n-r, 5n-r, 11n+r;\ 8n+1, 5$
	18, 0, 29	“ ; “		20, 0, 33	“ ; “
−525	1, 0, 525	$3n+r, 5n+r, 7n+r;\ 8n+1, 5$		11, 0, 60	$3n-r, 5n+r, 11n+r;\ 8n+3, 7$
	21, 0, 25	“ ; “		15, 0, 44	“ ; “
	3, 0, 175	$3n+r, 5n-r, 7n-r;\ 8n+3, 7$	−690	1, 0, 690	$3n+r, 5n+r, 23n+r;\ 8n+1, 3$
	7, 0, 75	“ ; “		6, 0, 115	“ ; “
−528	1, 0, 528	$3n+r, 11n+r;\ 8n+1$		2, 0, 345	$3n-r, 5n-r, 23n+r;\ 8n+1, 3$
	16, 0, 33	“ ; “		3, 0, 230	“ ; “
	3, 0, 176	$3n-r, 11n+r;\ 8n+3$		5, 0, 138	$3n-r, 5n-r, 23n-r;\ 8n+5, 7$
	11, 0, 48	“ ; “		23, 0, 30	“ ; “
−532	1, 0, 532	$7n+r, 19n+r;\ 8n+1, 5$		10, 0, 69	$3n+r, 5n+r, 23n-r;\ 8n+5, 7$
	4, 0, 133	“ ; “		15, 0, 46	“ ; “
	7, 0, 76	$7n-r, 19n+r;\ 8n+3, 7$	−742	1, 0, 742	$7n+r, 53n+r;\ 8n+1, 7$
	19, 0, 28	“ ; “		7, 0, 106	“ ; “
−570	1, 0, 570	$3n+r, 5n+r, 19n+r;\ 8n+1, 3$		2, 0, 371	$7n+r, 53n-r;\ 8n+3, 5$
	19, 0, 30	“ ; “		14, 0, 53	“ ; “
	2, 0, 285	$3n-r, 5n-r, 19n-r;\ 8n+5, 7$	−760	1, 0, 760	$5n+r, 19n+r;\ 8n+1$
	15, 0, 38	“ ; “		5, 0, 152	$5n-r, 19n+r;\ 8n+5$
	3, 0, 190	$3n+r, 5n-r, 19n-r;\ 8n+1, 3$		8, 0, 95	$5n-r, 19n-r;\ 8n+7$
	10, 0, 57	“ ; “		19, 0, 40	$5n+r, 19n-r;\ 8n+3$
	5, 0, 114	$3n-r, 5n+r, 19n+r;\ 8n+5, 7$	−765	1, 0, 765	$3n+r, 5n+r, 17n+r;\ 8n+1, 5$
	6, 0, 95	“ ; “		9, 0, 85	“ ; “
−580	1, 0, 580	$5n+r, 29n+r;\ 8n+1, 5$		5, 0, 153	$3n-r, 5n-r, 17n-r;\ 8n+1, 5$
	4, 0, 145	“ ; “		17, 0, 45	“ ; “
	5, 0, 116	“ ; “	−777	1, 0, 777	$3n+r, 7n+r, 37n+r;\ 8n+1, 5$
	20, 0, 29	“ ; “		21, 0, 37	“ ; “
−598	1, 0, 598	$13n+r, 23n+r;\ 8n+1, 7$		3, 0, 259	$3n+r, 7n-r, 37n+r;\ 8n+3, 7$
	23, 0, 26	“ ; “		7, 0, 111	“ ; “
	2, 0, 299	$13n-r, 23n+r;\ 8n+3, 5$	−793	1, 0, 793	$13n+r, 61n+r;\ 8n+1, 5$
	13, 0, 46	“ ; “		13, 0, 61	“ ; “

D	Formen.	Charaktere.	D	Formen.	Charaktere.
−798	1, 0, 798	3n+r, 7n+r, 19n+r ; 8n+1, 7	− 912	3, 0, 304	3n+r, 19n−r ; 8n+3
	7, 0, 114	“ ; “		19, 0, 48	“ ; “
	2, 0, 399	3n−r, 7n+r, 19n−r ; 8n+1, 7	−1012	1, 0, 1012	11n+r, 23n+r ; 8n+1, 5
	14, 0, 57	“ ; “		4, 0, 253	“ ; “
	3, 0, 266	3n−r, 7n−r, 19n−r ; 8n+3, 5		11, 0, 92	11n+r, 23n−r ; 8n+3, 7
	21, 0, 38	“ ; “		23, 0, 44	“ ; “
	6, 0, 133	3n+r, 7n−r, 19n+r ; 8n+3, 5	−1045	1, 0, 1045	5n+r, 11n+r, 19n+r ; 8n+1, 5
	19, 0, 42	“ ; “		5, 0, 209	“ ; “
−840	1, 0, 840	3n+r, 5n+r, 7n+r ; 8n+1		11, 0, 95	5n+r, 11n−r, 19n+r ; 8n+3, 7
	3, 0, 280	3n+r, 5n−r, 7n−r ; 8n+3		19, 0, 55	“ ; “
	5, 0, 168	3n−r, 5n−r, 7n−r ; 8n+5	−1092	1, 0, 1092	3n+r, 7n+r, 13n+r ; 8n+1, 5
	7, 0, 120	3n+r, 5n−r, 7n+r ; 8n+7		4, 0, 273	“ ; “
	8, 0, 105	3n−r, 5n−r, 7n+r ; 8n+1		3, 0, 364	3n+r, 7n−r, 13n+r ; 8n+3, 7
	15, 0, 56	3n−r, 5n+r, 7n+r ; 8n+7		12, 0, 91	“ ; “
	21, 0, 40	3n+r, 5n+r, 7n−r ; 8n+5		7, 0, 156	3n+r, 7n+r, 13n−r ; 8n+3, 7
	24, 0, 35	3n−r, 5n+r, 7n−r ; 8n+3		28, 0, 39	“ ; “
−858	1, 0, 858	3n+r, 11n+r, 13n+r ; 8n+1, 3		13, 0, 84	3n+r, 7n−r, 13n−r ; 8n+1, 5
	3, 0, 286	“ ; “		21, 0, 52	“ ; “
	2, 0, 429	3n−r, 11n−r, 13n−r ; 8n+5, 7	−1110	1, 0, 1110	3n+r, 5n+r, 37n+r ; 8n+1, 3
	6, 0, 143	“ ; “		10, 0, 111	“ ; “
	11, 0, 78	3n−r, 11n+r, 13n−r ; 8n+1, 3		2, 0, 555	3n−r, 5n−r, 37n−r ; 8n+3, 5
	26, 0, 33	“ ; “		5, 0, 222	“ ; “
	13, 0, 66	3n+r, 11n−r, 13n+r ; 8n+5, 7		3, 0, 370	3n+r, 5n−r, 37n+r ; 8n+3, 5
	22, 0, 39	“ ; “		30, 0, 37	“ ; “
−870	1, 0, 870	3n+r, 5n+r, 29n+r ; 8n+1, 7		6, 0, 185	3n−r, 5n+r, 37n−r ; 8n+1, 7
	6, 0, 145	“ ; “		15, 0, 74	“ ; “
	2, 0, 435	3n−r, 5n−r, 29n−r ; 8n+3, 5	−1122	1, 0, 1122	3n+r, 11n+r, 17n+r ; 8n+1, 3
	3, 0, 290	“ ; “		33, 0, 34	“ ; “
	5, 0, 174	3n−r, 5n+r, 29n+r ; 8n+3, 5		2, 0, 561	3n−r, 11n−r, 17n+r ; 8n+1, 3
	29, 0, 30	“ ; “		17, 0, 66	“ ; “
	10, 0, 87	3n+r, 5n−r, 29n+r ; 8n+1, 7		3, 0, 374	3n−r, 11n+r, 17n−r ; 8n+1, 3
	15, 0, 58	“ ; “		11, 0, 102	“ ; “
−897	1, 0, 897	3n+r, 13n+r, 23n+r ; 8n+1, 5		6, 0, 187	3n+r, 11n−r, 17n−r ; 8n+1, 3
	13, 0, 69	“ ; “		22, 0, 51	“ ; “
	3, 0, 299	3n−r, 13n+r, 23n+r ; 8n+3, 7	−1170	1, 0, 1170	3n+r, 5n+r, 13n+r ; 8n+1, 3
	23, 0, 39	“ ; “		9, 0, 130	“ ; “
−900	1, 0, 900	3n+r, 5n+r ; 8n+1, 5		2, 0, 585	3n−r, 5n−r, 13n−r ; 8n+1, 3
	4, 0, 225	“ ; “		18, 0, 65	“ ; “
	9, 0, 100	“ ; “		5, 0, 234	3n−r, 5n+r, 13n−r ; 8n+5, 7
	25, 0, 36	“ ; “		26, 0, 45	“ ; “
−910	1, 0, 910	5n+r, 7n+r, 13n+r ; 8n+1, 7		10, 0, 117	3n+r, 5n−r, 13n+r ; 8n+5, 7
	14, 0, 65	“ ; “		13, 0, 90	“ ; “
	2, 0, 455	5n−r, 7n+r, 13n−r ; 8n+1, 7	−1197	1, 0, 1197	3n+r, 7n+r, 19n+r ; 8n+1, 5
	7, 0, 130	“ ; “		9, 0, 133	“ ; “
	5, 0, 182	5n−r, 7n−r, 13n−r ; 8n+3, 5		7, 0, 171	3n+r, 7n−r, 19n+r ; 8n+3, 7
	13, 0, 70	“ ; “		19, 0, 63	“ ; “
	10, 0, 91	5n+r, 7n−r, 13n+r ; 8n+3, 5	−1290	1, 0, 1290	3n+r, 5n+r, 43n+r ; 8n+1, 3
	26, 0, 35	“ ; “		10, 0, 129	“ ; “
−912	1, 0, 912	3n+r, 19n+r ; 8n+1		2, 0, 645	3n−r, 5n−r, 43n−r ; 8n+5, 7
	16, 0, 57	“ ; “		5, 0, 258	“ ; “

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−1290	3, 0, 430	3n+r, 5n−r, 43n−r ; 8n+1, 3
	30, 0, 43	“ ; “
	6, 0, 215	3n−r, 5n+r, 43n+r ; 8n+5, 7
	15, 0, 86	“ ; “
−1302	1, 0, 1302	3n+r, 7n+r, 31n+r ; 8n+1, 7
	7, 0, 186	“ ; “
	2, 0, 651	3n−r, 7n+r, 31n+r ; 8n+3, 5
	14, 0, 93	“ ; “
	3, 0, 434	3n−r, 7n−r, 31n−r ; 8n+3, 5
	21, 0, 62	“ ; “
	6, 0, 217	3n+r, 7n−r, 31n−r ; 8n+1, 7
	31, 0, 42	“ ; “
−1320	1, 0, 1320	3n+r, 5n+r, 11n+r ; 8n+1
	3, 0, 440	3n−r, 5n−r, 11n+r ; 8n+3
	5, 0, 264	3n−r, 5n+r, 11n+r ; 8n+5
	8, 0, 165	3n−r, 5n−r, 11n−r ; 8n+5
	11, 0, 120	3n−r, 5n+r, 11n−r ; 8n+3
	15, 0, 88	3n+r, 5n−r, 11n+r ; 8n+7
	24, 0, 55	3n+r, 5n+r, 11n−r ; 8n+7
	33, 0, 40	3n+r, 5n−r, 11n−r ; 8n+1
−1365	1, 0, 1365	3n+r, 5n+r, 7n+r, 13n+r ; 8n+1, 5
	3, 0, 455	3n−r, 5n−r, 7n−r, 13n+r ; 8n+3, 7
	5, 0, 273	3n−r, 5n−r, 7n−r, 13n−r ; 8n+1, 5
	7, 0, 195	3n+r, 5n−r, 7n−r, 13n−r ; 8n+3, 7
	13, 0, 105	3n+r, 5n−r, 7n−r, 13n+r ; 8n+1, 5
	15, 0, 91	3n+r, 5n+r, 7n+r, 13n−r ; 8n+3, 7
	21, 0, 65	3n−r, 5n+r, 7n+r, 13n−r ; 8n+1, 5
	35, 0, 39	3n−r, 5n+r, 7n+r, 13n+r ; 8n+3, 7
−1380	1, 0, 1380	3n+r, 5n+r, 23n+r ; 8n+1, 5
	4, 0, 345	“ ; “
	3, 0, 460	3n+r, 5n−r, 23n+r ; 8n+3, 7
	12, 0, 115	“ ; “
	5, 0, 276	3n−r, 5n+r, 23n−r ; 8n+1, 5
	20, 0, 69	“ ; “
	15, 0, 92	3n−r, 5n−r, 23n−r ; 8n+3, 7
	23, 0, 60	“ ; “
−1582	1, 0, 1582	7n+r, 113n+r ; 8n+1, 7
	2, 0, 791	“ ; “
	7, 0, 226	“ ; “
	14, 0, 113	“ ; “
−1590	1, 0, 1590	3n+r, 5n+r, 53n+r ; 8n+1, 7
	6, 0, 265	“ ; “
	10, 0, 159	“ ; “
	15, 0, 106	“ ; “
	2, 0, 795	3n−r, 5n−r, 53n−r ; 8n+3, 5
	3, 0, 530	“ ; “
	5, 0, 318	“ ; “
	30, 0, 53	“ ; “
−1677	1, 0, 1677	3n+r, 13n+r, 43n+r ; 8n+1, 5
	13, 0, 129	“ ; “
−1677	3, 0, 559	3n+r, 13n+r, 43n−r ; 8n+3, 7
	39, 0, 43	“ ; “
−1680	1, 0, 1680	3n+r, 5n+r, 7n+r ; 8n+1
	16, 0, 105	“ ; “
	3, 0, 560	3n−r, 5n−r, 7n−r ; 8n+3
	35, 0, 48	“ ; “
	5, 0, 336	3n−r, 5n+r, 7n−r ; 8n+5
	21, 0, 80	“ ; “
	7, 0, 240	3n+r, 5n−r, 7n+r ; 8n+7
	15, 0, 112	“ ; “
−1705	1, 0, 1705	5n+r, 11n+r, 31n+r ; 8n+1, 5
	5, 0, 341	“ ; “
	11, 0, 155	5n+r, 11n+r, 31n−r ; 8n+3, 7
	31, 0, 55	“ ; “
−1710	1, 0, 1710	3n+r, 5n+r, 19n+r ; 8n+1, 7
	9, 0, 190	“ ; “
	2, 0, 855	3n−r, 5n−r, 19n−r ; 8n+1, 7
	18, 0, 95	“ ; “
	5, 0, 342	3n−r, 5n−r, 19n+r ; 8n+3, 5
	38, 0, 45	“ ; “
	19, 0, 90	3n+r, 5n+r, 19n−r ; 8n+3, 5
	10, 0, 171	“ ; “
−1768	1, 0, 1768	13n+r, 17n+r ; 8n+1
	17, 0, 104	“ ; “
	8, 0, 221	13n−r, 17n+r ; 8n+5
	13, 0, 136	“ ; “
−1785	1, 0, 1785	3n+r, 5n+r, 7n+r, 17n+r ; 8n+1, 5
	21, 0, 85	“ ; “
	3, 0, 595	3n+r, 5n−r, 7n−r, 17n−r ; 8n+3, 7
	7, 0, 255	“ ; “
	5, 0, 357	3n−r, 5n−r, 7n−r, 17n−r ; 8n+1, 5
	17, 0, 105	“ ; “
	15, 0, 119	3n−r, 5n+r, 7n+r, 17n+r ; 8n+3, 7
	35, 0, 51	“ ; “
−1813	1, 0, 1813	7n+r, 37n+r ; 8n+1, 5
	37, 0, 49	“ ; “
−1848	1, 0, 1848	3n+r, 7n+r, 11n+r ; 8n+1
	3, 0, 616	3n+r, 7n−r, 11n+r ; 8n+3
	7, 0, 264	3n+r, 7n−r, 11n−r ; 8n+7
	8, 0, 231	3n−r, 7n+r, 11n−r ; 8n+7
	11, 0, 168	3n−r, 7n+r, 11n+r ; 8n+3
	21, 0, 88	3n+r, 7n+r, 11n−r ; 8n+5
	24, 0, 77	3n−r, 7n−r, 11n−r ; 8n+5
	33, 0, 56	3n−r, 7n−r, 11n+r ; 8n+1
−1885	1, 0, 1885	5n+r, 13n+r, 29n+r ; 8n+1, 5
	29, 0, 65	“ ; “
	5, 0, 377	5n−r, 13n−r, 29n+r ; 8n+1, 5
	13, 0, 145	“ ; “

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—1918	1, 0, 1918	$7n+r$, $137n+r$; $8n+1$, 7	—2542	1, 0, 2542	$31n+r$, $41n+r$; $8n+1$, 7
	2, 0, 959	“ ; “		2, 0, 1271	“ ; “
	7, 0, 274	“ ; “		31, 0, 82	“ ; “
	14, 0, 137	“ ; “		41, 0, 62	“ ; “
—2002	1, 0, 2002	$7n+r$, $11n+r$, $13n+r$; $8n+1$, 3	—2632	1, 0, 2632	$7n+r$, $47n+r$; $8n+1$
	22, 0, 91	“ ; “		8, 0, 329	“ ; “
	2, 0, 1001	$7n+r$, $11n-r$, $13n-r$; $8n+1$, 3		7, 0, 376	$7n-r$, $47n+r$; $8n+7$
	11, 0, 182	“ ; “		47, 0, 56	“ ; “
	7, 0, 286	$7n-r$, $11n-r$, $13n-r$; $8n+5$, 7	—2730	1, 0, 2730	$3n+r$, $5n+r$, $7n+r$, $13n+r$; $8n+1$, 3
	13, 0, 154	“ ; “		30, 0, 91	“ ; “
	14, 0, 143	$7n-r$, $11n+r$, $13n+r$; $8n+5$, 7		2, 0, 1365	$3n-r$, $5n-r$, $7n+r$, $13n-r$; $8n+5$, 7
	26, 0, 77	“ ; “		15, 0, 182	“ ; “
—2020	1, 0, 2020	$5n+r$, $101n+r$; $8n+1$, 5		3, 0, 910	$3n+r$, $5n-r$, $7n-r$, $13n+r$; $8n+1$, 3
	4, 0, 505	“ ; “		10, 0, 273	“ ; “
	5, 0, 404	“ ; “		5, 0, 546	$3n-r$, $5n+r$, $7n-r$, $13n-r$; $8n+5$, 7
	20, 0, 101	“ ; “		6, 0, 455	“ ; “
—2088	1, 0, 2088	$3n+r$, $29n+r$; $8n+1$		7, 0, 390	$3n+r$, $5n-r$, $7n-r$, $13n-r$; $8n+5$, 7
	9, 0, 232	“ ; “		13, 0, 210	“ ; “
	8, 0, 261	$3n-r$, $29n-r$; $8n+5$		14, 0, 195	$3n-r$, $5n+r$, $7n-r$, $13n+r$; $8n+1$, 3
	29, 0, 72	“ ; “		26, 0, 105	“ ; “
—2170	1, 0, 2170	$5n+r$, $7n+r$, $31n+r$; $8n+1$, 3		21, 0, 130	$3n+r$, $5n+r$, $7n+r$, $13n-r$; $8n+5$, 7
	14, 0, 155	“ ; “		39, 0, 70	“ ; “
	2, 0, 1085	$5n-r$, $7n+r$, $31n+r$; $8n+5$, 7		35, 0, 78	$3n-r$, $5n-r$, $7n+r$, $13n+r$; $8n+1$, 3
	7, 0, 310	“ ; “		42, 0, 65	“ ; “
	5, 0, 434	$5n+r$, $7n-r$, $31n+r$; $8n+5$, 7	—2790	1, 0, 2790	$3n+r$, $5n+r$, $31n+r$; $8n+1$, 7
	31, 0, 70	“ ; “		9, 0, 310	“ ; “
	10, 0, 217	$5n-r$, $7n-r$, $31n+r$; $8n+1$, 3		10, 0, 279	“ ; “
	35, 0, 62	“ ; “		31, 0, 90	“ ; “
—2277	1, 0, 2277	$3n+r$, $11n+r$, $23n+r$; $8n+1$, 5		2, 0, 1395	$3n-r$, $5n-r$, $31n+r$; $8n+3$, 5
	9, 0, 253	“ ; “		5, 0, 558	“ ; “
	11, 0, 207	$3n-r$, $11n+r$, $23n-r$; $8n+3$, 7		18, 0, 155	“ ; “
	23, 0, 99	“ ; “		45, 0, 62	“ ; “
—2310	1, 0, 2310	$3n+r$, $5n+r$, $7n+r$, $11n+r$; $8n+1$, 7	—3108	1, 0, 3108	$3n+r$, $7n+r$, $37n+r$; $8n+1$, 5
	15, 0, 154	“ ; “		4, 0, 777	“ ; “
	2, 0, 1155	$3n-r$, $5n-r$, $7n+r$, $11n-r$; $8n+3$, 5		21, 0, 148	“ ; “
	30, 0, 77	“ ; “		37, 0, 84	“ ; “
	3, 0, 770	$3n-r$, $5n-r$, $7n-r$, $11n+r$; $8n+3$, 5		3, 0, 1036	$3n+r$, $7n-r$, $37n+r$; $8n+3$, 7
	5, 0, 462	“ ; “		7, 0, 444	“ ; “
	6, 0, 385	$3n+r$, $5n+r$, $7n-r$, $11n-r$; $8n+1$, 7		12, 0, 259	“ ; “
	10, 0, 231	“ ; “		28, 0, 111	“ ; “
	7, 0, 330	$3n+r$, $5n-r$, $7n+r$, $11n-r$; $8n+1$, 7	—3172	1, 0, 3172	$13n+r$, $61n+r$; $8n+1$, 5
	22, 0, 105	“ ; “		4, 0, 793	“ ; “
	11, 0, 210	$3n-r$, $5n+r$, $7n+r$, $11n+r$; $8n+3$, 5		13, 0, 244	“ ; “
	14, 0, 165	“ ; “		52, 0, 61	“ ; “
	21, 0, 110	$3n-r$, $5n+r$, $7n-r$, $11n-r$; $8n+3$, 5	—3465	1, 0, 3465	$3n+r$, $5n+r$, $7n+r$, $11n+r$; $8n+1$, 5
	35, 0, 66	“ ; “		9, 0, 385	“ ; “
	33, 0, 70	$3n+r$, $5n-r$, $7n-r$, $11n+r$; $8n+1$, 7		5, 0, 693	$3n-r$, $5n-r$, $7n-r$, $11n+r$; $8n+1$, 5
	42, 0, 55	“ ; “		45, 0, 77	“ ; “
				7, 0, 495	$3n+r$, $5n-r$, $7n-r$, $11n-r$; $8n+3$, 7
				55, 0, 63	“ ; “
				11, 0, 315	$3n-r$, $5n+r$, $7n+r$, $11n-r$; $8n+3$, 7
				35, 0, 99	“ ; “

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—3480	1, 0, 3480	$3n+r,\ 5n+r,\ 29n+r\ ;\ 8n+1$
	24, 0, 145	" ; "
	3, 0, 1160	$3n-r,\ 5n-r,\ 29n-r\ ;\ 8n+3$
	8, 0, 435	" ; "
	5, 0, 696	$3n-r,\ 5n+r,\ 29n+r\ ;\ 8n+5$
	29, 0, 120	" ; "
	15, 0, 232	$3n+r,\ 5n-r,\ 29n-r\ ;\ 8n+7$
	40, 0, 87	" ; "
—3570	1, 0, 3570	$3n+r,\ 5n+r,\ 7n+r,\ 17n+r\ ;\ 8n+1, 3$
	51, 0, 70	" ; "
	2, 0, 1785	$3n-r,\ 5n-r,\ 7n+r,\ 17n+r\ ;\ 8n+1, 3$
	35, 0, 102	" ; "
	3, 0, 1190	$3n-r,\ 5n-r,\ 7n-r,\ 17n-r\ ;\ 8n+1, 3$
	17, 0, 210	" ; "
	5, 0, 714	$3n-r,\ 5n+r,\ 7n-r,\ 17n-r\ ;\ 8n+5, 7$
	14, 0, 255	" ; "
	6, 0, 595	$3n+r,\ 5n+r,\ 7n-r,\ 17n-r\ ;\ 8n+1, 3$
	34, 0, 105	" ; "
	7, 0, 510	$3n+r,\ 5n-r,\ 7n-r,\ 17n-r\ ;\ 8n+5, 7$
	10, 0, 357	" ; "
	15, 0, 238	$3n+r,\ 5n-r,\ 7n+r,\ 17n+r\ ;\ 8n+5, 7$
	42, 0, 85	" ; "
	21, 0, 170	$3n-r,\ 5n+r,\ 7n+r,\ 17n+r\ ;\ 8n+5, 7$
	30, 0, 119	" ; "
—3588	1, 0, 3588	$3n+r,\ 13n+r,\ 23n+r\ ;\ 8n+1, 5$
	4, 0, 897	" ; "
	13, 0, 276	" ; "
	52, 0, 69	" ; "
	3, 0, 1196	$3n-r,\ 13n+r,\ 23n+r\ ;\ 8n+3, 7$
	12, 0, 299	" ; "
	23, 0, 156	" ; "
	39, 0, 92	" ; "
—3640	1, 0, 3640	$5n+r,\ 7n+r,\ 13n+r\ ;\ 8n+1$
	56, 0, 65	" ; "
	5, 0, 728	$5n-r,\ 7n-r,\ 13n-r\ ;\ 8n+5$
	13, 0, 280	" ; "
	7, 0, 520	$5n-r,\ 7n+r,\ 13n-r\ ;\ 8n+7$
	8, 0, 455	" ; "
	35, 0, 104	$5n+r,\ 7n-r,\ 13n+r\ ;\ 8n+3$
	40, 0, 91	" ; "
—4180	1, 0, 4180	$5n+r,\ 11n+r,\ 19n+r\ ;\ 8n+1, 5$
	4, 0, 1045	" ; "
	5, 0, 836	" ; "
	20, 0, 209	" ; "
	11, 0, 380	$5n+r,\ 11n-r,\ 19n+r\ ;\ 8n+3, 7$
	19, 0, 220	" ; "
	44, 0, 95	" ; "
	55, 0, 76	" ; "
—4305	1, 0, 4305	$3n+r,\ 5n+r,\ 7n+r,\ 41n+r\ ;\ 8n+1, 5$
	21, 0, 205	" ; "
—4305	3, 0, 1435	$3n+r,\ 5n-r,\ 7n-r,\ 41n-r\ ;\ 8n+3, 7$
	7, 0, 615	" ; "
	5, 0, 861	$3n-r,\ 5n+r,\ 7n-r,\ 41n+r\ ;\ 8n+1, 5$
	41, 0, 105	" ; "
	15, 0, 287	$3n-r,\ 5n-r,\ 7n+r,\ 41n-r\ ;\ 8n+3, 7$
	35, 0, 123	" ; "
—4488	1, 0, 4488	$3n+r,\ 11n+r,\ 17n+r\ ;\ 8n+1$
	33, 0, 136	" ; "
	3, 0, 1496	$3n-r,\ 11n+r,\ 17n-r\ ;\ 8n+3$
	11, 0, 408	" ; "
	8, 0, 561	$3n-r,\ 11n-r,\ 17n+r\ ;\ 8n+1$
	17, 0, 264	" ; "
	24, 0, 187	$3n+r,\ 11n-r,\ 17n-r\ ;\ 8n+3$
	51, 0, 88	" ; "
—4680	1, 0, 4680	$3n+r,\ 5n+r,\ 13n+r\ ;\ 8n+1$
	9, 0, 520	" ; "
	5, 0, 936	$3n-r,\ 5n+r,\ 13n-r\ ;\ 8n+5$
	45, 0, 104	" ; "
	8, 0, 585	$3n-r,\ 5n-r,\ 13n-r\ ;\ 8n+1$
	65, 0, 72	" ; "
	13, 0, 360	$3n+r,\ 5n-r,\ 13n+r\ ;\ 8n+5$
	40, 0, 117	" ; "
—4810	1, 0, 4810	$5n+r,\ 13n+r,\ 37n+r\ ;\ 8n+1, 3$
	10, 0, 481	" ; "
	26, 0, 185	" ; "
	65, 0, 74	" ; "
	2, 0, 2405	$5n-r,\ 13n-r,\ 37n-r\ ;\ 8n+5, 7$
	5, 0, 962	" ; "
	13, 0, 370	" ; "
	37, 0, 130	" ; "
—4830	1, 0, 4830	$3n+r,\ 5n+r,\ 7n+r,\ 23n+r\ ;\ 8n+1, 7$
	46, 0, 105	" ; "
	2, 0, 2415	$3n-r,\ 5n-r,\ 7n+r,\ 23n+r\ ;\ 8n+1, 7$
	23, 0, 210	" ; "
	3, 0, 1610	$3n-r,\ 5n-r,\ 7n-r,\ 23n+r\ ;\ 8n+3, 5$
	35, 0, 138	" ; "
	5, 0, 966	$3n-r,\ 5n+r,\ 7n-r,\ 23n-r\ ;\ 8n+3, 5$
	21, 0, 230	" ; "
	6, 0, 805	$3n+r,\ 5n+r,\ 7n-r,\ 23n+r\ ;\ 8n+3, 5$
	69, 0, 70	" ; "
	7, 0, 690	$3n+r,\ 5n-r,\ 7n+r,\ 23n-r\ ;\ 8n+1, 7$
	15, 0, 322	" ; "
	10, 0, 483	$3n+r,\ 5n-r,\ 7n-r,\ 23n-r\ ;\ 8n+3, 5$
	42, 0, 115	" ; "
	14, 0, 345	$3n-r,\ 5n+r,\ 7n+r,\ 23n-r\ ;\ 8n+1, 7$
	30, 0, 161	" ; "
—5160	1, 0, 5160	$3n+r,\ 5n+r,\ 43n+r\ ;\ 8n+1$
	40, 0, 129	" ; "
	3, 0, 1720	$3n+r,\ 5n-r,\ 43n-r\ ;\ 8n+3$
	43, 0, 120	" ; "

D	Formen.	Charaktere.
−5160	5, 0, 1032	$3n-r, 5n-r, 43n-r; 8n+5$
	8, 0, 645	“ ; “
	15, 0, 344	$3n-r, 5n+r, 43n+r; 8n+7$
	24, 0, 215	“ ; “
−5460	1, 0, 5460	$3n+r, 5n+r, 7n+r, 13n+r; 8n+1, 5$
	4, 0, 1365	“ ; “
	3, 0, 1820	$3n-r, 5n-r, 7n-r, 13n+r; 8n+3, 7$
	12, 0, 455	“ ; “
	5, 0, 1092	$3n-r, 5n-r, 7n-r, 13n-r; 8n+1, 5$
	20, 0, 273	“ ; “
	7, 0, 780	$3n+r, 5n-r, 7n-r, 13n-r; 8n+3, 7$
	28, 0, 195	“ ; “
	13, 0, 420	$3n+r, 5n-r, 7n-r, 13n+r; 8n+1, 5$
	52, 0, 105	“ ; “
	15, 0, 364	$3n+r, 5n+r, 7n+r, 13n-r; 8n+3, 7$
	60, 0, 91	“ ; “
	21, 0, 260	$3n-r, 5n+r, 7n+r, 13n-r; 8n+1, 5$
	65, 0, 84	“ ; “
	35, 0, 156	$3n-r, 5n+r, 7n+r, 13n+r; 8n+3, 7$
	39, 0, 140	“ ; “
−6402	1, 0, 6402	$3n+r, 11n+r, 97n+r; 8n+1, 3$
	3, 0, 2134	“ ; “
	22, 0, 291	“ ; “
	66, 0, 97	“ ; “
	2, 0, 3201	$3n-r, 11n-r, 97n+r; 8n+1, 3$
	6, 0, 1067	“ ; “
	11, 0, 582	“ ; “
	33, 0, 194	“ ; “
−6708	1, 0, 6708	$3n+r, 13n+r, 43n+r; 8n+1, 5$
	4, 0, 1677	“ ; “
	13, 0, 516	“ ; “
	52, 0, 129	“ ; “
	3, 0, 2236	$3n+r, 13n+r, 43n-r; 8n+3, 7$
	12, 0, 559	“ ; “
	39, 0, 172	“ ; “
	43, 0, 156	“ ; “
−7137	1, 0, 7137	$3n+r, 13n+r, 61n+r; 8n+1, 5$
	9, 0, 793	“ ; “
	13, 0, 549	“ ; “
	61, 0, 117	“ ; “
−7140	1, 0, 7140	$3n+r, 5n+r, 7n+r, 17n+r; 8n+1, 5$
	4, 0, 1785	“ ; “
	21, 0, 340	“ ; “
	84, 0, 85	“ ; “
	3, 0, 2380	$3n+r, 5n-r, 7n-r, 17n-r; 8n+3, 7$
	7, 0, 1020	“ ; “
	12, 0, 595	“ ; “
	28, 0, 255	“ ; “
−7140	5, 0, 1428	$3n-r, 5n-r, 7n-r, 17n-r; 8n+1, 5$
	17, 0, 420	“ ; “
	20, 0, 357	“ ; “
	68, 0, 105	“ ; “
	15, 0, 476	$3n-r, 5n+r, 7n+r, 17n+r; 8n+3, 7$
	35, 0, 204	“ ; “
	51, 0, 140	“ ; “
	60, 0, 119	“ ; “
−8710	1, 0, 8710	$5n+r, 13n+r, 67n+r; 8n+1, 7$
	10, 0, 871	“ ; “
	26, 0, 335	“ ; “
	65, 0, 134	“ ; “
	2, 0, 4355	$5n-r, 13n-r, 67n-r; 8n+3, 5$
	5, 0, 1742	“ ; “
	13, 0, 670	“ ; “
	67, 0, 130	“ ; “
−11718	1, 0, 11718	$13n+r, 17n+r, 53n+r; 8n+1, 5$
	13, 0, 901	“ ; “
	17, 0, 689	“ ; “
	53, 0, 221	“ ; “
−13398	1, 0, 13398	$3n+r, 7n+r, 11n+r, 29n+r; 8n+1, 7$
	22, 0, 609	“ ; “
	42, 0, 319	“ ; “
	58, 0, 231	“ ; “
	2, 0, 6699	$3n-r, 7n+r, 11n-r, 29n-r; 8n+3, 5$
	11, 0, 1218	“ ; “
	21, 0, 638	“ ; “
	29, 0, 462	“ ; “
	3, 0, 4466	$3n-r, 7n-r, 11n+r, 29n-r; 8n+3, 5$
	14, 0, 957	“ ; “
	66, 0, 203	“ ; “
	77, 0, 174	“ ; “
	6, 0, 2233	$3n+r, 7n-r, 11n-r, 29n+r; 8n+1, 7$
	7, 0, 1914	“ ; “
	33, 0, 406	“ ; “
	87, 0, 154	“ ; “
−13860	1, 0, 13860	$3n+r, 5n+r, 7n+r, 11n+r; 8n+1, 5$
	4, 0, 3465	“ ; “
	9, 0, 1540	“ ; “
	36, 0, 385	“ ; “
	5, 0, 2772	$3n-r, 5n-r, 7n-r, 11n+r; 8n+1, 5$
	20, 0, 693	“ ; “
	45, 0, 308	“ ; “
	77, 0, 180	“ ; “
	7, 0, 1980	$3n+r, 5n-r, 7n-r, 11n-r; 8n+3, 7$
	28, 0, 495	“ ; “
	55, 0, 252	“ ; “
	63, 0, 220	“ ; “

D	Formen.	Charaktere.	D	Formen.	Charaktere.
−13860	11, 0, 1260	$3n-r, 5n+r, 7n+r, 11n-r; 8n+3, 7$	−46852	1, 0, 46852	$13n+r, 17n+r, 53n+r; 8n+1, 5$
	35, 0, 396	“ ; “		4, 0, 11713	“ ; “
	44, 0, 315	“ ; “		13, 0, 3604	“ ; “
	99, 0, 140	“ ; “		17, 0, 2756	“ ; “
−17220	1, 0, 17220	$3n+r, 5n+r, 7n+r, 41n+r; 8n+1, 5$		52, 0, 901	“ ; “
	4, 0, 4305	“ ; “		53, 0, 884	“ ; “
	21, 0, 820	“ ; “		68, 0, 689	“ ; “
	84, 0, 205	“ ; “		212, 0, 221	“ ; “
	3, 0, 5740	$3n+r, 5n-r, 7n-r, 41n-r; 8n+3, 7$			
	7, 0, 2460	“ ; “			
	12, 0, 1435	“ ; “			
	28, 0, 615	“ ; “			
	5, 0, 3444	$3n-r, 5n+r, 7n-r, 41n+r; 8n+1, 5$			
	20, 0, 861	“ ; “			
	41, 0, 420	“ ; “			
	105, 0, 164	“ ; “			
	15, 0, 1148	$3n-r, 5n-r, 7n+r, 41n-r; 8n+3, 7$			
	35, 0, 492	“ ; “			
	60, 0, 287	“ ; “			
	123, 0, 140	“ ; “			
−19240	1, 0, 19240	$5n+r, 13n+r, 37n+r; 8n+1$			
	40, 0, 481	“ ; “			
	65, 0, 296	“ ; “			
	104, 0, 185	“ ; “			
	5, 0, 3848	$5n-r, 13n-r, 37n-r; 8n+5$			
	8, 0, 2405	“ ; “			
	13, 0, 1480	“ ; “			
	37, 0, 520	“ ; “			
−19320	1, 0, 19320	$3n+r, 5n+r, 7n+r, 23n+r; 8n+1$			
	105, 0, 184	“ ; “			
	3, 0, 6440	$3n-r, 5n-r, 7n-r, 23n+r; 8n+3$			
	35, 0, 552	“ ; “			
	5, 0, 3864	$3n-r, 5n+r, 7n-r, 23n-r; 8n+5$			
	21, 0, 920	“ ; “			
	7, 0, 2760	$3n+r, 5n-r, 7n+r, 23n-r; 8n+7$			
	15, 0, 1288	“ ; “			
	8, 0, 2415	$3n-r, 5n-r, 7n+r, 23n+r; 8n+7$			
	23, 0, 840	“ ; “			
	24, 0, 805	$3n+r, 5n+r, 7n-r, 23n+r; 8n+5$			
	69, 0, 280	“ ; “			
	40, 0, 483	$3n+r, 5n-r, 7n-r, 23n-r; 8n+3$			
	115, 0, 168	“ ; “			
	56, 0, 345	$3n-r, 5n+r, 7n+r, 23n-r; 8n+1$			
	120, 0, 161	“ ; “			

Nova methodus numeros compositos a primis dignoscendi illorumque factores inveniendi.

P. Seelhoff.

Quaeruntur divisores numeri N.

$$\text{Sit } N = w^2 + r$$

atque $N \equiv \rho\,(p)$, ρ significante residuum aliquod quadrati cum ipsius p, numeri primi, ita ut $w_1^2 \equiv \rho\,(p)$ existat.

Sumatur $N = w_1^2 + (w + w_1)(w - w_1) + r$ et designetur $(w + w_1)(w - w_1) + r$ litera b, unde sequitur $b = w^2 + r - w_1^2$.

$$\begin{array}{lrcl} \text{At} & w^2 + r & \equiv & \rho\,(p) \\ & - w_1^2 & \equiv & -\rho\,(p) \\ \hline \text{hinc} & b = w^2 + r - w_1^2 & \equiv & 0\,(p) \end{array}$$

Radix w_1 in $w_1 + py$ amplificata dat

$$N = (w_1 + py)^2 + \{w + (w_1 + py)\}\{w - (w_1 + py)\} + r.$$

Repertis ergo valoribus w, pro numeris primis usque ad 97 circiter, nisi N nimis magnus est (15 figuras non excedens) et pro binariis illorum potestatibus (pro 2, 3, 5 altiores etiam potestates adhibendae sunt), sin autem N major est, modulo congruentiarum pari passu extenso, plures simplices binariae quadratae repraesentationes comparando illos valores evadent et sequentia statui possunt.

Si numerus N compositus est, mox aut duas repraesentationes ejusdem determinantis aut plures adipisceris, e quibus elimininandis communibus factoribus duae ut

$$a_1^2 + mc_1^2 = \mu N$$
$$\text{et} \quad a_2^2 + mc_2^2 = \nu N$$

sequuntur, quae ad dispares radices congruentiae $z^2 \equiv -m\,(N)$ pertinent itaque duos divisores ipsius N producunt.

Sin vero numerus N est primus, haud secus facile ad tales eliminationes pervenies, quae e contrario ad eandem radicem $\pm z$ perducunt. N numerum primum esse pluribus determinantibus unius factoris evadentibus aut ambobus determinantibus $+\Delta$ et $-\Delta$ saepius occurrentibus affirmatur. Certitudinis causa auxilio determinantium repertorum omnes illi numeri primi quorum hi non-residua sunt quasi inepti ad divisionem excludi possunt.

Variatio quaedam utilis erit, nisi N formam $8n+1$ praebet. Sit *e. g.*, $N=8n+3$; jam ponatur $N=3w^2+r$ et $w_1^2 \equiv \frac{\rho+px}{3}\,(p)$, ita ut aliis numeris primis opus sit. Hoc modo factor 2^n pro b non omittitur.

Habemus similiter atque prius

$$N = 3w_1^2 + 3(w+w_1)(w-w_1) + r.$$

$$\text{At}\quad 3w^2 + r \equiv \rho\,(p)$$

$$-3w_1^2 \equiv -(\rho+px) \equiv -\rho\,(p),\ \text{unde}$$

$$b = 3(w+w_1)(w-w_1) + r \equiv 0\,(p).$$

Pro calculo ipso ponatur

$$w \mp (w_1+py) = \alpha,\ \text{unde}$$

$$w_1 + py = \pm(w-\alpha)\ \text{et}$$

$$w + (w_1+py)\ \text{aut}\ w-(w_1+py) = 2w-\alpha$$

$$N = (w-\alpha)^2 + 2(w-\alpha)\alpha + r.$$

Sit praeterea

$$2w \equiv \pm 2\beta\,(p)$$

$$r \equiv \gamma\,(p),$$

tum solvenda est congruentia

$$(\pm 2\beta - \alpha)\alpha \equiv -\gamma\,(p)\ \text{sive}$$

$$\alpha^2 \mp 2\beta\alpha \equiv \gamma\,(p)\ \text{et ponendo}$$

$$\alpha = \pm\beta + z$$

$$z^2 - (\beta^2+\gamma) \equiv 0\,(p).$$

Est autem

$$\beta^2 \equiv w^2$$

$$\gamma \equiv r$$

$$\beta^2 + \gamma \equiv w^2 + r \equiv \rho\,(p),\ \text{sive ut antea}$$

$$z^2 - \rho \equiv 0\ \text{et}\ z = w_1.$$

Sit, ut ad finem perveniam

$$\beta = \pm(w-py),\ \text{habetur atque prius}$$

$$\alpha = w \mp (w_1+py).$$

Congruentiae igitur et aequationes, quibus tota methodus nititur, hae sunt:

$N = w^2 + r$ $\qquad N \equiv \rho_1 (p),\ w_1^2 \equiv \rho_1 (p)$

$$w \equiv \pm \beta_1 (p)$$

$$a = \pm \beta_1 + w_1$$

praeterea $\qquad N \equiv \rho_2 (p^2),\ w_2^2 \equiv \rho_2 (p^2)$

$$w \equiv \pm \beta_2 (p^2)$$

$$a = \pm \beta_2 + w_2$$

pro 2, 3, 5 denique $\qquad N \equiv \rho_n (p^n),\ w_n^2 \equiv \rho_n (p^n)$

$$w \equiv \pm \beta_n (p^n)$$

$$a = \pm \beta_n + w_n$$

$$N = (w - a)^2 + \overline{(2w - a)\, a + r}^{\,b}.$$

Si numerus $N = 8n + 3$, etc., ponendum est $N = 3w^2 + \rho$, et loco congruentiarum

$$w_1^2 \equiv \rho_1 (p),\ w_2^2 \equiv \rho_2 (p^2),\ w_n^2 \equiv \rho_n (p^n)$$

ponendae sunt

$$w_1^2 \equiv \frac{\rho + px}{3} (p),\ w_2^2 \equiv \frac{\rho_2 + p^2}{3} (p^2),\ w_n^2 \equiv \frac{\rho_n + p^n}{3} (p^n) \text{ etc.}$$

et loco $\qquad N = (w - a)^2 + (2w - a)\, a + r$ aequatio

$$N = 3\,(w - a)^2 + 3\,\overline{(2w - a) + r}^{\,b}$$

ponenda est, etc.; reliqua intacta remanent.

Dentur exempla:

I. $\qquad N = 7.2^{34} + 1 = 120259084289$

$N = 346783^2 + 635200$, unde

$w = 346783$

$N = (346783 - a)^2 + (693566 - a)\, a + 635200$

$N \equiv 20\,(31),\ \rho_1 = 20;\ w \equiv +\,17\,(31),\ \beta_1 \equiv -\,14$

$w_1^2 \equiv 20\,(31),\ w_1 = \pm\,12$

$$a = -\,14 \pm 12 = 5 \text{ et } 29$$

$$\text{sive } \ a = 31y + 5,\ 29$$

$N \equiv 764\,(31^2),\ \rho_2 = 764 \qquad w \equiv +\,823\,(31^2),\ \beta_2 = -\,128$

$w_2^2 \equiv 764\,(31^2),\ w_2 = \pm\,198$

$$a = -\,128 \pm 198 = 60 \text{ et } 625$$

$$\text{sive } \ a = 31^2 y + 60,\ 625.$$

Hoc modo reperitur

$a = 2^3y + 0, 2, 4, 6;\ 2^4y + 0, 6, 8, 14;\ 2^5y + 0, 14, 16, 30;$
$2^6y + 0, 30, 32, 62;\ 2^7y + 30, 32, 94, 96;\ 2^8y + 30, 32, 158, 160;$
$2^9y + 158, 160, 414, 416;\ 2^{10}y + 158, 160, 670, 672.$

$a = 5y + 0, 1;\ 5^2y + 0, 16;\ 5^3y + 16, 50;\ 5^4y + 141, 300.$
$a = 7y + 2, 4;\ 7^2y + 2, 18.$
$a = 11y + 2, 3;\ 11^2y + 47, 68.$
$a = 19y + 1, 8;\ 19^2y + 115, 331.$
$a = 31y + 5, 29;\ 31^2y + 60, 625.$
$a = 37y + 12, 26;\ 37^2y + 271, 581.$ (1369)
$a = 47y + 10, 24;\ 47^2y + 762, 1387.$ (2209)
$a = 53y + 12, 49;\ 53^2y + 261, 2291.$ (2809)
$a = 67y + 2, 47;\ 67^2y + 114, 2146.$ (4489)
$a = 71y + 1, 37;\ 71^2y + 3871, 4119.$ (5041)
$a = 97y + 45, 68;\ 97^2y + 1911, 4798.$ (9409)
$a = 127y + 49, 97;\ 127^2y + 1748, 14400.$ (16129)

Habetur

(1) $N = 344833^2 + 2.7.11.2960^2$ (Ex $a = 1950$, $5y + 0$ cum $37^2y + 581$)
(2) $N = 203351^2 + 7.106172^2$ (Ex $a = 143432$, $11y + 3$ cum $127^2y + 14400$)
(3) $N = 350619^2 - 2.11.11026^2$ (Ex $a = -3836$, $11y + 3$ cum $37^2y + 271$)

Ex (1) et (2) sequitur (4) $11.832082029^2 - 2.150479740^2 = 62953059.N$ unde, comparando cum (3),

$$50459950484647^2 - 26380527979530^2 = \mu.N.$$

Maximus communis divisor differentiae $50459950484647 - 26380527979530$ et ipsius N, *i. e.* 317306291 est factor quaesitus, alter est 379.

II. Membrum quadragesimum octavum seriei 0, 1, 1, 2, 3, 5 ... est

$$N = 2971215073 = 54508^2 + 93009, \text{ et}$$
$$w = 54508$$
$$N = (54508 - a)^2 + \overbrace{(10916 - a)a + 93009}^{b}.$$

Simili modo atque in antecedente exemplo habebitur

pro 1, $a = 59$ $b = 2.7.17.72^2$
2, $a = 4109$ $b = 2.3.7.3204^2$
3, $a = -1$ $b = 2.3.23.29.2^2$
4, $a = -387$ $b = -3.7.17.344^2$

5,	$a = -\ 831$	$b = -\ 2.3.23.31.146^2$
6,	$a = -\ 5987$	$b = -\ 2.7.97.712^2$
7,	$a = 93$	$b = 17.29.144^2$
8,	$a = -\ 7519$	$b = -\ 2.31.37.618^2$
9,	$a = -\ 3187$	$b = -\ 2.3.7.31.524^2$
10,	$a = 1517$	$b = 2.7.17.828^2$
11,	$a = 3323$	$b = 3.7.17.992^2$
12,	$a = 3827$	$b = 3.7.29.43.124^2$
13,	$a = -\ 7051$	$b = -\ 7.10812^2$
14,	$a = 15421$	$b = 7.31.37.424^2$
15,	$a = -\ 28707$	$b = -\ 2.7.23.3504^2$
16,	$a = 31143$	$b = 2.3.43.3066^2$
17,	$a = 20561$	$b = 2.17.7314^2$
18,	$a = -\ 5891$	$b = -\ 23.37.43.136^2$
19,	$a = -\ 13573$	$b = -\ 3.7.23.1856^2$
20,	$a = 18305$	$b = 2.3.7.23.73.406^2$
21,	$a = -\ 94257$	$b = -\ 2.3.23.3204^2$
22,	$a = 21801$	$b = 2.3.17802^2$
23,	$a = -\ 24383$	$b = -\ 2.7.23.29.31.106^2$
24,	$a = 19$	$b = 7.556^2$
25,	$a = -\ 99$	$b = -\ 2.3.1336^2$, etc. etc.

(*a*) Ex 15 habemus $83215^2 - 2.7.23.3504^2 = N$

" 19 " $68081^2 - 3.7.23.1856^2 = N$, unde sequitur

$$3.4969913^2 - 2.4826470^2 = 9259.N \text{ et}$$

$$1670196456^2 \equiv 6\,(N).$$

Eadem congruentia ex 25

$$54607^2 - 2.3.1336^2 = N$$

derivari potest. Idem attingit in aliis casibus.

(*b*) Perspicuum est, multas repraesentationes atque $x^2 + cy^2 = \mu N$ climinandis communibus factoribus formari posse, quarum determinans ex uno factore constat.

(*c*) Habentur determinantes $+7\,(13)$ et $-7\,(24)$; $+6\,(25)$ et $-6\,(22)$ etc. Unde concludi potest, numerum N esse primum. Revera auxilio determinantium repertorum cuncti numeri primi usque ad $\sqrt{N}$ quasi inepti ad divisionem excludendi sunt; numerus 2971215073 est igitur numerus primus.

Ut valor ipsius α quam facillime obtineatur, tabulas composui, exhibentes radices congruentiae

$$w_1^2 \equiv \rho_1 (p)$$

pro numeris a 7 usque ad 199, radices congruentiae $w_2^2 \equiv \rho_2 (p^2)$ pro numeris a 7^2 usque ad 47^2, radices congruentiae $w_n^2 \equiv \rho_n (p^n)$ pro 2^3 usque ad 2^{10}, $3'$ usque ad 3^6, $5'$ usque ad 5^4.

Praeterea autem tabulas auxiliares construxi pro modulo p^2 a 53^2 usque ad 199^2.

Nam

$$\rho_2 \equiv \rho_1 (p) \text{ sive } \rho_2 = q \cdot p + \rho_1$$
$$w_1^2 \equiv \rho_1 (p) \text{ sive } w_1^2 = q_0 p + \rho_1$$
$$2\rho_1 u \equiv 1 (p) \text{ et}$$
$$(q - q_0) u \equiv \delta (p)$$

sequitur $w_2 = \pm \delta + w_1$.

Tabulae auxiliares amplectuntur igitur quatuor columnas, quarum inscriptiones sunt

$$\rho_1 \;.\; q_0 \;.\; u \;.\; w_1.$$

BREMEN, Mai 1885.

Analysis of Quintic Equations.

By Emory McClintock, LL.D., F. I. A.

1. The solution of equations of degrees below the fifth depends in each case on that of an auxiliary equation, or resolvent, one degree lower than the given equation. On applying the same principles towards the solution of the general quintic, its resolvent is found to be of the sixth degree instead of the fourth, for reasons which it is not necessary to recite. Abandoning therefore the idea of a solution, mathematicians have nevertheless considered it an interesting problem to accomplish in the simplest manner what is known as the resolution of the quintic. This has been defined by Sir James Cockle* as "the expression of its roots in terms of those of its resolvent sextic." Assuming that a sextic is discovered, appropriate for the purpose, the object to be attained is the expression of the roots of the quintic as a function of those of the sextic. Methods of resolution may be divided into two classes: those in which the usual theory of equations is illustrated by the intervenience of four auxiliary quantities which I shall herein call the elements, and those in which it is not. In methods of the first class each root is regarded as a sum of elements affected by powers of roots of unity as coefficients, and the problem consists in finding the simplest expressions for the elements as functions of the roots of the resolvent. In methods of the second class the intervening elements are omitted. If the chief object of effecting the resolution be considered, as I think it must, to be the illustration of the general theory of equations in the case of the lowest degree for which algebraic solution is impossible, the methods of the first class will be preferred. Before closing, I shall present one or two methods of the second class; but my leading purpose, apart from the indication of suitable resolvents, is to exhibit the elements most directly as functions of the roots of resolvents, by what I shall for brevity call element-formulæ.

* "On the Resolution of Quintics," Quarterly Journal, June, 1860.

2. The only resolvent hitherto* known is that of Malfatti. It has been independently rediscovered by several analysts, and has been shown in various different forms. The simplest rational function of the roots of the quintic which can be determined by Malfatti's resolvent is a quantity which I denote by v. The importance of this quantity has, I think very singularly, been wholly overlooked. I find that, analogous to Malfatti's resolvent, which I call the dexter resolvent, there is another which I call the sinister resolvent, and a third, so to speak between the two, and of greater importance as well as greater analytic symmetry than either, which I call the central resolvent. The simplest rational functions of the roots determined by the sinister and central resolvents, respectively, I denote by s and t. Between the three auxiliary quantities just mentioned exists the relation $s = t^2 v$. I shall state what seem to be the most direct formulæ for expressing t and v as rational functions of each other, and indicate means for similarly expressing s and v. Whenever, therefore, one of the three auxiliary quantities is found by means of one of the three resolvents, the others become known. The element-formulæ to be shown in the next paragraph contain only the auxiliary quantities and the coefficients of the quintic, and are thus available whichever of the three resolvents is employed. Beginning with a summary statement of what appears the best attainable method of resolution, with illustrations, I shall proceed to give a sketch of the history of the subject, in which I shall endeavor to give due credit to each discoverer. Entering then upon a more general discussion, I shall develop the theory of the three resolvents, followed by that of the element-formulæ, and of the expression as functions of each other of t and v. After treating of various subsidiary matters, I shall conclude with a number of suggestions relating to methods of resolution of the second class, and to solutions of the quintic by means of tabulated functions.

3. If the general equation of the fifth degree be

$$ax^5 + 5bx^4 + 10cx^3 + 10dx^2 + 5ex + f = 0, \tag{1}$$

let

$$\left.\begin{aligned} \gamma &= a^{-2}(ac - b^2), \\ \delta &= a^{-3}(a^2d - 3abc + 2b^3), \\ \varepsilon &= a^{-4}(a^3e - 4a^2bd + 6ab^2c - 3b^4), \\ \zeta &= a^{-5}(a^4f - 5a^3be + 10a^2b^2d - 10ab^3c + 4b^5). \end{aligned}\right\} \tag{2}$$

* Except as given in a recent paper by the present writer, "On the Resolution of Equations of the Fifth Degree" (*American Journal of Mathematics*, 1884, Vol. VI, pp. 301–315). Such parts of that paper as appear to possess permanent value will be embodied herein as original, with footnote references to "Vol. VI." I take this opportunity to point out certain *errata*. Page 302, line 2, for p^5 read p_5. Page 310, line 12, for "and ϕ" read "when ϕ is, and sometimes when ϕ". Page 311, last line, for $-2\delta\varepsilon\zeta$ read $+2\delta\varepsilon\zeta$.

Let there be an auxiliary quintic, $A_y = t^5 + 10\gamma t^3 + 10\delta t^2 + 5\varepsilon t + \zeta$, with its canonizant, $C_y = c_0 t^3 + c_1 t^2 + c_2 t + c_3$, and its simplest linear covariant, $L_y = l_0 t + l_1$, where

$$\begin{aligned}
c_0 &= -\gamma^3 + \gamma\varepsilon - \delta^2,\\
c_1 &= -\gamma^2\delta + \gamma\zeta - \delta\varepsilon,\\
c_2 &= -\gamma\delta^2 + \gamma^2\varepsilon + \delta\zeta - \varepsilon^2,\\
c_3 &= 2\gamma\delta\varepsilon - \gamma^2\zeta - \delta^3,\\
l_0 &= -15\gamma^4\varepsilon + 10\gamma^3\delta^2 - 2\gamma^2\delta\zeta + 14\gamma^2\varepsilon^2 - 22\gamma\delta^2\varepsilon + \gamma\zeta^2 + 9\delta^4 - 2\delta\varepsilon\zeta + \varepsilon^3,\\
l_1 &= 9\gamma^4\zeta - 20\gamma^3\delta\varepsilon + 10\gamma^2\delta^3 + 8\gamma^2\varepsilon\zeta - 12\gamma\delta\varepsilon^2 - 2\gamma\delta^2\zeta + 6\delta^3\varepsilon + \delta\zeta^2 - \varepsilon^2\zeta.
\end{aligned}$$

Let a value of t be found by the numerical solution of the central resolvent*

$$A_y L_y - 25C_y^2 = 0, \tag{3}$$

which may be written thus:

$$\left.\begin{aligned}
&(l_0 - 25c_0^2)\,t^6 + (l_1 - 50c_0c_1)\,t^5 + 5\,(2\gamma l_0 - 5c_1^2 - 10c_0c_2)\,t^4\\
&+ 10\,(\gamma l_1 + \delta l_0 - 5c_0c_3 - 5c_1c_2)\,t^3 + 5\,(2\delta l_1 + \varepsilon l_0 - 5c_2^2 - 10c_1c_3)\,t^2\\
&+ (5\varepsilon l_1 + \zeta l_0 - 50c_2c_3)\,t + \zeta l_1 - 25c_3^2 = 0.
\end{aligned}\right\} \tag{4}$$

Also, after t is known, let v be found by using either of these expressions:

$$v = -C_y A_y^{-1} = -\tfrac{1}{25} L_y C_y^{-1}. \tag{5}$$

Then the five roots of (1), say x_1, x_2, x_3, x_4, x_5, are severally determined by the equation†

$$x_{r+1} = \omega^{4r}u_1 + \omega^{3r}u_2 + \omega^{2r}u_3 + \omega^r u_4 - ba^{-1}, \tag{6}$$

where ω is any unreal fifth root of unity, and r is 0, 1, 2, 3, or 4; and where u_1, u_2, u_3, u_4, are determined by the element-formulæ

$$\left.\begin{aligned}
u_1^5 &= \tfrac14 r_1 + \tfrac14 r_2 + \sqrt{(s_1 + s_2)},\\
u_2^5 &= \tfrac14 r_1 - \tfrac14 r_2 + \sqrt{(s_1 - s_2)},\\
u_3^5 &= \tfrac14 r_1 - \tfrac14 r_2 - \sqrt{(s_1 - s_2)},\\
u_4^5 &= \tfrac14 r_1 + \tfrac14 r_2 - \sqrt{(s_1 + s_2)},
\end{aligned}\right\} \tag{7}$$

where

$$r_1 = -\zeta - 20tv, \tag{8}$$

$$\left.\begin{aligned}
r_2 = (\gamma^2 - v)^{-1}\,(&12\gamma tv^2 - \delta v^2 - t^3v^2 + 4\gamma^3tv + 2\gamma^2\delta v\\
&+ \gamma\delta t^2v + \delta^2tv - 2\gamma\varepsilon tv + \delta\varepsilon v - \gamma^4\delta + \gamma^2\delta\varepsilon - \gamma\delta^3)\,v^{-\frac12},
\end{aligned}\right\} \tag{9}$$

$$s_1 = \tfrac{1}{16} r_1^2 + \tfrac{1}{16} r_2^2 + \gamma^5 + 10\gamma^3 v + 5\gamma v^2, \tag{10}$$

$$s_2 = \tfrac18 r_1 r_2 - (5\gamma^4 + 10\gamma^2 v + v^2)\,v^{\frac12}. \tag{11}$$

The result is unaffected by any change in the signs given to the square-roots.

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*Vol. VI, p. 315. The quantity t was there taken with the contrary sign. Its existence and properties appear to have remained unnoticed by previous writers.

† According to Euler's well-known theory; the quantities u_1, u_2, u_3, u_4, remaining to be ascertained.

4. As an illustration let us take the equation

$$x^5 - 5x^4 - 25x^3 + 125x^2 - 36x - 180 = 0. \qquad (12)$$

Here $a = 1$, $b = -1$, $c = -\frac{5}{2}$, $d = \frac{25}{2}$, $e = -\frac{36}{5}$, $f = -180$. Hence, by (2), $\gamma = -\frac{7}{2}$, $\delta = 3$, $\varepsilon = \frac{124}{5}$, $\zeta = -120$. We have therefore $c_0 = -\frac{2117}{40}$, $c_1 = \frac{6177}{20}$, $c_2 = -\frac{31987}{80}$, $c_3 = \frac{4611}{5}$, $l_0 = \frac{6906219}{125}$, $l_1 = -\frac{4891314}{25}$, and when these values are substituted in (4), and the result cleared of fractions, the resolvent is

$$\left.\begin{array}{l} -118277109t^6 + 4973134020t^5 - 48088650860t^4 + 166597190880t^3 \\ -187938837936t^2 - 11132130240t + 17735889600 = 0. \end{array}\right\} \qquad (13)$$

One root of this is $t = 30$. Using this value in (5), we find that $v = \frac{1}{20}$. The other values of t, with the corresponding values of v, are

t	6	$\frac{130}{51}$	$\frac{66}{19}$	$\frac{22}{69}$	$-\frac{18}{61}$
v	$\frac{841}{500}$	$\frac{2601}{500}$	$\frac{361}{500}$	$\frac{4761}{500}$	$\frac{3721}{500}$

Taking $t = 30$, $v = \frac{1}{20}$, we have $r_1 = 90$, $r_2 = 6\sqrt{5}$, $s_1 = -\frac{146}{5}$, $s_2 = -\frac{1018}{125}\sqrt{5}$, so that by (7)

$$\left.\begin{array}{l} u_1^5 = \frac{45}{2} + \frac{3}{2}\sqrt{5} + \sqrt{(-\frac{146}{5} - \frac{1018}{125}\sqrt{5})}, \\ u_2^5 = \frac{45}{2} - \frac{3}{2}\sqrt{5} + \sqrt{(-\frac{146}{5} + \frac{1018}{125}\sqrt{5})}, \\ u_3^5 = \frac{45}{2} - \frac{3}{2}\sqrt{5} - \sqrt{(-\frac{146}{5} + \frac{1018}{125}\sqrt{5})}, \\ u_4^5 = \frac{45}{2} + \frac{3}{2}\sqrt{5} - \sqrt{(-\frac{146}{5} - \frac{1018}{125}\sqrt{5})}. \end{array}\right\} \qquad (14)$$

The arithmetical evaluation of these quantities is not possible, since they are imaginary. There are five real roots in the example selected, and the same difficulty arises as in the corresponding ("irreducible") case in cubic equations, where there are three real roots. Yet we have here the correct analytic expression for the elements of the roots.

5. As another example—this time with but one real root—let

$$x^5 + 5x^4 + 9x^3 + 5x^2 - 4x - 5 = 0. \qquad (15)$$

Here $a = 1$, $b = 1$, $c = \frac{9}{10}$, $d = \frac{1}{2}$, $e = -\frac{4}{5}$, $f = -5$, so that, by (2), $\gamma = -\frac{1}{10}$, $\delta = -\frac{1}{5}$, $\varepsilon = -\frac{2}{5}$, $\zeta = -1$; whence $c_0 = .001$, $c_1 = .022$, $c_2 = .04$, $c_3 = .002$, $l_0 = -.0062$, $l_1 = -.0353$, and by (4), after multiplying by 2500,

$$-15\tfrac{9}{16}t^6 - 91t^5 - 19\tfrac{3}{4}t^4 + 9t^3 + 102t^2 + 182t + 88 = 0. \qquad (16)$$

One root is $t = \frac{4}{5}$, and this substituted in (5) gives $v = .018$. Hence $r_1 = .52$, $r_2 = .104\sqrt{5}$, $s_1 = .019928$, $s_2 = .00660256\sqrt{5}$, and by (7)

$$\left.\begin{array}{l} u_1^5 = .13 + .026\sqrt{5} + \sqrt{(.019928 + .00660256\sqrt{5})}, \\ u_2^5 = .13 - .026\sqrt{5} + \sqrt{(.019928 - .00660256\sqrt{5})}, \\ u_3^5 = .13 - .026\sqrt{5} - \sqrt{(.019928 - .00660256\sqrt{5})}, \\ u_4^5 = .13 + .026\sqrt{5} - \sqrt{(.019928 + .00660256\sqrt{5})}, \end{array}\right\} \qquad (17)$$

and from (6), putting $r = 0$, we have as the real root of the quintic,

$$\left.\begin{aligned} x_1 &= u_1 + u_2 + u_3 + u_4 - 1 \\ &= 0.82161 + 0.67843 - 0.05036 + 0.28501 - 1 = 0.73469. \end{aligned}\right\} \quad (18)$$

6. It was proved by Lagrange that in determining the elements the extraction of more than one fifth-root is unnecessary. We shall see later that $u_1 u_4 = -\gamma + v^{\frac{1}{2}}$, and $u_2 u_3 = -\gamma - v^{\frac{1}{2}}$, so that u_4 and u_3 are readily found when u_1 and u_2 are known. For finding u_2 we may note that $u_2 = (u_2 u_3)\, u_1^2 (u_1^2 u_3)^{-1}$, and we shall see that $u_1^2 u_3 = -\tfrac{1}{2}\delta - \tfrac{1}{2} t v^{\frac{1}{2}} + \sqrt{(s_3 + s_4)}$, where $s_3 = \gamma^3 - \gamma v + \tfrac{1}{4}\delta^2 + \tfrac{1}{4} t^2 v$, and $s_4 = (v - \gamma^2 + \tfrac{1}{2}\delta t)\, v^{\frac{1}{2}}$. No perceptible benefit is derived, however, from the latter variation. To illustrate the determination of u_3 and u_4 when u_2 and u_1 are known, let us take the example in the preceding paragraph. Here $\gamma = -.1$, $v^{\frac{1}{2}} = .13416$, $u_1 u_4 = .23416$, $u_2 u_3 = -.03416$, $u_1 = .82161$, $u_2 = .67843$; hence $u_4 = .28501$, $u_3 = -.05036$, as before.

7. The foregoing method of resolution is available for all equations of the fifth degree wherein (5) is not indeterminate, except those in which $v = \gamma^2$ and those in which $v = 0$. In the former case, $u_1 u_2 u_3 u_4 = 0$, or say $u_4 = 0$, since the final result is the same when one element vanishes, whichever be chosen. In the latter case, $u_1 u_4 = u_2 u_3$. The quantity $u_1 u_2 u_3 u_4$ was called "the resolvent product" by Sir James Cockle, who pointed out* that all quintics are solvable in which the resolvent product vanishes, and furnished a criterion by which to recognize a quintic of this class when met with. The following is a modification† of the criterion in question: $v = \gamma^2$ when

$$\left.\begin{aligned} &[1600\gamma^6 - 640\gamma^4\varepsilon + 160\gamma^3\delta^2 - 52\gamma^2\delta\zeta + 64\gamma^2\varepsilon^2 + \gamma\zeta^2 \\ &\qquad + 28\gamma\delta^2\varepsilon - 2\delta\varepsilon\zeta - 16\delta^4 + \varepsilon^3]^2 = \gamma^2 \mathrm{D}, \end{aligned}\right\} \quad (19)$$

where $\mathrm{D} = \text{discriminant} \times a^{-8} = 8\,[432\gamma^5\zeta^2 - 1440\gamma^4\delta\varepsilon\zeta + 800\gamma^4\varepsilon^3 + 640\gamma^3\delta^3\zeta - 400\gamma^3\delta^2\varepsilon^2 - 180\gamma^3\varepsilon\zeta^2 + 330\gamma^2\delta^2\zeta^2 + 560\gamma^2\delta\varepsilon^2\zeta - 320\gamma^2\varepsilon^4 - 1260\gamma\delta^3\varepsilon\zeta + 720\gamma\delta^2\varepsilon^3 - 15\gamma\delta\zeta^3 + 20\gamma\varepsilon^2\zeta^2 + 432\delta^5\zeta - 270\delta^4\varepsilon^2 + 45\delta^2\varepsilon\zeta^2 - 80\delta\varepsilon^3\zeta + 32\varepsilon^5 + \tfrac{1}{8}\zeta^4]$. In any such case, one or two critical cases excepted, the process of solution about to be described will no doubt be found satisfactory. When $v = \gamma^2$,

$$t = \frac{800\gamma^5\delta + 16\gamma^4\zeta + 920\gamma^3\delta\varepsilon - 560\gamma^2\delta^3 - 8\gamma^2\varepsilon\zeta - 38\gamma\delta\varepsilon^2 - 48\gamma\delta^2\zeta + 44\delta^3\varepsilon - \delta\zeta^2 + \varepsilon^2\zeta}{-10400\gamma^6 + 1960\gamma^4\varepsilon - 240\gamma^3\delta^2 + 48\gamma^2\delta\zeta - 86\gamma^2\varepsilon^2 + 28\gamma\delta^2\varepsilon + \gamma\zeta^2 - 16\delta^4 - 2\delta\varepsilon\zeta + \varepsilon^3}. \quad (20)$$

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* Appendix to Lady's and Gentleman's Diary, 1858, p. 82, as quoted by Harley, Manchester Memoirs, Vol. XV. Mr. Harley gave a convenient process of solution, but different from that now presented, and more complex.

† Vol. VI, p. 812.

Supposing $u_4 = 0$, we have $v^2 = \gamma$, and $u_2 u_3 = -2\gamma$, giving u_3 when u_2 is known. Then

$$u_1^5 = \tfrac{1}{4}\gamma^{-2}(\gamma t + \delta)^2(\gamma t - \delta) = \tfrac{1}{4}(\gamma t^3 + \delta t^2 - \gamma^{-1}\delta^2 t - \gamma^{-2}\delta^3), \qquad (21)$$

$$u_2 = -u_1^{-2}(\gamma t + \delta). \qquad (22)$$

Take for example this equation,

$$x^5 + 5x^3 + 5x^2 + 5x - \tfrac{35}{2} = 0. \qquad (23)$$

Here $\gamma = \frac{1}{2}$, $\delta = \frac{1}{2}$, $\varepsilon = 1$, $\zeta = -\frac{35}{2}$, and by (20) $t = 3$, so that $u_1 = 4^{\frac{1}{5}}$, $u_2 = -2^{\frac{1}{5}}$, $u_3 = 2^{-\frac{1}{5}}$, $u_4 = 0$, and hence a root of (23), the only real root, is $1.31951 + 0.87055 - 1.14870 = 1.04136$. Among the exceptional or critical cases are that of De Moivre, wherein $\delta = 0$ and $\varepsilon = 4\gamma^2$, and that of Euler, wherein $\gamma = 0$ and $16\delta^4 + 2\delta\varepsilon\zeta - \varepsilon^3 = 0$, of which solutions are well known.

8. The second class of quintics spoken of in the last paragraph as requiring special treatment is that wherein $u_1 u_4 = u_2 u_3$, a class first separately discussed, so far as my knowledge goes, by Mr. G. P. Young,* who noted that within it are included, with others, such quintics as might be employed, under Gauss's theory, in the determination of roots of unity. I gave somewhat later† a method of solution for this class of cases, less circuitous than that indicated by Mr. Young, but much more so than that which will now be presented. I also supplied a criterion‡ by which to recognize such cases when they occur, as follows: $u_1 u_4 = u_2 u_3$ when $v = 0$, and $v = 0$ when

$$25\gamma^6 - 35\gamma^4\varepsilon + 40\gamma^3\delta^2 + 2\gamma^2\delta\zeta + 11\gamma^2\varepsilon^2 - \gamma\zeta^2 - 28\gamma\delta^2\varepsilon + 2\delta\varepsilon\zeta + 16\delta^4 - \varepsilon^3 = 0. \qquad (24)$$

In any such case, not critical, the quintic may be solved instantly by employing the normal formulæ (7) for the elements, but putting $r_1 = -\zeta$, $r_2 = (3\gamma^4 - 4\gamma^2\varepsilon + 4\gamma\delta^2 - \delta\zeta + \varepsilon^2)(\gamma^2 - \gamma\varepsilon + \delta^2)^{-\frac{1}{2}}$, $s_1 = \frac{1}{16}r_1^2 + \frac{1}{16}r_2^2 + \gamma^5$, $s_2 = \frac{1}{8}r_1 r_2$. For example, let

$$x^5 + 10x^3 - 80x^2 + 145x - 480 = 0. \qquad (25)$$

Here $\gamma = 1$, $\delta = -8$, $\varepsilon = 29$, $\zeta = -480$, $r_1 = 480$, $r_2 = -476$, $s_1 = 28562$, $s_2 = -28560$, and these may be substituted at once in (7). When $\gamma^2 - \gamma\varepsilon + \delta^2 = 0$, the expression for r_2 must be replaced by $-(\gamma^{-2}\delta^3 + 4\gamma\delta)$. Thus, if

$$x^5 + 20x^3 + 20x^2 + 30x + 10 = 0, \qquad (26)$$

we have $\gamma = 2$, $\delta = 2$, $\varepsilon = 6$, $\zeta = 10$, $r_1 = -10$, $r_2 = -18$, $s_1 = \frac{117}{2}$, $s_2 = \frac{45}{2}$, and by (7) we have $u_1^5 = 2$, $u_2^5 = -4$, $u_3^5 = 8$, $u_4^5 = -16$, whence $x_1 = 2^{\frac{1}{5}} - 2^{\frac{2}{5}} + 2^{\frac{3}{5}} - 2^{\frac{4}{5}}$.

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* "Resolution of Solvable Equations of the Fifth Degree," *American Journal of Mathematics*, VI, 103 (1883).

† Vol. VI, p. 313. ‡ Vol. VI, p. 311.

9. Thus far we have had only a statement of certain results without proofs, brought together for the convenience of those who may have occasion to deal practically with the problem of the resolution of the quintic. Let us now consider the subject in due order.

10. Referring to the general equation (1), let $y = x + ba^{-1}$; then substituting,

$$y^5 + 10\gamma y^3 + 10\delta y^2 + 5\varepsilon x + \zeta = 0, \tag{27}$$

where the coefficients have the values shown in (2). Let

$$x_1 + \omega^m x_2 + \omega^{2m} x_3 + \omega^{3m} x_4 + \omega^{4m} x_5 = 5u_m, \tag{28}$$

where m has any integral value from 1 to 4 inclusive. From this, bearing in mind that $x_1 + x_2 + x_3 + x_4 + x_5 = -5ba^{-1}$, and that $\omega^m + \omega^{2m} + \omega^{3m} + \omega^{4m} = -1$, we have at once the proof of (6). We may write y for x in (28) without changing the result, since the addition of ba^{-1} to each root in the first member adds merely to the second member the expression $ba^{-1}(1 + \omega^m + \omega^{2m} + \omega^{3m} + \omega^{4m}) = 0$. Then, on development,

$$\begin{aligned}
\Sigma y^2 &= 10\,(u_1u_4 + u_2u_3),\\
\Sigma y^3 &= 15\,(u_1^2u_3 + u_4^2u_2 + u_2^2u_1 + u_3^2u_4),\\
\Sigma y^4 &= 30\,(u_1^2u_4^2 + u_2^2u_3^2) + 20\,(u_1^3u_2 + u_4^3u_3 + u_2^3u_4 + u_3^3u_1) + 120u_1u_2u_3u_4,\\
\Sigma y^5 &= 5\Sigma u^5 + 100\,(u_1^3u_3u_4 + u_4^3u_2u_1 + u_2^3u_1u_3 + u_3^3u_4u_2)\\
&\quad + 150\,(u_1^2u_2^2u_4 + u_4^2u_3^2u_1 + u_2^2u_4^2u_3 + u_3^2u_1^2u_2).
\end{aligned}$$

But $\Sigma y^2 = -20\gamma$, $\Sigma y^3 = -30\delta$, $\Sigma y^4 = 200\gamma^2 - 20\varepsilon$, $\Sigma y^5 = 500\gamma\delta - 5\zeta$. Hence, by comparison with the above, we obtain four equations, which contain as unknown quantities the four elements, and which we may refer to hereafter as the Eulerian equations,* namely,

$$-2\gamma = u_1u_4 + u_2u_3, \tag{29}$$

$$-2\delta = u_1^2u_3 + u_4^2u_2 + u_2^2u_1 + u_3^2u_4, \tag{30}$$

$$-\varepsilon = u_1^3u_2 + u_4^3u_3 + u_2^3u_4 + u_3^3u_1 + 3u_1u_2u_3u_4 - 4\gamma^2, \tag{31}$$

$$-\zeta = \Sigma u^5 - 10\,(u_1^3u_3u_4 + u_4^3u_2u_1 + u_2^3u_1u_3 + u_3^3u_4u_2) + 20\gamma\delta. \tag{32}$$

11. The elements (u_1, u_2, u_3, u_4) were discovered and discussed independently by Euler and Bezout,† and further light was thrown on the theory by Lagrange and Vandermonde, also independently.‡ Such of their results as we shall need

* This succinct exposition of the theory as it stood in 1771 is substantially copied from Mr. Harley's "Theory of Quintics," Quarterly Journal of Mathematics, January, 1860.

† Euler, "De Resolutione Æquationum cuiusvis Gradus," Novi Comm. Petrop. IX (1762–63); Bezout, "Sur la Résolution générale des Équations de tous les Degrés," Paris Mémoires, 1765.

‡ The best known paper of Lagrange on algebraic equations and Vandermonde's paper on the resolution of equations were read about the same time, in 1771, before the academies of Berlin and Paris respectively.

to make use of are embodied in the preceding paragraph. Lagrange was the first* to recommend the resolution of the quintic by the use of an auxiliary sextic. He showed that Σu^5, or $u_1^5 + u_2^5 + u_3^5 + u_4^5$, has six values, which he proposed to take as the roots of a resolvent sextic, whose coefficients might be determined in terms of those of the quintic. Supposing that by such a resolvent there were once ascertained a value of Σu^5, say $-p_3$, he showed that a biquadratic might be taken,

$$w^4 + p_3 w^3 + p_2 w^2 + p_1 w + p_0 = 0, \tag{33}$$

of which the four roots would be u_1^5, u_2^5, u_3^5, and u_4^5,† and of which the remaining coefficients, p_2, p_1, p_0, might be determined by a process which would have been mathematically feasible, though in practice involving much labor, and which was never carried out in detail by any one, so far as appears. It is but a momentary digression from our course to inquire into the real form of the unnecessary biquadratic in Lagrange's theory, as a mere matter of historic interest. The four values of w are given in (7). Putting these together to form (33), we see that $p_3 = -r_1$, $p_2 = \frac{1}{4} r_1^2 - \frac{1}{4} r_2^2 - 2r_3$, $p_1 = r_1 r_3 + r_2 r_4$, $p_0 = r_3^2 - r_4^2$, where r_1 and r_2 have the same values as in (7), and $r_3 = \gamma^5 + 10\gamma^3 v + 5\gamma v^2$, $r_4 = (5\gamma^4 + 10\gamma^2 v + v^2) v^{\frac{1}{2}}$. Since p_0 is the product of the roots of (33), it is the fifth power of the "resolvent product" $u_1 u_2 u_3 u_4$, and Lagrange's method of resolution was so far modified by Messrs. Cockle, Harley, and Cayley‡ as to start with p_0 as the first coefficient of the biquadratic to be determined, for employment instead of p_3 as the basis for finding the others; a proceeding equally laborious with that suggested by Lagrange, and apparently never undertaken. A slight improvement upon Lagrange's proposal for a resolvent was suggested by Meyer Hirsch in 1808, without practical result.§ We may illustrate (33) by the first example (12) before chosen, wherein $v = \frac{1}{20}$, $\gamma = -\frac{7}{2}$, $r_1 = 90$, $r_2 = 6\sqrt{5}$, so that $r_3 = -\frac{5467}{10}$, $r_4 = \frac{18911}{250}\sqrt{5}$, and the biquadratic is

$$w^4 - 90w^3 + \frac{15367}{5} w^2 - \frac{1173342}{25} y + \frac{844596301}{3125} = 0. \tag{34}$$

* For a historical sketch of the views of different writers on the solvability of the quintic, see Cockle, "On the Method of Symmetric Products," Philosophical Magazine, Feb. 1854. The impossibility of a solution was not conceded by all—Jerrard being apparently the last disputant—before 1868; see Cockle, "Concluding Remarks," Philosophical Magazine, Sept., 1868. Schulenburg's "solution" was translated and printed in the Analyst in 1877. Meyer Hirsch's "solution" was once famous.

† The elements discussed by Lagrange were five times as large as these, with corresponding differences throughout.

‡ Cayley, "On a New Auxiliary Equation in the Theory of Equations of the Fifth Order," Philosophical Transactions, Vol. 151, p. 263 (1861). This paper contained, as will be stated later, an unexceptionable method for determining p_0.

§ Ross, Translation of Hirsch's Sammlung, p. 282.

12. Lagrange afterwards reviewed his theory of equations in comparison with the closely similar theory of Vandermonde. Of this review we need to note but a single point.* Any one who takes the trouble to develop u_1^5, u_2^5, u_3^5, u_4^5, as the elements are defined in (28), will find that $u_1^5 + u_2^5 + u_3^5 + u_4^5$, say r_1, is a rational function of the roots, and that $u_1^5 + u_4^5 - u_2^5 - u_3^5$, say r_2, is another rational function multiplied by $\sqrt{5}$. In carrying out a certain numerical example, Lagrange found the fifth-powers of the elements, according to his theory, to be of a form which we may write briefly thus:

$$\left.\begin{aligned} u_1^5 &= \tfrac{1}{4} r_1 + \tfrac{1}{4} r_2 + q_1, \qquad & u_2^5 &= \tfrac{1}{4} r_1 - \tfrac{1}{4} r_2 + q_2, \\ u_4^5 &= \tfrac{1}{4} r_1 + \tfrac{1}{4} r_2 - q_1, \qquad & u_3^5 &= \tfrac{1}{4} r_1 - \tfrac{1}{4} r_2 - q_2. \end{aligned}\right\} \tag{35}$$

This arrangement is indeed a necessary one, if we define q_1 to be $\tfrac{1}{2}(u_1^5 - u_4^5)$, and q_2 to be $\tfrac{1}{2}(u_2^5 - u_3^5)$. But Vandermonde had worked out the same example thus:

$$\begin{aligned} u_1^5 &= \tfrac{1}{4} r_1 + \tfrac{1}{4} r_2 + q_1, \qquad & u_2^5 &= \tfrac{1}{4} r_1 - \tfrac{1}{4} r_2 - q_1, \\ u_4^5 &= \tfrac{1}{4} r_1 + \tfrac{1}{4} r_2 - q_2, \qquad & u_3^5 &= \tfrac{1}{4} r_1 - \tfrac{1}{4} r_2 + q_2. \end{aligned}$$

In comparing Vandermonde's result with that obtained by his own method, Lagrange pointed out that the former was erroneous even according to Vandermonde's theory, as he showed by revising Vandermonde's work and producing the correct result (35).

13. It was not until 1858 that any one noticed that Gian Francisco Malfatti† had, in the very same year that Lagrange and Vandermonde brought out their almost identical theories, done more than either of them towards the resolution of quintic equations.‡ For eighty seven years his successful method of resolution was looked upon merely as a fruitless effort towards algebraic solution, as indeed Malfatti himself would seem to have regarded it. He pursued a method for

* Lagrange, Théorie des Équations Numériques, Note XIV, paragraphs 30–36.

† My authorities concerning Malfatti are Harley, "On Recent Researches in the Theory of Equations," Manchester Proceedings, Nov. 3, 1863 (who quotes from Brioschi, "Sulla Risolvente di Malfatti per le Equationi del Quinto Grado," Reale Istituto Lombardo—Rendiconti, IX, 215); Brioschi, "Ueber die Auflösung der Gleichungen vom fünften Grade," Mathematische Annalen, XIII, 109; and Minich, "Sulle Equazioni del Quinto Grado" (I have seen only the second chapter of the "first memoir"), Atti del Reale Istituto Veneto (5th series), VIII, 898. Mr. Harley's paper contains a striking statement of the enormous difficulties attending the solution of the Eulerian equations by any usual method of elimination.

‡ Like most sweeping statements, this remark, that no one noticed what Malfatti had done, has its exception. Ruffini wrote in 1805 (Memorie della Società Italiana, XII, 322): "L'illustre Autore ha così il vanto di avere, senza punto conoscere le corrispondenti osservazioni del Lagrange, ottenuta completamente una trasformazione, rapporto alla quale l'immortale Matematico di Torino non aveva infine determinato che il grado, ed un solo coefficiente." The last four words might well have been omitted. Malfatti's memoir, "De Æquationibus Quadrato-cubicis Disquisitio Analytica," appeared in the Siena transactions for 1771.

solving equations of the lower degrees by means of a resolvent, the resolvent in each case being one degree lower than the equation to be solved. In applying this method to the quintic, he actually obtained, to his disappointment, a resolvent sextic, in which the unknown quantity was what I should denote by $25v + 5\gamma^2 + \frac{5}{8}\varepsilon$. I presume that he did this by the solution of the Eulerian equations, or by some equivalent process, and it is certain that he indicated some simpler method for the final determination of the elements than the solution of a biquadratic. But with all this he had a sense of failure, for he continued during many years to avow himself a believer in the solvability of the quintic. Possessed of high reputation in Italy, he was chiefly known in other countries* as taking a conspicuous part on the wrong side of that question. The result even in Italy was to deter inquirers from study of his writings on the quintic, which are indeed said to have been superciliously disparaged and before long overlooked and forgotten. The existence of Malfatti's resolvent was brought to the notice of the Venetian Institute, in 1858, by Professor Minich, and a paper concerning it was read in 1863 by Signor Brioschi at Milan.

14. Meantime, all went on as if Malfatti's work had never been done. A resolvent was wanted. In 1835 Jacobi† showed that a resolvent can be constructed having for its roots the six values of a quantity which we may, following Professor Cayley, denote by ϕ, equal in our notation to $10\sqrt{(5v)}$. His reasoning would seem to have been substantially as follows, or at any rate the following reasoning will justify his conclusions.

15. Let c represent a cycle, by which term let us understand the sum of five similar functions of the roots $(x_1, x_2, x_3, x_4, x_5)$, in which the subscript numbers differ cyclically according to a fixed sequence. If, for example, the sequence is 12345, we may have

$$\mathrm{c}x_1^2x_4 = x_1^2x_4 + x_2^2x_5 + x_3^2x_1 + x_4^2x_2 + x_5^2x_3,$$

while if the sequence is 12354,

$$\mathrm{c}x_1^2x_4 = x_1^2x_4 + x_2^2x_1 + x_3^2x_2 + x_5^2x_3 + x_4^2x_5.$$

When necessary to distinguish different sequences, let 12345, 13425, 14235, 12534, 12453, 13254, be denoted respectively by the symbols c_1, c_2, c_3, c_4, c_5, c_6, and 13524, 12354, 12543, 13245, 14325, 12435, by c_1', c_2', c_3', c_4', c_5', c_6'. (I have

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* Peacock, Report to the British Association, 1833, p. 312.

† Jacobi, "Observatio de Æquatione Sexti Gradus ad quam Æquationes Quinti Gradus Revocari Possunt," in a paper entitled "Observatiunculæ ad Theoriam Æquationum Pertinentes," Crelle, XIII, 340, as cited by Cayley, "On Tschirnhausen's Transformation," Philosophical Transactions, v. 152, p. 578.

here followed Mr. Cayley's order.) These twelve comprise all possible cycles, since reversing the sequence—as 15432 for 12345—does not change the cycle. We may look on $\text{c}_1 x_1 x_2$ as a group of five products, each of two different roots, in which group each root appears twice. Conversely, every such group is a cycle comprising the products of adjacent roots, and the number of possible cycles (12) is the number of possible groups. Similarly, $\text{c}_1 x_1 x_3$ is a group of the same sort. Conversely, every such group is a cycle comprising the products of non-adjacent roots, the possible number being 12. In fact, $\text{c}_1 x_1 x_2 = \text{c}_1' x_1 x_3$, and so on, the roots being adjacent in the former cycle and non-adjacent in the latter. Let $\phi_1 = \text{c}_1 x_1 (x_2 - x_3)$, a cycle comprising the difference between the products of adjacent and non-adjacent roots. It will be found that $\text{c}_1' x_1 (x_3 - x_5) = -\phi_1$. Taking all possible combinations, we have

$$\left.\begin{array}{ll} \phi_1 = \text{c}_1 x_1 (x_2 - x_3) = -\text{c}_1' x_1 (x_3 - x_5), & \phi_4 = \text{c}_4 x_1 (x_2 - x_5) = -\text{c}_4' x_1 (x_3 - x_2), \\ \phi_2 = \text{c}_2 x_1 (x_3 - x_4) = -\text{c}_2' x_1 (x_2 - x_3), & \phi_5 = \text{c}_5 x_1 (x_2 - x_4) = -\text{c}_5' x_1 (x_4 - x_3), \\ \phi_3 = \text{c}_3 x_1 (x_4 - x_2) = -\text{c}_3' x_1 (x_2 - x_5), & \phi_6 = \text{c}_6 x_1 (x_3 - x_2) = -\text{c}_6' x_1 (x_2 - x_4). \end{array}\right\} \quad (36)$$

The function ϕ has therefore 12 values, of which six are respectively identical with the other six, but with contrary signs. Hence ϕ^2 is a six-valued function, and if its values be taken as the roots of a sextic, the coefficients of the sextic will be rational and integral symmetric functions of the roots of the quintic, and therefore rational and integral functions of the coefficients of the quintic. Regarding this sextic as the product of a sextic whose roots are the values of ϕ by another whose roots are the values of $-\phi$, we have for ϕ a sextic whose coefficients can contain only rational and integral functions of the coefficients of the quintic or square roots of such functions. Suppose that two roots are equal, say $x_5 = x_4$, and that the discriminant ($\square$) therefore vanishes, by the usual theory. Then $\text{c}_1 = \text{c}_2'$, so that $\phi_1 = -\phi_2$, and similarly $\phi_3 = -\phi_5$ and $\phi_4 = -\phi_6$. Hence ϕ^2 has in this case but three values, the roots of a cubic, so that the odd powers of the sextic severally vanish, whence we must infer that some power of the discriminant is a factor of each coefficient of an odd power in the sextic for ϕ. In this sextic, since ϕ is a function of the roots of two dimensions, the coefficient of ϕ must be a function of 10 dimensions, that of ϕ^3 of 6 dimensions, and that of ϕ^5 of 2 dimensions, while $\sqrt{\square}$ has ten dimensions, and this can appear only in the coefficient of ϕ, whence it follows that

$$\phi^6 + a_2\phi^4 + a_4\phi^2 \pm n\sqrt{\square}\,\phi + a_6 = 0. \quad (37)$$

This is the form given by Jacobi, n being a numerical factor whose value he determined correctly, leaving the other coefficients to be investigated by others.

16. Jacobi's resolvent is substantially the same as Malfatti's, the one being the result of an endeavor to improve on Lagrange's proposal by building up a sextic from a chosen function of the roots, the other the result of elimination from the Eulerian equations or their equivalent. The identity of the two resolvents is shown by an equation which, as v has not yet been formally defined, we shall adopt as a definition, namely,

$$v = \frac{1}{500}\phi^2. \tag{38}$$

The following might, however, be taken as the definition of the same symbol:

$$v = \gamma^2 - u_1u_2u_3u_4. \tag{39}$$

Or, we might define $v^{\frac{1}{2}}$ by a separate symbol, as has already been done by at least one writer.* It will then be easy to define v as the square of $v^{\frac{1}{2}}$. Thus, observing (29), we may put

$$v^{\frac{1}{2}} = \gamma + u_1u_4 = -\gamma - u_2u_3. \tag{40}$$

The connection between (39) and (40) is obvious, since (39) may be derived from (40) by multiplying the value of u_1u_4 by that of u_2u_3. The connection between (38) and (40) becomes clear when we substitute for the elements their values as defined by (28), aided by one or two of the most elementary formulæ in the theory of symmetric functions. We find, for instance, that

$$25u_1u_4 = C_1(x_1^2 + \omega x_1x_2 + \omega^2x_1x_3 + \omega^3x_1x_3 + \omega^4x_1x_2) = -25\gamma \pm \tfrac{1}{2}\sqrt{5}\phi.$$

17. The use of a special symbol for the quantity v may not seem to constitute an important step in advance in the theory of this well-known resolvent. It is to be observed, however, that the relations of this quantity with t, as shown in (5), make the introduction and discussion of it a necessity in any event; that either $v^{\frac{1}{2}}$ or some other symbol for the same quantity is needed in (40) and elsewhere, and that $v^{\frac{1}{2}}$ is to be preferred as marking the fact that it is an irrational function of the roots; and that the use of the quantity ϕ instead of v is not indicated, after the rudiments are left behind, by any analytic advantage. It will be observed that v is rational whenever t is, and *vice versa*; that v is rational whenever ϕ is, and sometimes when ϕ is not. Finally, we shall see that the use of a symbol for the quantity v facilitates materially the derivation of the three chief resolvents from the Eulerian equations.†

18. Unaware of what had been done by Malfatti and Jacobi, three English analysts, by one step after another, went on improving the theory as then known

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* Schläfli, in a paper referred to later.

† I observe that in one place Signor Brioschi employs a symbol for v, but only momentarily, as a stepping stone from $\gamma^2 - v$ to $v^{\frac{1}{2}}$. See his paper previously cited.

until the same resolvent, in Jacobi's form, was brought out in the fullest detail. Mr. (now Sir James) Cockle discovered a resolvent for the special case $x^5 + 10dx^2 + f = 0$, identical, as far as it goes, with that of Malfatti; Mr. Harley worked up the subject further, and Mr. Cayley, carrying out elaborately a hint given by Mr. Harley, calculated the complete resolvent.* About the same time Captain von der Schulenburg gave the same resolvent in the course of what he supposed to be a solution of the quintic.† He made no claim to novelty, and may be presumed to have seen Jacobi's paper. Finally, the same form of Malfatti's resolvent was rediscovered about 1863 by Professor Minich,‡ who was then unaware of what had been done in Germany and England.

19. Some account of the reinstatement of Malfatti's resolvent has already been given. In its presentation by Signor Brioschi, the work was simplified by suppressing δ and afterwards showing that the result was not thereby affected. For solving the Eulerian equations two auxiliary quantities (besides seven others afterwards eliminated) were introduced, namely, r, which I call $-\delta - tv^4$, and v, which I call $\gamma^2 - v$. Both were finally eliminated to produce the (dexter) resolvent in x, which I call v^4, substantially identical with that of Jacobi. Whether these details were taken from Malfatti or introduced by Signor Brioschi, I do not know, but infer that several simplifications were made by the latter. The relations given by the Eulerian theory were made use of to determine the values of u_1u_4, u_2u_3, $u_1^2u_3$, etc., and the values of the elements were obtained by the formulæ

$$\left.\begin{aligned} u_1^5 &= (u_1^2 u_3)^2 u_2^2 u_1 (u_2u_3)^{-2}, & u_3^5 &= (u_3^2 u_4)^2 u_1^2 u_3 (u_1u_4)^{-2}, \\ u_2^5 &= (u_2^2 u_1)^2 u_4^2 u_2 (u_1u_4)^{-2}, & u_4^5 &= (u_4^2 u_2)^2 u_3^2 u_4 (u_2u_3)^{-2}. \end{aligned}\right\} \qquad (41)$$

A somewhat different statement of Malfatti's method was published in 1869 by Professor Schläfli.§ The two quantities taken by him for discussion, after reducing the Eulerian equations to manageable form in substantially the same manner as that exhibited by Signor Brioschi, are t, which I call v^4, and another

* Cockle, "Researches in the Higher Algebra," Manchester Memoirs, XV (1858), 131; "Theory of Equations of the Fifth Degree," Philosophical Magazine, July, 1859; Harley, "On the Method of Symmetric Products," Manchester Memoirs, XV (1859); "On the Theory of Quintics," Quarterly Journal, Jan., 1860; Cayley, "On a New Auxiliary Equation in the Theory of Equations of the Fifth Order," Philosophical Transactions, 151 (1861), p. 263. Mr. Cayley's statement (1862) of Jacobi's prior discovery has already been cited.

† Already referred to: Auflösung der Gleichungen Fünften Grades, Halle, 1861. Mr. Cayley's memoir had been read in February of the same year.

‡ See his statement, Atti del Reale Istituto Veneto, VIII, 906–908 (1862).

§ "La Risolvente dell' Equazione di Quinto Grado sotto la Forma di un Determinante Simmetrico a Quattro Linee," Annali di Matematica, 2d Series, III, 171.

which I call $s^{\frac{1}{2}}$, and which—see (30)—may be explained thus:

$$s^{\frac{1}{2}} = \delta + u_2^2 u_1 + u_3^2 u_4 = -\delta - u_1^2 u_3 - u_4^2 u_2. \tag{42}$$

He finally obtains a single equation of the twelfth degree in t, which I call $v^{\frac{1}{2}}$, in the form of a symmetric determinant. Of this the first row and first column contain only odd powers of the variable, and the other constituents only even powers, so that the determinant becomes, on evaluation, a sextic in t^2, equivalent in substance to Malfatti's resolvent. It is only another instance of the simplification resulting everywhere from the use of a symbol for the quantity v that, if we change the notation and divide one row and one column by $v^{\frac{1}{2}}$, Professor Schläfli's determinant becomes expressly a sextic, namely,

$$\begin{vmatrix} \mathrm{A}, & \mathrm{H}, & \mathrm{G}, & \zeta \\ \mathrm{H}, & \mathrm{B}, & \mathrm{F}, & \mathrm{L} \\ \mathrm{G}, & \mathrm{F}, & \mathrm{C}, & \delta \\ \zeta, & \mathrm{L}, & \delta, & -\gamma \end{vmatrix} v = 0, \tag{43}$$

where

$$\mathrm{A} = 625v^2 - 25(20\gamma^2 + 7\varepsilon)v + 100\gamma^4 + 45\gamma^2\varepsilon + 31\varepsilon^2 + 25\gamma\delta^2 - 16\delta\zeta + (25\gamma^6 - 35\gamma^4\varepsilon + 40\gamma^3\delta^2 + 2\gamma^2\delta\zeta + 11\gamma^2\varepsilon^2 - \gamma\zeta^2 - 28\gamma\delta^2\varepsilon + 2\delta\varepsilon\zeta + 16\delta^4 - \varepsilon^3)v^{-1},$$

$$\mathrm{B} = -250\gamma v + 25\gamma^3 - 10\gamma\varepsilon + 16\delta^2,$$

$$\mathrm{C} = 25v + \gamma^2 - \varepsilon,$$

$$\mathrm{F} = -\gamma\delta + \zeta,$$

$$\mathrm{G} = -25\gamma v + 35\gamma^3 - 19\gamma\varepsilon + 25\delta^2,$$

$$\mathrm{H} = 25\delta v + 15\gamma^2\delta + 19\delta\varepsilon - 24\gamma\zeta,$$

$$\mathrm{L} = 25v - \varepsilon.$$

That he did not himself put it in this shape can only be due, I think, to his being embarrassed by the second irrational, $s^{\frac{1}{2}}$, the value of which he proceeds to determine, after finding $v^{\frac{1}{2}}$, by means of the minors of his determinant. (We shall see further on how, after introducing the quantity t, which, like v and s, is a rational function of the roots, the same minors, in simpler form, may be used, if any one chooses that course, to better purpose.) Having ascertained $v^{\frac{1}{2}}$ and $s^{\frac{1}{2}}$, he employs, like Signor Brioschi, the relations given in (29), (30), (40), and (42) to determine the values of $u_1 u_4$, $u_2 u_3$, $u_1^2 u_3$, $u_2^2 u_1$, $u_4^2 u_2$, and thence he derives those of the elements by these formulæ: $u_1^5 = (u_1^2 u_3)^2 u_2^2 u_1 (u_2 u_3)^{-2}$, $u_2 = u_1^2 . u_4^2 u_2 (u_1 u_4)^{-2}$, $u_3 = u_1^2 u_3 . u_1^{-2}$, $u_4 = u_1 u_4 . u_1^{-1}$.

20. Two lines of progress will have been noticed in the historical sketch just given. The building up of a sextic upon a simple function of the roots was

suggested by Lagrange and accomplished finally by Mr. Cayley, but this did not result in expressing the elements in terms of that function.* On the other hand, Malfatti and his followers obtained substantially the same sextic by elimination, without regard to the relation of the variable to the roots, but with the advantage of determining the elements as the immediate result of a solution of the sextic. It will presently be found that the function of the roots, ϕ, upon which Jacobi's form of Malfatti's resolvent is built up, is only one, and not in all respects the most important, of three such functions, and we shall accordingly proceed to discuss the two other resolvents. Examining the theory deduced by elimination, we shall find that the introduction of a new quantity, t, and of a special symbol for a quantity hitherto neglected, v, lead to notable simplifications in the derivation and form of the two fundamental equations referred to in the preceding paragraph, and that from these equations each of the three resolvents can at once be derived. We shall find it possible also to deduce simple expressions by which to determine the elements in terms of t and v, or t and s, and the coefficients of the quintic.

21. Any six-valued function of the roots may be used for a resolvent sextic, including, in certain cases, six values of a twelve-valued function. Of such functions the number is unlimited, because the simpler six-valued functions admit of any number of six-valued combinations. Comparatively few six-valued functions have been definitely pointed out. That first suggested by Lagrange, Σu^5, is complex. A slightly simpler function for the same purpose was suggested by Hirsch, viz., $2\text{c}_1 x_1^3 (x_2 x_5 + x_3 x_4) + 3\text{c}_1 x_1^2 x_2 (x_3^2 + x_2 x_4)$. But it may be remarked incidentally that the only sextic needed for the immediate determination of the values of Σu^5 is one in which the roots are of the form $\text{c}_1 y_1^2 y_2 (y_3^2 + y_2 y_4)$, where $y = x + ba^{-1}$, as in (27). The function $\text{c}_1 x_1 (x_2 - x_3) x_5$ was observed by Schulenburg as having six values. M. Hermite suggested, and Dr. Salmon† carried out, the calculation of a sextic whose roots are squares of functions of the form $\text{c}_1 x_1^2 (x_2^2 x_3 - x_3^2 x_5 + x_4^2 x_2 - x_5^2 x_4 + 2x_3 x_4 x_5 - 2x_2 x_3 x_4)$, or $(x_1 - x_2)(x_2 - x_3)(x_3 - x_4)(x_4 - x_5)(x_5 - x_1)$. This function does not seem well suited to serve as a resolvent. Of all six-valued functions of the roots of the quintic, I find three pre-eminently suitable for the formation of resolvent sextics. These are $\text{c}_1 x_1 (x_2 - x_3)$, which we call ϕ after Mr. Cayley; $\text{c}_1 x_1 (x_2 - x_3) x_5$, which let us call ψ; and $\phi\psi^{-1}$,

* Attention was called to this defect in a paper by Cockle, "On the Theory of Equations," Manchester Proceedings, Nov., 1863, p. 171.

† Modern Higher Algebra, 3d Ed., pp. 228–230.

which let us call τ. The separate values of ϕ have been shown in (36). Those of τ are $\tau_1 = \phi_1\psi^{-1}$, $\tau_2 = \phi_2\psi^{-2}$, and so on, and those of ψ are

$$\left.\begin{aligned}
&\psi_1 = c_1x_1(x_2-x_3)x_5 = -c_1'x_1(x_3-x_5)x_4, \quad \psi_4 = c_4x_1(x_2-x_5)x_4 = -c_4'x_1(x_3-x_2)x_5,\\
&\psi_2 = c_2x_1(x_3-x_4)x_5 = -c_2'x_1(x_2-x_3)x_4, \quad \psi_5 = c_5x_1(x_2-x_4)x_3 = -c_5'x_1(x_4-x_3)x_5,\\
&\psi_3 = c_3x_1(x_4-x_2)x_5 = -c_3'x_1(x_2-x_5)x_3, \quad \psi_6 = c_6x_1(x_3-x_2)x_4 = -c_6'x_1(x_2-x_4)x_5.
\end{aligned}\right\} \quad (44)$$

22. Of the three functions just mentioned, only ϕ has, I think, been suggested heretofore as suitable for a resolvent. It underlies that resolvent whose history, under various forms, has been given above, the only resolvent hitherto found feasible.* In his investigation of the sextic in ϕ, Mr. Cayley proved that function to be a seminvariant, or function which remains unaltered when any given quantity is added to each root. (This fact is made instantly visible by the use of our notation, since $c_1[x_1+k][x_2+k-x_3-k] = c_1x_1[x_2-x_3] + kc_1[x_2-x_3]$, the last term vanishing.) The coefficients, he remarked, must therefore all be seminvariants, as follows from their being functions of the several values of ϕ. He then shortened the calculation of them by assuming that one root vanished, whence $f = 0$, and afterwards filling out the missing terms by reference to the known properties of seminvariants. The full values of the coefficients as computed by him will be given presently. The independent term, equal to $\phi_1\phi_2\phi_3\phi_4\phi_5\phi_6$, is stated by him at length, and I find it reducible to this shorter expression:

$$\phi_1\phi_2\phi_3\phi_4\phi_5\phi_6 = 40000a^{-6}(al_0' - 25c_0'^2). \quad (45)$$

Here l_0' is the leading coefficient of the simplest linear covariant of the quintic, say $l_0'x + l_1'$, the full expression for which will be stated further on, as given in various standard works; and $c_0' = ace - ad^2 - b^2e + 2bcd - c^3$, the leading coefficient of the canonizant. Or,

$$\phi_1\phi_2\phi_3\phi_4\phi_5\phi_6 = 40000(l_0 - 25c_0^2), \quad (46)$$

where l_0 and c_0 have the same values as in paragraph 2. The coefficients of the sextic in ϕ, which for distinction I will refer to as the dexter resolvent, are best exhibited, I think, in the following form:

$$\left.\begin{aligned}
&\phi^6 + 100(-3\gamma^2 - \varepsilon)\phi^4 + 2000(15\gamma^4 - 2\gamma^2\varepsilon + 8\gamma\delta^2 - 2\delta\zeta + 3\varepsilon^2)\phi^2\\
&\qquad \pm 800\sqrt{(5\text{D})} + 40000(l_0 - 25c_0^2) = 0.
\end{aligned}\right\} \quad (47)$$

Here D, the discriminant $\times\, a^{-8}$, has the same value as in (19). For brevity, let us express (47) thus:

$$\phi^6 + 100d_2\phi^4 + 2000d_4\phi^2 \pm 800\sqrt{(5\text{D})}\,\phi + 40000d_6 = 0. \quad (48)$$

* I exclude, of course, sextics called resolvents in the solution of the quintic by elliptic functions.

Since by (38) we put $v = \frac{1}{500}\phi^2$, we have $\phi = 2\sqrt{5}\,(5v^{\frac{1}{2}})$, and (48) becomes, after dividing by 40000,

$$\tfrac{1}{5}(5v^{\frac{1}{2}})^6 + d_2(5v^{\frac{1}{2}})^4 + d_4(5v^{\frac{1}{2}})^2 \pm \tfrac{1}{5}\sqrt{\mathrm{D}}\,(5v^{\frac{1}{2}}) + d_6 = 0. \tag{49}$$

Transposing the odd power, and squaring,

$$\left.\begin{aligned} &5^{10}v^6 + 2.5^9 d_2 v^5 + (2.5^7 d_4 + 5^8 d_2^2)\,v^4 + 2\,(5^5 d_6 + 5^6 d_2 d_4)\,v^3 \\ &\quad + (2.5^4 d_2 d_6 + 5^4 d_4^2)\,v^2 + (2.5^2 d_4 d_6 - \mathrm{D})\,v + d_6^2 = 0. \end{aligned}\right\} \tag{50}$$

A very slight modification of this will give the sextic of Malfatti, mentioned in paragraph 13. It will be seen that (50) coincides with the determinant (43), and we may remark that

$$\begin{vmatrix} 25\gamma^2 - 10\gamma\varepsilon + 16\delta^2, & \gamma\delta - \zeta, & \varepsilon \\ \gamma\delta - \zeta, & \gamma^2 - \varepsilon, & \delta \\ \varepsilon, & \delta, & -\gamma \end{vmatrix} = d_6. \tag{51}$$

For the present, we may consider the coefficients of the dexter resolvent, so far as they were left undetermined by Jacobi (see paragraph 15), as having been proved for us by the process followed by Mr. Cayley, of which it is not necessary to repeat the details. A different proof will be given when we discuss the Eulerian equations. When $v = \gamma^2$, (50) becomes (19). When $v = 0$, $d_6 = 0$, which proves (24).

23. For the second function named at the close of paragraph 21 we are now enabled to obtain at once what I call the sinister resolvent, which is merely the dexter resolvent of the quintic when the order of the coefficients is reversed, f being interchanged with a, e with b, and so on. This is equivalent to supposing that x is replaced by x^{-1}, when ϕ_1, or $\mathrm{C}_1 x_1 (x_2 - x_3)$, is replaced by $\mathrm{C}_1 x_1^{-1}(x_2^{-1} - x_3^{-1})$, which is equal to $-af^{-1}\psi_1$, since $af^{-1} = -x_1^{-1}x_2^{-1}x_3^{-1}x_4^{-1}x_5^{-1}$ and $\psi_1 = \mathrm{C}_1 x_4 (x_3 - x_2)\, x_5$, as will be seen on comparison. The other values of ϕ are similarly equal to the corresponding values of $-af^{-1}\psi$. The dexter resolvent of the reversed quintic is therefore a sextic in ψ, and we may write it out by reversing the several letters as stated. Mr. Cayley's determination of the *dexter resolvent* is as follows, except that I abbreviate the last term as already explained:

$$\left.\begin{aligned} &a^6\phi^6 - 100a^4(ae - 4bd + 3c^2)\,\phi^4 + 2000a^2(-2a^2df + 3a^2e^2 + 6abcf \\ &\quad - 14abde - 2ac^2e + 8acd^2 - 4b^3f + 10b^2ce + 20b^2d^2 - 40bc^2d \\ &\quad + 15c^4)\,\phi^2 \pm 800a^2\sqrt{(5\square)}\,\phi + 40000\,(al_0' - 25c_0'^2) = 0. \end{aligned}\right\} \tag{52}$$

Here $\square$ = discriminant, and l_0' and c_0' have the same meaning as in the preceding paragraph, namely,

$$l_0' = a^2cf^2 - 2a^2def + a^2e^3 - ab^2f^2 - 4abcef + 8abd^2f - 2abde^2 - 2ac^2df + 14ac^2e^2$$
$$- 22acd^2e^* + 9ad^4 + 6b^3ef - 12b^2cdf - 15b^2ce^2 + 10b^2d^2e + 6bc^3f + 30bc^2de$$
$$- 20bcd^3 - 15c^4e + 10c^3d^2,$$
$$c_0' = ace - ad^2 - b^2e + 2bcd - c^3.$$

These are respectively the leading coefficients of the simplest linear and cubic covariants of the quintic, the other coefficients being

$$l_1' = a^2df^2{}' - a^2e^2f - 2abcf^2 - 4abdef + 6abe^3 + 8ac^2ef - 2acd^2f - 12acde^2 + 6ad^3e$$
$$+ b^3f^2 - 2b^2cef + 14b^2d^2f - 15b^2de^2 - 22bc^2df + 10bc^2e^2 + 30bcd^2e$$
$$- 15bd^4 + 9c^4f - 20c^3de + 10c^2d^3,$$
$$c_1' = acf - ade - b^2f + bce + bd^2 - c^2d,$$
$$c_2' = adf - ae^2 - bcf + bde + c^2e - cd^2,$$
$$c_3' = bdf - be^2 - c^2f + 2cde - d^3.$$

Reversing the letters in (52) as just explained, we have the ***sinister resolvent,***

$$\left.\begin{aligned} & a^6\psi^6 - 100a^4(bf - 4ce + 3d^2)\psi^4 + 2000a^2(-2acf^2 + 6adef - 4ae^3 \\ & + 3b^2f^2 - 14bcef - 2bd^2f + 10bde^2 + 8c^2df + 20c^2e^2 - 40cd^2e \\ & + 15d^4)\psi^2 \mp 800af\surd(5\square)\psi + 40000(fl_1' - 25c_3'^2) = 0. \end{aligned}\right\} \qquad (53)$$

24. The elements were shown in paragraph 10 to be seminvariants, so that in using any resolvent we shall find it well to employ functions of the roots that are themselves seminvariants. This was done necessarily in paragraph 22, in discussing the dexter resolvent, since ϕ and v are seminvariants. It is better to modify the sinister resolvent somewhat, since ψ is not a seminvariant.† Any function of the roots can be transmuted into a seminvariant by deducting from each root one-fifth of the sum of all, that is, by adding ba^{-1} to each root. In lieu of ψ, equal to $\mathrm{C}_1x_1(x_2 - x_3)x_5$, let us take σ, denoting $\mathrm{C}_1(x_1 + ba^{-1})(x_2 - x_3)(x_5 + ba^{-1})$. This is the same as $\mathrm{C}_1x_1(x_2 - x_3)(x_5 + ba^{-1}) + \mathrm{C}_1ba^{-1}(x_2 - x_3)(x_5 + ba^{-1})$. The latter term vanishes, since $\mathrm{C}_1ba^{-1}x_2x_5 - \mathrm{C}_1ba^{-1}x_5x_3 = 0$ and $\mathrm{C}_1b^2a^{-2}(x_2 - x_3) = 0$. Hence $\sigma = \mathrm{C}_1x_1(x_2 - x_3)x_5 + \mathrm{C}_1x_1(x_2 - x_3)ba^{-1}$, or

$$\sigma = \psi + \phi ba^{-1}. \qquad (54)$$

Now σ is that function of the roots of the quintic (27) which ψ is of the general quintic. As a special case of (53), therefore, we have this modified form of the sinister resolvent:

$$\left.\begin{aligned} & \sigma^6 + 100(4\gamma\varepsilon - 3\delta^2)\sigma^4 + 2000(8\gamma^2\delta\zeta + 20\gamma^2\varepsilon^2 - 2\gamma\zeta^2 - 40\gamma\delta^2\varepsilon + 6\delta\varepsilon\zeta \\ & + 15\delta^4 - 4\varepsilon^3)\sigma^2 \mp 800\zeta\surd(5\mathrm{D})\sigma + 40000(\zeta l_1 - 25c_3^2) = 0. \end{aligned}\right\} \qquad (55)$$

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* The wrong sign is given to this term in Faà de Bruno's table.

† I have devised formulæ for deducing the values of t and v from ψ, but they are omitted as circuitous and, in comparison, useless.

Here D, l_1, and C_3 have the same values as in (19) and (4). The result thus obtained corresponds closely to (47). Let us write (55) in this form:

$$\sigma^6 + 100e_2\sigma^4 + 2000e_4\sigma^3 \mp 800\zeta\sqrt{(5\text{D})}\,\sigma + 40000e_6 = 0. \tag{56}$$

This corresponds to (48). Let

$$s = \frac{1}{500}\sigma^2. \tag{57}$$

This definition corresponds to (38). Then $\sigma = 2\sqrt{5}\,(5s^{\frac12})$, and (56) becomes, after dividing by 40000,

$$\tfrac15(5s^{\frac12})^6 + e_2(5s^{\frac12})^4 + e_4(5s^{\frac12})^3 \mp \tfrac15\zeta\sqrt{\text{D}}\,(5s^{\frac12}) + e_6 = 0. \tag{58}$$

Transposing the odd power, and squaring, we have, as a final variation of the sinister resolvent, this sextic in s:

$$\left.\begin{aligned} 5^{10}s^6 + 2.5^9e_2s^5 + (2.5^7e_4 + 5^8e_2^2)s^4 + 2\,(5^5e_6 + 5^6e_2e_4)\,s^3 \\ + (2.5^4e_2e_6 + 5^4e_4^2)\,s^2 + (2.5^3e_4e_6 - \zeta^2\text{D})\,s + e_6^2 = 0. \end{aligned}\right\} \tag{59}$$

The last two equations correspond respectively to (49) and (50).

25. The third function of the roots named in paragraph 21 as suitable for a resolvent is of all three the easiest to deal with. Let

$$\tau = \psi\phi^{-1}, \tag{60}$$

$$t = \sigma\phi^{-1} = \tau + ba^{-1}. \tag{61}$$

Thus $\tau = \frac{C_1x_1(x_2 - x_3)\,x_5}{C_1x_1(x_2 - x_3)}$. If we divide both numerator and denominator by $x_1x_2x_3x_4x_5$, we have $\tau = \frac{C_1x_3^{-1}x_4^{-1} - C_1x_2^{-1}x_4^{-1}}{C_1x_3^{-1}x_4^{-1}x_5^{-1} - C_1x_2^{-1}x_4^{-1}x_5^{-1}}$. Now $C_1x_3^{-1}x_4^{-1} = C_1x_1^{-1}x_2^{-1}$, $C_1x_2^{-1}x_4^{-1} = C_1x_1^{-1}x_3^{-1}$, $C_1x_3^{-1}x_4^{-1}x_5^{-1} = C_1x_1^{-1}x_2^{-1}x_5^{-1}$, $C_1x_2^{-1}x_4^{-1}x_5^{-1} = C_1x_1^{-1}x_3^{-1}x_5^{-1}$. Hence $\tau = \frac{C_1x_1^{-1}(x_2^{-1} - x_3^{-1})}{C_1x_1^{-1}(x_2^{-1} - x_3^{-1})x_5^{-1}}$, the reciprocal of the same function of the reciprocals of the roots. It is therefore a covariant function, and having six values is to be determined by a covariant sextic, say

$$(\phi_1\tau - \psi_1)(\phi_2\tau - \psi_2)(\phi_3\tau - \psi_3)(\phi_4\tau - \psi_4)(\phi_5\tau - \psi_5)(\phi_6\tau - \psi_6) = 0. \tag{62}$$

To determine the coefficients we need to know but one, either the first or the last, the rest being obtainable at once by the usual rules for the formation of covariants. Considering (62) under this form,

$$\phi_1\phi_2\phi_3\phi_4\phi_5\phi_6\tau^6 + \ldots + \psi_1\psi_2\psi_3\psi_4\psi_5\psi_6 = 0, \tag{63}$$

we see that this is the correct form of the covariant, the leading coefficient being a function of the roots of twelve dimensions, the final eighteen, and the weight eighteen. The covariant is therefore of the sixth degree and sixth order. Examining the list of covariants of the quintic, we find that any covariant of the sixth degree and sixth order must be of the form $m_1\text{AL} + m_2\text{Q}^2 + m_3\text{BQ} + m_4\text{C}^3$, where m_1, m_2, m_3, m_4, are unknown numerical multipliers, A is the quintic, L its simplest linear, Q its simplest quadratic, C its simplest cubic, and B its simplest biquadratic or quartic covariant. Ascertaining in any manner the value of $\phi_1\phi_2\phi_3\phi_4\phi_5\phi_6$, we find that $m_1 = 1$, $m_2 = m_3 = 0$, $m_4 = -25$. This can be done by Mr. Cayley's method, by symmetric functions, assuming that one root vanishes, or by discussion of the Eulerian equations, as will be shown further on. The latter method is no doubt the simpler. We thus derive the *central resolvent*,

$$\mathrm{AL} - 25\mathrm{C}^2 = 0. \tag{64}$$

Here

$$\mathrm{A} = a\tau^5 + 5b\tau^4 + 10c\tau^3 + 10d\tau^2 + 5e\tau + f,$$

$$\mathrm{L} = l_0'\tau + l_1',$$

$$\mathrm{C} = c_0'\tau^3 + c_1'\tau^2 + c_2'\tau + c_3',$$

where the coefficients have the same values as in paragraph 23. If we obtain a value of τ by this resolvent, the corresponding value of t is at once found by (61). In most cases it will be easier to use a variation of (64), being that special case of it which results when the quintic has the form given in (27). In this case $\tau = t$, and we have

$$\mathrm{A}_y\mathrm{L}_y - 25\mathrm{C}_y^2 = 0. \tag{65}$$

This is the resolvent (3) explained and exemplified near the beginning.

26. Of the different forms which the dexter and sinister resolvents may assume, (47) and (55) are usually most convenient in practice. To illustrate them we may take one of the examples (15) employed for the central resolvent (3). In that case we have, for determining ϕ by the dexter resolvent,

$$\phi^6 + 37\phi^4 + 115\phi^2 \pm 1504\phi - 249 = 0, \tag{66}$$

and for determining σ by the sinister resolvent,

$$\sigma^6 + 4\sigma^4 - 32\sigma^2 \pm 1504\sigma + 1408 = 0. \tag{67}$$

A value of ϕ by (66) is ∓ 3, and a value of σ by (67) is ∓ 4. It happens that these values are consistent with each other, as is shown by the result of (16), $t = \frac{1}{2}$, but we could not assume it without some such verification. There is little choice in point of convenience between the dexter and sinister resolvents. As between either and the central resolvent, it may be remarked that with the latter the chief difficulty lies in the calculation of l_0 and l_1, while either of the others requires the calculation of one of these quantities, besides the more laborious evaluation of the discriminant, though on the other hand there are fewer coefficients to be determined. Were nothing further involved, we might find one as convenient as another; but it is necessary to determine two quantities, t and v or t and s, and we shall see that on the whole, both for theoretic symmetry and practical availability, the preference is to be given to the central resolvent.

27. Observing (38), (57), and (60), we see that the three quantities t, v, and s, obtainable by the central, dexter, and sinister resolvents respectively, are connected by the necessary relation

$$s^{\frac{1}{2}} = tv^{\frac{1}{2}}. \tag{68}$$

For each value of t, six in all, there are corresponding values of v and s, and *vice versa*. By the simultaneous use of all three resolvents, with (60), the values corresponding to each other could be picked out; but the determination and minute comparison of eighteen roots would be most laborious. Means of determining, from a single root of one resolvent, any other necessary quantities, are therefore requisite, but before entering on that question it is desirable to inquire which of these quantities are needed for expressing the elements.

28. From (41) we know that for expressing the fifth powers of the elements it is sufficient to know the values of the combinations u_1u_4, u_2u_3, $u_1^2u_3$, $u_2^2u_1$, $u_3^2u_4$, $u_4^2u_2$. We shall shortly find that the relations stated in (40) and (42) enable us to express these combinations in terms of the twelve-valued quantities $v^{\frac{1}{2}}$ and $s^{\frac{1}{2}}$ and of the coefficients. We need a value of $v^{\frac{1}{2}}$, and a corresponding value of $s^{\frac{1}{2}}$. It is obvious from (68) that we get these when we have corresponding values of v and t, or of s and t, all six-valued functions. We may therefore dismiss either v or s from further consideration. Let us retain v as the simpler function of the two, merely remembering that in all places where v occurs it may be replaced by st^{-2}.

29. That the $s^{\frac{1}{2}}$ appearing in (42) is the same quantity as the $s^{\frac{1}{2}}$, equal to $tv^{\frac{1}{2}}$, defined in (57), is proved on replacing the elements in (42) by their equivalents as given in (28). By (40) and (42),

$$\left.\begin{aligned} u_1u_4 &= -\gamma + v^{\frac{1}{2}}, & u_2u_3 &= -\gamma - v^{\frac{1}{2}} \\ u_1^2u_3 + u_4^2u_2 &= -\delta - tv^{\frac{1}{2}}, & u_2^2u_1 + u_3^2u_4 &= -\delta + tv^{\frac{1}{2}}. \end{aligned}\right\} \tag{69}$$

Hence

$$\left.\begin{aligned} 2u_1^2u_3 &= -\delta - tv^{\frac{1}{2}} + (u_1^2u_3 - u_4^2u_2), \\ 2u_4^2u_2 &= -\delta - tv^{\frac{1}{2}} - (u_1^2u_3 - u_4^2u_2), \\ 2u_2^2u_1 &= -\delta + tv^{\frac{1}{2}} + (u_2^2u_1 - u_3^2u_4), \\ 2u_3^2u_4 &= -\delta + tv^{\frac{1}{2}} - (u_2^2u_1 - u_3^2u_4). \end{aligned}\right\} \tag{70}$$

Multiplying the first pair together, and also the second pair, and remembering that $u_1^2u_4^2u_2u_3 = (\gamma^2 - v)(-\gamma + v^{\frac{1}{2}})$, $u_1u_4u_2^2u_3^2 = (\gamma^2 - v)(-\gamma - v^{\frac{1}{2}})$, we find that

$$\left.\begin{aligned}(u_1^2u_3 - u_4^2u_2)^2 &= (\delta + tv^{\frac{1}{2}})^2 + 4\,(\gamma^2 - v)(\gamma - v^{\frac{1}{2}}),\\ (u_2^2u_1 - u_3^2u_4)^2 &= (\delta - tv^{\frac{1}{2}})^2 + 4\,(\gamma^2 - v)(\gamma + v^{\frac{1}{2}}).\end{aligned}\right\} \tag{71}$$

If the square roots of these quantities be substituted in (70), the combinations forming the left hand members are expressed in terms of t, $v^{\frac{1}{2}}$, and the coefficients. Then, since

$$\left.\begin{aligned}u_1^5 &= (u_1^2u_3)^2\,u_2^2u_1\,(u_2u_3)^{-2}, & u_2^5 &= (u_2^2u_1)^2\,u_4^2u_2\,(u_1u_4)^{-2},\\ u_4^5 &= (u_4^2u_2)^2\,u_3^2u_4\,(u_2u_3)^{-2}, & u_3^5 &= (u_3^2u_4)^2\,u_1^2u_3\,(u_1u_4)^{-2},\end{aligned}\right\} \tag{72}$$

we have all that is needed to express the fifth-powers of the elements in terms of t, $v^{\frac{1}{2}}$, and the coefficients. There is nothing new, except the quantity t and the use of a symbol for v, in this paragraph, which merely states the reasoning which justifies the formulæ of Signor Brioschi. We can, however, improve upon these expressions, which are simple only in appearance.

30. We have seen in (35) the form of the fifth-powers of the elements as arrived at by Lagrange and Vandermonde, after investigating those powers as functions of the roots. If we put $2p_1 = u_1^5 + u_4^5$, $2p_2 = u_2^5 + u_3^5$, $2q_1 = u_1^5 - u_4^5$, $2q_2 = u_2^5 - u_3^5$, we have

$$\left.\begin{aligned}u_1^5 &= p_1 + q_1, & u_2^5 &= p_2 + q_2,\\ u_4^5 &= p_1 - q_1, & u_3^5 &= p_2 - q_2.\end{aligned}\right\} \tag{73}$$

Multiplying by pairs, we have $q_1^2 = p_1^2 - u_1^5u_4^5$, $q_2^2 = p_2^2 - u_2^5u_3^5$. This is substantially the form reached by Sir James Cockle.* If we combine (35) and (73) by putting $2p_1 + 2p_2 = \Sigma u^5 = r_1$, $2p_1 - 2p_2 = r_2$, we find that $p_1 = \frac{1}{4}r_1 + \frac{1}{4}r_1$, and $p_2 = \frac{1}{4}r_1 - \frac{1}{4}r_2$. It remains to determine the values ef r_1 and r_2 in terms of t, $v^{\frac{1}{2}}$, and the coefficients. We are obviously able to do this by substitution, observing (72). The results, however, are unnecessarily complex, and to simplify them we shall find it advantageous to resort to the Eulerian equations (31) and (32).

31. For brevity in handling the expressions in (70), let $2m_1 = -\delta - tv^{\frac{1}{2}}$, $2m_2 = -\delta + tv^{\frac{1}{2}}$, $2n_1 = u_1^2u_3 - u_4^2u_2$, $2n_2 = u_2^2u_1 - u_3^2u_4$. Then $u_1^2u_3 = m_1 + n_1$, $u_4^2u_2 = m_1 - n_1$, $u_2^2u_1 = m_2 + n_2$, $u_3^2u_4 = m_2 - n_2$. Since $u_1u_2u_3u_4 = u_1u_4 . u_2u_3 = (-\gamma + v^{\frac{1}{2}})(-\gamma - v^{\frac{1}{2}}) = \gamma^2 - v$, we may write the third Eulerian equation (31) thus:

$$\varepsilon = \gamma^2 + 3v + (v - \gamma^2)^{-1}\left[u_1^4u_2^2u_3u_4 + u_4^4u_3^2u_1u_2 + u_2^4u_4^2u_1u_3 + u_3^4u_1^2u_2u_4\right]. \tag{74}$$

The expressions within the bracket are severally equal to $(m_1 + n_1)(m_2 + n_2)$

* "On the Resolution of Quintics," Quarterly Journal, June, 1860.

$(-\gamma + v^{\frac{1}{2}})$, $(m_1 - n_1)(m_2 - n_2)(-\gamma + v^{\frac{1}{2}})$, $(m_1 - n_1)(m_2 + n_2)(-\gamma - v^{\frac{1}{2}})$, $(m_1 + n_1)$ $(m_2 - n_2)(-\gamma - v^{\frac{1}{2}})$, and their sum is $-4\gamma m_1 m_2 + 4n_1 n_2 v^{\frac{1}{2}}$. Therefore

$$\varepsilon = \gamma^2 + 3v + (v - \gamma^2)^{-1}(\gamma t^2 v - \gamma\delta^2 + p), \tag{75}$$

where $p = v^{\frac{1}{2}}(u_1^2 u_3 - u_4^2 u_2)(u_2^2 u_1 - u_3^2 u_4)$, and hence also

$$p = \gamma\delta^2 - \gamma t^2 v + (v - \gamma^2)(\varepsilon - \gamma^2 - 3v). \tag{76}$$

Again, we may write the fourth Eulerian equation (32) thus:

$$\begin{aligned}\Sigma u^5 &= -\zeta - 20\gamma\delta + 10\left[(m_1+n_1)u_1u_4 + (m_1-n_1)u_1u_4 + (m_2+n_2)u_2u_3 + (m_2-n_2)u_2u_3\right]\\ &= -\zeta - 20\gamma\delta + 10\left[(-\delta - tv^{\frac{1}{2}})(-\gamma + v^{\frac{1}{2}}) + (-\delta + tv^{\frac{1}{2}})(-\gamma - v^{\frac{1}{2}})\right]\\ &= -\zeta - 20tv.\end{aligned} \tag{77}$$

This equation coincides with one given by Professor Schläfli, except, of course, that instead of tv he writes what corresponds in his notation to $v^{\frac{1}{2}}s^{\frac{1}{2}}$.

32. We are now able to proceed with the subject of paragraph 30. Since $r_1 = u_1^5 + u_2^5 + u_3^5 + u_4^5$, we have from (77)

$$r_1 = -\zeta - 20tv. \tag{78}$$

For determining r_2, equal to $u_1^5 + u_4^5 - u_2^5 - u_3^5$, we must expand these powers separately by employing (72). To avoid fractions, let us multiply each by $u_1^2u_2^2u_3^2u_4^2 = (\gamma^2 - v)^2$. Thus,

$$\left.\begin{aligned}(\gamma^2 - v)^2 u_1^5 &= (m_1^2 + 2m_1n_1 + n_1^2)(m_2 + n_2)(\gamma^2 - 2\gamma v^{\frac{1}{2}} + v),\\ (\gamma^2 - v)^2 u_4^5 &= (m_1^2 - 2m_1n_1 + n_1^2)(m_2 - n_2)(\gamma^2 - 2\gamma v^{\frac{1}{2}} + v),\\ (\gamma^2 - v)^2 u_2^5 &= (m_2^2 + 2m_2n_2 + n_2^2)(m_1 - n_1)(\gamma^2 + 2\gamma v^{\frac{1}{2}} + v),\\ (\gamma^2 - v)^2 u_3^5 &= (m_2^2 - 2m_2n_2 + n_2^2)(m_1 + n_1)(\gamma^2 + 2\gamma v^{\frac{1}{2}} + v).\end{aligned}\right\} \tag{79}$$

The sum of the first two, less the sum of the last two, is (remembering that $4n_1n_2v^{\frac{1}{2}} = p$)

$$\left.\begin{aligned}(\gamma^2 - v)^2 r_2 = (m_1 + m_2)&\left[pv^{-\frac{1}{2}}(\gamma^2 + v) - 4m_1m_2\gamma v^{\frac{1}{2}}\right]\\ &+ 2(m_1 - m_2)\left[m_1m_2(\gamma^2 + v) - p\gamma\right]\\ &+ 2(\gamma^2 + v)(n_1^2m_2 - n_2^2m_1) - 4\gamma v^{\frac{1}{2}}(n_1^2m_2 + n_2^2m_1).\end{aligned}\right\} \tag{80}$$

The values of $4n_1^2$ and $4n_2^2$ are given by (71), and that of p by (76), while $m_1 + m_2 = -\delta$, $m_1 - m_2 = -tv^{\frac{1}{2}}$, $m_1m_2 = \frac{1}{4}(\delta^2 - t^2v)$. Substituting these values, and dividing throughout by $(\gamma^2 - v)^2$, we have

$$\left.\begin{aligned}r_2 = (\gamma^2 - v)^{-1}(12\gamma tv^2 &- \delta v^2 - t^3v^2 + 4\gamma^3tv + 2\gamma^2\delta v + \gamma\delta t^2v\\ &+ \delta^2tv - 2\gamma\varepsilon tv + \delta\varepsilon v - \gamma^4\delta + \gamma^2\delta\varepsilon - \gamma\delta^3)\,v^{-\frac{1}{2}}.\end{aligned}\right\} \tag{81}$$

For determining q_1 and q_2, we have—see (73)—

$$\left.\begin{aligned}q_1^2 &= (\tfrac{1}{4}r_1 + \tfrac{1}{4}r_2)^2 + (\gamma - v^{\frac{1}{2}})^5,\\ q_2^2 &= (\tfrac{1}{4}r_1 - \tfrac{1}{4}r_2)^2 + (\gamma + v^{\frac{1}{2}})^5.\end{aligned}\right\} \tag{82}$$

Let $2s_1 = q_1^2 + q_2^2$, and $2s_2 = q_1^2 - q_2^2$; then $q_1 = \sqrt{(s_1 + s_2)}$, $q_2 = \sqrt{(s_1 - s_2)}$, where

$$\left.\begin{aligned} s_1 &= \tfrac{1}{16} r_1^2 + \tfrac{1}{16} r_2^2 + \gamma^5 + 10\gamma^3 v + 5\gamma v^2, \\ s_2 &= \tfrac{1}{8} r_1 r_2 - (5\gamma^4 + 10\gamma^2 v + v^2) v^{\frac{1}{2}}. \end{aligned}\right\} \qquad (83)$$

And (73) becomes

$$\left.\begin{aligned} u_1^5 &= \tfrac{1}{4} r_1 + \tfrac{1}{4} r_2 + \sqrt{(s_1 + s_2)}, \qquad & u_2^5 &= \tfrac{1}{4} r_1 - \tfrac{1}{4} r_2 + \sqrt{(s_1 - s_2)}, \\ u_4^5 &= \tfrac{1}{4} r_1 + \tfrac{1}{4} r_2 - \sqrt{(s_1 + s_2)}, \qquad & u_3^5 &= \tfrac{1}{4} r_1 - \tfrac{1}{4} r_2 - \sqrt{(s_1 - s_2)}. \end{aligned}\right\} \qquad (84)$$

We have thus demonstrated the element-formulæ first given, numbered (7) to (12).

33. For the critical cases $v = \gamma^2$ and $v = 0$ separate element-formulæ were presented in paragraphs 7 and 8, and these will now be proved. For the case $v = \gamma^2$, we had $u_4 = 0$, $v^{\frac{1}{2}} = \gamma$, $u_2 u_3 = -2\gamma$. From (69) we find that $u_1^2 u_3 = -(\gamma t + \delta)$, $u_2^2 u_1 = \gamma t - \delta$. Hence, by (72), $u_1^5 = (\gamma t + \delta)^2 (\gamma t - \delta)(4\gamma^3)^{-1}$, which proves (21). Also, $u_3 = u_1^2 u_3 . u_1^{-2} = -(\gamma t + \delta) u_1^{-2}$, which proves (22). It is to be remarked that the substitution of these simpler expressions does not impugn the generality of the element-formulæ (7), which are applicable to this as to all other cases. The expression given in (9) for the quantity r_2 is indeterminate in the present instance. Its value is $u_1^5 - u_2^5 - u_3^5$. We know what u_1^5 is, and that $u_3^5 = -32\gamma^5 u_2^{-5}$. To determine u_2^5, we have $u_2^5 = u_2 u_3 (u_2^2 u_1)^2 (u_1^2 u_3)^{-1} = 2\gamma (\gamma t - \delta)^2 (\gamma t + \delta)^{-1}$. For DeMoivre's quintic, $x^5 + 10\gamma x^3 + 20\gamma^2 x + \zeta = 0$, we find $v = \gamma^2$, $t = 0$, $u_1 = u_4 = 0$. In this case $r_2 = -u_2^5 - u_3^5 = -r_1 = \zeta$, and the solution follows by (7). For Euler's quintic, $x^5 + 10\delta x^3 + 5\varepsilon x + \zeta = 0$, where $16\delta^4 + \varepsilon^3 = 2\delta\varepsilon\zeta$, we find $v = \gamma^2 = 0$, whence by (7) $u_1^5 = \tfrac{1}{2} r_1 + \tfrac{1}{2} r_2$, $u_2^5 = \tfrac{1}{2} r_1 - \tfrac{1}{2} r_2$, $u_3^5 = u_4^5 = 0$. Here $r_1 = -\zeta$. By (30) and (31), $u_2^2 u_1 = -2\delta$, $u_1^2 u_2 = -\varepsilon$, whence u_1^5 and u_2^5 are found, and $r_2 = u_1^5 - u_2^5$. The binomial quintic $x^5 + \zeta = 0$ may be regarded as a special case of DeMoivre's, with u_3 vanishing, or of Euler's, with u_2 vanishing.

34. When $v = 0$, the element-formulæ are valid, but r_2 becomes indeterminate. In this case the quantity t must be dispensed with, being usually infinite, while s, equal to $t^2 v$, is finite. Where t appears, we must substitute $s^{\frac{1}{2}} v^{-\frac{1}{2}}$. Then $tv = s^{\frac{1}{2}} v^{\frac{1}{2}} = 0$, and by (8) or (78) $r_1 = -\zeta$. Putting $v = 0$ in (79), we obtain, by appropriate summation,

$$\left.\begin{aligned} \gamma^3 r_1 &= 2(m_1 + m_2) m_1 m_2 + 4(m_1 - m_2) n_1 n_2 + 2n_1^2 m_2 + 2n_2^2 m_1, \\ \gamma^3 r_2 &= 2(m_1 - m_2) m_1 m_2 + 4(m_1 + m_2) n_1 n_2 + 2n_1^2 m_2 - 2n_2^2 m_1. \end{aligned}\right\} \qquad (85)$$

Here $r_1 = -\zeta$, $m_1 + m_2 = -\delta$, $m_1 - m_2 = -s^{\frac{1}{2}}$, $4m_1 m_2 = \delta^2 - s$, $n_1^2 = m_1^2 + \gamma^3$, $n_2^2 = m_2^2 + \gamma^3$, whence $2n_1^2 m_2 + 2n_2^2 m_1 = 2(m_1 + m_2)(m_1 m_2 + \gamma^3)$, and $2n_1^2 m_2 - 2n_2^2 m_1 = 2(m_1 - m_2)(m_1 m_2 - \gamma^3)$. Substituting these values,

$$\left.\begin{aligned} -\gamma^2\zeta &= -\delta(\delta^2 - s + 2\gamma^3) - 4n_1n_2v^4 \\ \gamma^3 r_2 &= -s^4(\delta^2 - s - 2\gamma^3) - 4n_1n_2\delta. \end{aligned}\right\} \tag{86}$$

From (76), since $p = 4n_1n_2v^4 = 0$, we obtain

$$s = \gamma^3 - \gamma\varepsilon + \delta^2. \tag{87}$$

Eliminating n_1n_2 from (86),

$$r_2 = s^{-1}(3\gamma^4 - 4\gamma^2\varepsilon + 4\gamma\delta^2 - \delta\zeta + \varepsilon^2). \tag{88}$$

We have thus demonstrated the special element-formulæ of paragraph 8, except for the case $s = 0$. We may express (88) more briefly thus:

$$r_2 = s^{-1}(4\gamma s - \gamma^4 - \delta\zeta + \varepsilon^2). \tag{89}$$

When $s = 0$, we have from (86) $r_2 = -4n_1n_2\gamma^{-3}\delta$; also, by multiplying together the two equations (71), $16n_1^2n_2^2 = (\delta^2 + 4\gamma^3)^2$, or $4n_1n_2 = \delta^2 + 4\gamma^3$; so that $r_2 = -\gamma^{-3}\delta^3 - 4\gamma\delta$, as stated in paragraph 8. The contrary sign might be given to $4n_1n_2$ without affecting the elements otherwise than by changing their order.

35. Results not uninteresting may be obtained by assimilating the element-formulæ to the forms of the fifth roots of unity. We know that r_1 is a rational function of the roots of the quintic, and since $500v = \phi^2$ we see that $r_2\sqrt{5}$ is another. The quantities q in (35) were found by Lagrange to be of the form $a\sqrt{(-10 \pm 2\sqrt{5})} \pm b\sqrt{(-10 \mp \sqrt{5})}$, where a and b are rational functions of the roots; and this may be verified by expanding u^5 in terms of the roots. Taking (14) as an example, we may replace $\sqrt{(-\frac{146}{5} \pm \frac{1018}{125}\sqrt{5})}$ by $\frac{33}{25}\sqrt{(-10 \pm 2\sqrt{5})} \mp \frac{23}{25}\sqrt{(-10 \mp 2\sqrt{5})}$. The values of a and b may be readily ascertained from another form, which will be found equivalent to that just given as having been indicated by Lagrange, namely, $(a - \frac{1}{2}b \mp \frac{1}{2}b\sqrt{5})$ $\sqrt{(-10 \pm 2\sqrt{5})}$. Let us denote $a - \frac{1}{2}b$ by h_1, and $-\frac{1}{2}b\sqrt{5}$ by h_2; then h_1 and $h_2\sqrt{5}$ are rational functions of the roots, and it only remains to determine their values. We have $(h_1 \pm h_2)\sqrt{(-10 \pm 2\sqrt{5})} = \sqrt{(s_1 \pm s_2)}$, whence $(h_1^2 \pm 2h_1h_2 + h_2^2)(-10 \pm 2\sqrt{5}) = s_1 \pm s_2$, and $(h_1^2 - h_2^2)\,4\sqrt{5} = \sqrt{(s_1^2 - s_2^2)}$. From these equations we find that

$$\left.\begin{aligned} 400h_1^2 &= 10\sqrt{(5s_1^2 - 5s_2^2)} - 25s_1 - 5\sqrt{5}s_2, \\ 400h_2^2 &= -10\sqrt{(5s_1^2 - 5s_2^2)} - 25s_1 - 5\sqrt{5}s_2. \end{aligned}\right\} \tag{90}$$

Taking the same example (14) as before, we obtain $h_1^2 = (\frac{19}{10})^2$, $h_2^2 = (\frac{23}{50}\sqrt{5})^2$. The quantities q may therefore be represented by $(\frac{19}{10} \pm \frac{23}{50}\sqrt{5})\sqrt{(-10 \pm 2\sqrt{5})}$. A similar modification, subject to the same remarks, may be made in the form of n_1 and n_2, which have the form $\sqrt{(s_3 \pm s_4)}$, as may be seen from (71), where

s_3 and $s_4\sqrt{5}$ are rational functions of the roots. The values of these symbols are

$$\left.\begin{aligned} s_3 &= \tfrac{1}{4}\delta^2 + \tfrac{1}{4}t^2v + \gamma^3 - \gamma v, \\ s_4 &= (\tfrac{1}{2}\delta t - \gamma^2 + v)\,v^{\frac{1}{2}}. \end{aligned}\right\} \qquad (91)$$

The quantities n may be put under the form $(h_3 \pm h_4)\sqrt{(-10 \pm 2\sqrt{5})}$, by formulæ precisely similar to (90).

36. Assuming the coefficients of the quintic to be rational, t and v may be rational or irrational, but each is normally a rational function of the other, by Lagrange's theory, since each has six values respectively corresponding, and we shall, in fact, see each expressed rationally in terms of the other. Any rational function of t or v is a rational function of the roots, but the contrary does not hold. The quantities a and b, h_1 and $h_2\sqrt{5}$, for example, discussed in the preceding paragraph, are not necessarily rational functions of t and v. But by (76) p, or $4n_1n_2v^{\frac{1}{2}}$, is such a function, and so, it may be shown, is $q_1q_2v^{\frac{1}{2}}$, which let us denote by k. We note first that $n_2 = 4p^{-1}n_2^2v^{\frac{1}{2}}n_1$, and therefore is of the form $n_2 = (a_1 + a_2v^{\frac{1}{2}})\,n_1$, where a_1 and a_2 are rational functions of t and v. If we expand $q_1 = \frac{1}{2}(u_1^5 - u_4^5)$ by means of (79), it will take the form $q_1 = (b_1 + b_2v^{\frac{1}{2}})\,n_1$, where b_1 and b_2 are rational functions of t and v. Similarly, $q_2 = (b_1 - b_2v^{\frac{1}{2}})\,n_2$; hence $4k = p\,(b_1^2 - b_2^2v)$, and k is a rational function as stated.

37. Let $\text{E} = \sqrt{(s_1^2s_2^{-2} - 1)} = ks_2^{-1}v^{-\frac{1}{2}}$, $\text{Z} = \text{E}^2 + 1 = s_1^2s_2^{-2}$, $\text{H} = s_1^{-1}s_2^2$, $\text{B} = \frac{1}{4}r_1$, $\text{B}' = \frac{1}{4}r_2s_1^{-1}s_2$; then $\frac{1}{4}r_2 = \text{B}'\text{Z}^{\frac{1}{2}}$, $s_1 = \text{HZ}$, $s_2 = \text{HZ}^{\frac{1}{2}}$, and if these be substituted in (7) we have

$$u_1^5 = \text{B} + \text{B}'\text{Z}^{\frac{1}{2}} + \sqrt{(\text{HZ} + \text{HZ}^{\frac{1}{2}})}. \qquad (92)$$

The quantities here introduced seem arbitrary, and are in reality complex in character, yet they will all be recognized as rational functions of t and v. That the fifth-powers of the elements can be exhibited in the form shown in (92) was ascertained by Mr. G. P. Young, by a minute consideration of the possible surds involved in the roots of the general quintic.* The quantities B, B′, H, E, were shown by him to be rational functions, of unknown form, of each other. He indicated no means of determining them by a resolvent, nor did the quantities t and v enter into his discussion. We have now found the values of his undetermined symbols. Taking (12) as an example, where $v = \frac{1}{20}$, $r_1 = 90$, $r_2 = 6\sqrt{5}$, $s_1 = -\frac{146}{5}$, $s_2 = -\frac{1018}{125}\sqrt{5}$, it will be found that $\text{B} = \frac{45}{2}$, $\text{B}' = \frac{3054}{365}$, $\text{Z} = \frac{666125}{259081}$, and $\text{H} = -\frac{518162}{455625}$. By the same investigation, Mr. Young arrived at the following form, without determining the values of the quantities:

$$u_1^2u_3 = -\tfrac{1}{2}\delta + \text{CZ}^{\frac{1}{2}} + (\Theta + \Phi\text{Z}^{\frac{1}{2}})\sqrt{(\text{HZ} + \text{HZ}^{\frac{1}{2}})}. \qquad (93)$$

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* See note to paragraph 8.

We observe that $cz^4 = -\frac{1}{2}tv^4$, so that $c = -\frac{1}{2}ts_1^{-1}s_2v^4$. Using the same illustration, where $t = 30$, it follows that $c = -\frac{1527}{780}$.

38. Let us now consider the most important question remaining, namely, how to determine a value of v from the corresponding value of t, and the contrary. The theory of Lagrange, elaborated by Meyer Hirsch, supplies formulæ for the value of an n-valued function of roots in terms of the corresponding value of another n-valued function. It would be more troublesome, however, to pursue that method than to make use of the Eulerian equations in the manner which will now be exhibited.

39. The product of the left hand members of the equations numbered (71), multiplied by v, is the same as p^2, as defined in paragraph 31. Hence,

$$p^2 = (\delta^2 - t^2v)^2v + 8(\gamma^2 - v)(\gamma\delta^2 + \gamma t^2v + 2\delta tv)v + 16(\gamma^2 - v)^3v. \qquad (94)$$

If we set this equal to the square of p as given in (76), we see at once that the resulting equation is divisible by $(\gamma^2 - v)$. This done,

$$\left.\begin{aligned} 25v^3 &+ (-t^4 + 14\gamma t^2 + 16\delta t - 35\gamma^2 - 6\varepsilon)v^2 \\ &+ (-2c_0t^2 + 2\gamma\delta^2 + 4\gamma^2\varepsilon + 11\gamma^4 + \varepsilon^2)v - c_0^2 = 0. \end{aligned}\right\} \qquad (95)$$

Here $c_0 = -\gamma^3 + \gamma\varepsilon - \delta^2$, as in paragraph 2. Again, summing the expressions in (79), and dividing by $(\gamma^2 - v)$, we find that

$$\Sigma u^5 = (\gamma^2 - v)^{-1}[\gamma t^3v - \delta t^2v + \varepsilon tv + \gamma c_0t - 10\gamma^2tv - 5tv^2 + \delta^3 - 2\gamma\delta\varepsilon] = 0. \qquad (96)$$

Equating this value of Σu^5 with that in (77), and clearing of fractions, we find that

$$25tv^2 + (-\gamma t^3 + \delta t^2 - \varepsilon t - 10\gamma^2t + \zeta)v - \gamma c_0t + c_3 = 0. \qquad (97)$$

Here $c_3 = -\gamma^2\zeta + 2\gamma\delta\varepsilon - \delta^3$, as in paragraph 2. The two equations in t and v, (95) and (97),* are substantially equivalent to two of Signor Brioschi's and to two of Professor Schläfli's, though more simply deduced and expressed, owing to the advantages secured by employing the quantities t and v. By substituting st^{-2} for v, equations in t and s may be obtained upon occasion. The equations in t and s will be found useful for determining the value of t in any case where s has been ascertained by the use of the sinister resolvent. I have throughout considered it unnecessary to state separately the formulæ in terms of t and s which may be derived from those in t and v by replacing v by st^{-2}.

40. By subtracting (95) multiplied by t from (97) multiplied by v there results a second quadratic in v, say $b_0v^2 + b_1v + b_2 = 0$, where $b_0 = t^5 - 15\gamma t^3 - 15\delta t^2 + 25\gamma^2t + 5\varepsilon t + \zeta$, $b_1 = 2c_0t^3 - \gamma\delta^2t - 5\gamma^2\varepsilon t - 10\gamma^4t - \varepsilon^2t + c_3$, $b_2 = c_0^2t$.

* Vol. VI, pp. 314, 315. I supposed them entirely new.

We may write (97) as $a_0v^2 + a_1v + a_2 = 0$. From these two quadratics we have by elimination

$$v = \frac{a_2b_0 - a_0b_2}{a_0b_1 - a_1b_0} = \frac{a_1b_2 - a_2b_1}{a_2b_0 - a_0b_2}, \tag{98}$$

$$(a_2b_0 - a_2b_0)^2 + (a_1b_0 - a_0b_1)(a_1b_2 - a_2b_1) = 0. \tag{99}$$

The resultant (99) contains, as extraneous factors, the quantity t just introduced into one equation, and also the expression in t corresponding to the factor $\gamma^3 - v$, previously introduced. From (96) we find that when $\gamma^3 - v = 0$, $\gamma^3t^3 - \gamma^2\delta t^2 + (2\gamma^2\varepsilon - 16\gamma^4 - \gamma\delta^2)t - 2\gamma\delta\varepsilon + \delta^3 = 0$. Dividing (99), after expansion, by this extraneous factor, and by t, we obtain the central resolvent in t (3) already repeatedly mentioned. It was in this manner that I first obtained that resolvent,* but it may be demonstrated with much less labor, as follows.

41. Let us resume the discussion of the central resolvent as we left it when stating (64). The mere definition of τ as a function of the roots was sufficient to demonstrate the resolvent, assuming as determined the quantities m in the equation

$$m_1\text{AL} + m_2\text{Q}^3 + m_3\text{BQ} + m_4\text{C}^2 = 0. \tag{100}$$

To determine these it will suffice to take $a = 1$, $b = d = f = 0$, whence $\tau = t$, $\gamma = c$, $\varepsilon = e$, $\delta = \zeta = 0$. In this case, referring to the tables of covariants, the leading coefficient of AL is $-15\gamma^4\varepsilon + 14\gamma^2\varepsilon^2 + \varepsilon^3$; that of Q^3 is $27\gamma^6 + 27\gamma^4\varepsilon + 9\gamma^2\varepsilon^2 + \varepsilon^3$; that of BQ is $9\gamma^6 - 9\gamma^4\varepsilon - \gamma^2\varepsilon^2 + \varepsilon^3$; and that of C^2 is $\gamma^6 - 2\gamma^4\varepsilon + \gamma^2\varepsilon^2$. In (99) we have $a_0 = 25t$, $a_1 = -\gamma t^3 - 10\gamma^2t - \varepsilon t$, $a_2 = (-\gamma^4 + \gamma^2\varepsilon)t$, $b_0 = t^5 - 15\gamma t^3 + (25\gamma^2 + 5\varepsilon)t$, $b_1 = 2(\gamma\varepsilon - \gamma^3)t^3 - (10\gamma^4 + 5\gamma^2\varepsilon + \varepsilon^2)t$, $b_2 = (\gamma^5 - 2\gamma^4\varepsilon + \gamma^3\varepsilon^2)t$. The highest power of t in (99) which does not vanish is t^{10}, and the only terms of (99) in which t^{10} can appear are $a_2^2b_0^2 + a_1^2b_2b_0 - a_1a_2b_0b_1$. In this expression the coefficient of t^{10} is $-25\gamma^9 + 35\gamma^7\varepsilon - 11\gamma^5\varepsilon^2 + \gamma^3\varepsilon^3$. This is, as we have seen, to be divided by the extraneous factor γ^3, and the result must coincide with the leading coefficient of (100). Therefore

$$\left.\begin{aligned}(27m_2 + 9m_3 + m_4)\gamma^6 &= -25\gamma^6,\\ (-15m_1 + 27m_2 - 9m_3 - 2m_4)\gamma^4\varepsilon &= 35\gamma^4\varepsilon,\\ (14m_1 + 9m_2 - m_3 + m_4)\gamma^2\varepsilon^2 &= -11\gamma^2\varepsilon^2,\\ (m_1 + m_2 + m_3)\varepsilon^3 &= \varepsilon^3.\end{aligned}\right\} \tag{101}$$

From these, $m_1 = 1$, $m_2 = m_3 = 0$, $m_4 = -25$, as assumed in paragraph 25. The central resolvent is thus proved to have been correctly stated in (64).

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* Vol. VI, p. 315.

42. Either of the two expressions in (98) will determine a value of v when the corresponding value of t is once known. Simpler formulæ, presented in (5), remain to be demonstrated. Expansion of both sides suffices to establish the identity

$$\text{L}_y(a_2b_0 - a_0b_2) + 25\text{C}_y(a_1b_2 - a_2b_1) = a_2(\text{A}_y\text{L}_y - 25\text{C}_y^2). \tag{102}$$

The second member vanishes, by (65), and by (98) we know that $a_1b_2 - a_2b_1 = v(a_2b_0 - a_0b_2)$. Combining these, we derive the formulæ (5) for expressing v in terms of t,

$$v = -\tfrac{1}{25}\text{L}_y\text{C}_y^{-1} = -\text{C}_y\text{A}_y^{-1}. \tag{103}$$

In the statement of (27) it will be seen that the factor a was dropped from the original quantic, so that $\text{A}_y = \text{A}a^{-1}$, and hence also $\text{C}_y = \text{C}a^{-3}$, $\text{L}_y = \text{L}a^{-5}$; wherefore the corresponding expressions of v in terms of τ are

$$v = -\tfrac{1}{25}\text{LC}^{-1}a^{-2} = -\text{CA}^{-1}a^{-2}. \tag{104}$$

43. We might similarly eliminate t from (95) and (97), dividing the resultant by the extraneous factor $(v-\gamma^2)^3v^6$, and so obtaining the dexter resolvent (50);* or, after substituting st^{-2} for v in the same equations, we might eliminate t to produce the sinister resolvent (59).

44. To obtain an expression for t in terms of v, we may consider (95) as briefly written $a_0t^4 + a_2t^2 + a_3t + a_4 = 0$, and (97) as $b_0t^3 + b_1t^2 + b_2t + b_3 = 0$. By the usual theory for the determination of common roots,

$$t = \text{Numerator} \div \text{Denominator} \tag{105}$$

where the numerator and denominator are expressed by determinants thus:

Numerator.

$$\begin{vmatrix} a_0 & 0 & a_2 & a_3 & 0 \\ 0 & a_0 & 0 & a_2 & a_4 \\ b_0 & b_1 & b_2 & b_3 & 0 \\ 0 & b_0 & b_1 & b_2 & 0 \\ 0 & 0 & b_0 & b_1 & b_3 \end{vmatrix}$$

Denominator.

$$\begin{vmatrix} -a_0 & 0 & a_2 & a_3 & a_4 \\ 0 & a_0 & 0 & a_2 & a_3 \\ -b_0 & b_1 & b_2 & b_3 & 0 \\ 0 & b_0 & b_1 & b_2 & b_3 \\ 0 & 0 & b_0 & b_1 & b_2 \end{vmatrix}$$

Dividing both, after evaluation, by $(v-\gamma^2)^2v^5$, we obtain these formulæ,

$$\left.\begin{aligned} \text{Numerator} = {} & 5^4(2\gamma\delta + \zeta)v^2 - 5^2(20\gamma^3\delta + 22\gamma^2\zeta - 36\gamma\delta\varepsilon + 24\delta^3 + 2\varepsilon\zeta)v \\ & + 50\gamma^5\delta - 59\gamma^4\zeta + 20\gamma^3\delta\varepsilon + 40\gamma^2\delta^3 \\ & + 42\gamma^2\varepsilon\zeta - 48\gamma\delta^2\zeta - 38\gamma\delta\varepsilon^2 + 44\delta^3\varepsilon - \delta\zeta^2 + \varepsilon^2\zeta, \\ \text{Denominator} = {} & -5^6v^3 - 3.5^4d_2v^2 - 5^2d_4v + d^6. \end{aligned}\right\} \tag{106}$$

Here the symbols have the same meaning as in (48). If the dexter resolvent be

* I have actually performed this operation, and do not advise any one to repeat it unnecessarily.

used in any case, in preference to the central, these formulæ, with (7), will be found best fitted to complete the work. The proof of (20) is furnished by (106), when $v = \gamma^2$. The following expression is equivalent to (106), the origin of it being due to an examination of the minors of Professor Schläfli's determinant, referred to in paragraph 19:

$$t = \begin{vmatrix} \text{H}, & \text{F}, & -\text{L} \\ \text{G}, & \text{C}, & -\delta \\ \zeta, & \delta, & \gamma \end{vmatrix} \div \begin{vmatrix} \text{B}, & \text{F}, & \text{L} \\ \text{F}, & \text{C}, & \delta \\ \text{L}, & \delta, & -\gamma \end{vmatrix}. \tag{107}$$

It will be remembered that the quantity denoted by t was not discussed by or (so far as appears) known to the author just mentioned.

45. Let us, for present convenience, call γ, δ, v, and t the Constituents of the quintic. I have before* pointed out that for each such set of constituents there are two conjugate equations, and that "when v and t have been determined for any quintic, we can at once construct the conjugate quintic. Equation [95] is a quadratic in ε, one value being the given value of ε, the other being that of ε in the conjugate quintic. From the latter value, that of ζ may be obtained by [97]. For example, if the given quintic be $x^5 + \frac{625}{4}x + 3750 = 0$, the conjugate is $x^5 + \frac{3125}{4}x + 3125 = 0$." Here it is assumed as known that $t = -5$, $v = \frac{125}{4}$, $\gamma = 0$, $\delta = 0$. The solution of (95), regarded as a quadratic, is given by (75) and (94), namely,

$$\varepsilon = \gamma^2 + 3v + (v - \gamma^2)^{-1}(\gamma t^2 v - \gamma\delta^2 + p), \tag{108}$$

where p is such that

$$p^2 = (\delta^2 - t^2 v)^2 v + 8\,(\gamma^2 - v)(\gamma\delta^2 + \gamma t^2 v + 2\delta t v)v + 16(\gamma^2 - v)^3 v. \tag{109}$$

In the simple case taken for illustration, a trinomial of the form $x^5 + 5\varepsilon x + \zeta = 0$, this becomes

$$\varepsilon = 3v \pm \sqrt{(t^4 v - 16v^2)}. \tag{110}$$

Hence $\varepsilon = 3.\frac{125}{4} \pm \sqrt{(5^4.\frac{125}{4} - 5^6)} = (3 \pm 2)\frac{125}{4}$, namely, $\frac{125}{4}$ or $\frac{625}{4}$, as stated. The value of ζ by (97) is

$$\left.\begin{aligned} \zeta = (v - \gamma^2)^{-1}(&- 25tv^2 + \gamma t^3 v - \delta t^2 v + \varepsilon t v + 10\gamma^2 t v - \gamma^4 t \\ &+ \gamma^2 \varepsilon t - \gamma\delta^2 t - 2\gamma\delta\varepsilon + \delta^3). \end{aligned}\right\} \tag{111}$$

In the case of the trinomial in question, this becomes

$$\zeta = t(\varepsilon - 25v). \tag{112}$$

Hence $\zeta = -5\,(\frac{125}{4} - \frac{3125}{4}) = 3750$, or $\zeta = -5\,(\frac{625}{4} - \frac{3125}{4}) = 3125$, as stated.

46. In addition to the illustration just quoted, we may take equations (12)

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* Vol. VI, p. 315.

and (15), which give the following:

	γ	δ	t	v	ε	ζ
(12)	$-\frac{7}{2}$	3	30	$\frac{1}{20}$	$\frac{41 \pm 141}{250}$	$-\frac{891 \pm 141}{750}$
(15)	$-\frac{1}{10}$	$-\frac{1}{5}$	$\frac{4}{3}$	$\frac{9}{500}$	$\frac{6932 \pm 632}{5.61}$	$-\frac{200040 \pm 246480}{61^2}$

Since each quintic has six sets of constituents, it has six other quintics conjugate to it, and these have others, the only apparent limit to their formation being that they must all be of the same class as regards the number of real roots. This I infer from the fact that in (79) a change of sign in either n_1, n_2, or $v^{\frac{1}{2}}$, while not affecting the reality of the elements, will change the sign of p, and therefore lead to the elements of the conjugate. The construction of quintics from a given set of constituents is useful chiefly for obtaining examples wherein the constituents are simple in nature. To construct a quintic with rational coefficients and constituents, rational values must be found for the symbols employed in (109) such as will satisfy that equation, and the rest follows. For example, the values $\gamma = 0$, $\delta = 1$, $t = -2$, $v = 1$, $p = \pm 5$, satisfy (109), and it follows that $\varepsilon = 3 \pm 5$, $\zeta = 41 \mp 10$. Perhaps the simplest values possible for γ and δ are $\gamma = 0$, $\delta = 0$, producing the trinomial $x^5 + 5\varepsilon x + \zeta = 0$, for which (110) and (112) are available. In this case we have only to find values of the symbols consistent with the relation embodied in (110), $p^2 = t^4 v - 16 v^2$, or

$$v = t^4 (p^2 v^{-2} + 16)^{-1}, \tag{113}$$

and for this we may assign any values we please to the quantities t and pv^{-1}.*

47. For the trinomial just spoken of, $x^5 + 5\varepsilon x + \zeta = 0$, the element-formulæ (7) become most readily applicable. In this case

$$\left.\begin{array}{ll} r_1 = -\zeta - 20tv, & r_2 = t^3 v^{\frac{1}{2}}, \\ s_1 = \frac{1}{16}(r_1^2 + r_2^2), & s_2 = \frac{1}{8} r_1 r_2 - v^{\frac{5}{2}}. \end{array}\right\} \tag{114}$$

Let us take, for example, the case quoted in paragraph 45, wherein $t = -5$, $pv^{-1} = -2$, $v = \frac{125}{4}$, $\varepsilon = 3v + p = \frac{125}{4}$, $\zeta = t(\varepsilon - 25v) = 3750$. Then $r_1 = -5^4$, $r_2 = -\frac{1}{2} 5^4 \sqrt{5}$, $s_1 = \frac{9}{64} 5^8$, $s_2 = \frac{9}{32} 5^7 \sqrt{5}$, and these values, substituted in (7), complete the element-formulæ.

*** The construction of trinomials of this form by assigning values to quantities identifiable with the two here named has been effected by Mr. G. P. Young (*American Journal of Mathematics*, VII, 170) in a different manner—the quantity v not appearing at all, and the quantity t being introduced as the fourth-root of a fraction. He also gives a method (indirect, and requiring the solution of a biquadratic) for deducing the elements from the two arbitrary quantities. Mr. J. C. Glashan states that Mr. Young informed him of this discovery in the latter part of 1883, at which time my paper just quoted had not appeared.**

48. Since the element-formulæ (7) enable us to express the elements in terms of the constituents, the problem of solving any quintic is reduced to that of determining its constituents. For the general quintic this can only be done by the aid of a sextic resolvent. For such individual quintics as exhibit certain relations between the coefficients the task is easier, and the quintics are said to be solvable. I say "said to be solvable," for the word solvable has received no exact definition. The knowledge of a value of v or of t is enough to effect a solution. This knowledge must be obtained legitimately. We have no right, for example, to construct a quintic from arbitrary values of the constituents, as in the last paragraph, and then declare it solvable because we know what the values are. If we had that right, we might determine the constituents of any quintic by means of a resolvent, then reconstruct the quintic from the values of the constituents so obtained, and thereupon call it solvable. The same quintic cannot be solvable to one inquirer and non-solvable to another, although the first may have constructed it from arbitrary constituents of which the second does not possess the secret. Wherein, then, does the distinction lie?

49. Shall we say that the formulæ (108) and (111), which express ε and ζ in terms of the four constituents, are criteria of solvability, and that all quintics are solvable whose coefficients are of that form? Then every quintic would be solvable, for all are of that form. Shall we limit the statement to cases wherein the constituents are rational? On the one hand no good reason can be given for such a limitation, and on the other hand it might be urged that the boundary is still too extended. Where else can the line be drawn? I am unable to answer, and think that the use of the word "solvable," so far as it implies a distinction between two supposed classes of solvable and insolvable quintics, should be discontinued. To justify this opinion, I shall take up several cases wherein solvability is conceded, and proceed to others between which and the first no dividing line can be established.

50. Any equation is solvable if any relation is known between its roots. If it be ascertained, for example, that the function of roots known as v, or that known as t, has a given value, the quintic is solvable. If in (50) $v = \gamma^5$ and $\delta = 0$, that equation becomes reduced to $4\gamma^5 - \varepsilon = 0$. Therefore, $y^5 + 10\gamma y^3 + 20\gamma^2 y + \zeta = 0$ is solvable, since we know that $v = \gamma^5$, nor is it necessary that either γ or γ^5 be rational. This is the oldest of the solvable quintics, and was discovered by DeMoivre. If in (50) $v = \gamma^5$, without restricting the value of δ, we have the relation between the coefficients shown in (19), or if $v = 0$, we have

(24). Whenever we meet with a quintic whose coefficients are observed to be related in either of these modes, we know the value of v, and the equation is solvable. Let us go a step further. If in (59) we put $s = 0$, we have as the criterion for a new class of solvable quintics,

$$\zeta l_1 = 25 c_3^2. \tag{115}$$

Whenever this relation is observed to exist between the coefficients, we know that $s = 0$ and (unless $v = 0$) $t = 0$, and the quintic is solvable. It would be possible to form a table of criteria to any desired extent, for given successive values of t or v, so that the number of solvable classes is unlimited. But the argument can best be enforced by an illustration.

51. Let $t = -2$, and for simplicity let $\gamma = 0$. Then (4) becomes

$$\left.\begin{aligned} 1024\delta^4 + 128\delta\varepsilon\zeta - 64\varepsilon^3 + 32\delta\zeta^2 - 1408\delta^3\varepsilon - 32\varepsilon^2\zeta + 1200\delta^2\varepsilon^2 - 800\delta^3\zeta \\ + 320\delta^5 - 320\delta\varepsilon^3 + 240\delta^2\varepsilon\zeta + 60\delta^2\zeta^2 - 220\delta^4\varepsilon - 120\delta\varepsilon^2\zeta + 80\varepsilon^4 \\ + 6\delta\varepsilon\zeta^2 - 40\delta^3\varepsilon^2 + 118\delta^4\zeta - 8\varepsilon^3\zeta + 25\delta^6 - 6\delta^3\varepsilon\zeta - \delta\zeta^3 + \varepsilon^2\zeta^2 = 0. \end{aligned}\right\} \tag{116}$$

I say that any quintic whose coefficients are connected in this manner, when $\gamma = 0$, is solvable, because such coefficients indicate a known relation between the roots, namely, $t = -2$. In every such case, by (5),

$$v = -\frac{1}{25}\,\frac{-18\delta^4 + 4\delta\varepsilon\zeta - 2\varepsilon^3 + 6\delta^3\varepsilon + \delta\zeta^2 - \varepsilon^2\zeta}{8\delta^2 - 4\delta\varepsilon - 2\delta\zeta + 2\varepsilon^2 - \delta^3}. \tag{117}$$

Then the elements of the roots are known by (7). We have here a "class of solvable quintics," with a suitable criterion, and an immediate solution. For instance, if we meet the equation $y^5 + 10y^3 - 10y + 51 = 0$, and test it by the criterion (116), we find the latter satisfied, and by (117) we find $v = 1$, so that by means of (7) it is at once solved. We might take any value of t, or any value of v, no matter why chosen or how determined, and obtain for it an equation like (116), to be thereafter regarded as a criterion for a "class of solvable quintics." But if any equation is "solvable" after such a criterion is laid down, it must be "solvable" before. I am unable, therefore, to concur in the usual assumption that some quintics are intrinsically solvable and that all others are not. Solution in every case depends on the recognition of the quintic as one of a previously discussed class, by observation of certain relations between its coefficients. The number of such possible classes is not limited. "An infinitely large accumulation of solvable quintics would leave none of the other sort, by which to illustrate the fact that the quintic cannot be solved algebraically; yet it would not change the fact."*

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* Vol. VI, p. 313.

52. Quintics "solvable" by peculiar methods may be brought under the general theory now presented. Reciprocal equations form a class of such quintics, their form being

$$ax^5 + bx^4 + cx^3 \pm cx^2 \pm bx \pm a = 0. \tag{118}$$

Comparing (52) and (53), we find that in this case $\phi = \mp \psi$, whence by (60) $\tau = \mp 1$, and by (61) $t = \mp 1 + ba^{-1}$. The solution of each such equation follows at once by (97) and (7). For example, if

$$x^5 + x^4 + x^3 + x^2 + x + 1 = 0, \tag{119}$$

we know that the roots are -1, $\frac{1 \pm \sqrt{(-3)}}{2}$, and $\frac{-1 \pm \sqrt{(-3)}}{2}$. By the present theory, we should proceed as follows to discover these roots. Since $\tau = -1$, $t = -1 + ba^{-1} = -\frac{4}{5}$. This value substituted in (97) gives $v^2 - \frac{13}{2} 5^{-3} v + \frac{1}{16} 5^{-4} = 0$, and the same result is obtainable from (95), or from the "second quadratic" of paragraph 40. Hence $v = \frac{13}{500} \pm \frac{12}{500}$. Taking $v = \frac{1}{20}$, we have $r_1 = -104.5^{-5}$, $r_2 = -4.5^{-3}\sqrt{5}$, $s_1 = \frac{39}{2} 5^{-6}$, $s_2 = -\frac{237}{2} 5^{-8}\sqrt{5}$, and employing these values in (7),

$$\left.\begin{aligned}
5^5 u_1^5 &= -26 - 25\sqrt{5} + \tfrac{5}{2}\sqrt{(1950 - 474\sqrt{5})},\\
5^5 u_2^5 &= -26 + 25\sqrt{5} + \tfrac{5}{2}\sqrt{(1950 + 474\sqrt{5})},\\
5^5 u_3^5 &= -26 + 25\sqrt{5} - \tfrac{5}{2}\sqrt{(1950 + 474\sqrt{5})},\\
5^5 u_4^5 &= -26 - 25\sqrt{5} - \tfrac{5}{2}\sqrt{(1950 - 474\sqrt{5})}.
\end{aligned}\right\} \tag{120}$$

From this, by (6), the real root of the reciprocal equation (119) is

$$\left.\begin{aligned}
x_1 = u_1 + u_2 + u_3 + u_4 - \tfrac{1}{5} &= -0.2977659 + 0.5566772\\
&\quad - 0.5094635 - 0.5494478 - 0.2 = -1.
\end{aligned}\right\} \tag{121}$$

The other four values of v are $0.026 \pm 0.016\sqrt{(-3)}$ and $-0.022 \pm 0.008\sqrt{(-3)}$.

53. It is evident from (5) and (106), as well as from their mutual relations as functions of the roots, that v is rational whenever t is rational, and *vice versa*, assuming the coefficients rational. An apparent exception to this rule is found in some cases wherein v or t has two equal values, t or v having irrational values corresponding. In any such case, (5) or (106) becomes indeterminate, and recourse must be had, as in the preceding paragraph, and in other cases when t or v has equal values, to the equations (95) and (97). In the exceptional cases here spoken of, two equal values, say m, are to be regarded as a pair of irrational values, say $m \pm \sqrt{n}$, wherein n vanishes. Thus, if the roots be denoted by $a, b, c, d + \sqrt{e}, d - \sqrt{e}$, v (not t) will have two equal rational values if $e = d^2 - 2ad + a(b + c) - bc$. For instance, if the roots be -29, 21, -4, $6 + \sqrt{(-25)}$, $6 - \sqrt{(-25)}$, v has two equal values, namely, -125, which we may

regard as $-125 \pm \sqrt{0}$, and the corresponding values of t are $-29 \pm \sqrt{(-25)}$. By referring to the definition in (38) of v as a function of the roots, it will be found that all of its values are real when all five roots are real, that some of its values are real when only one root is real, and that when three roots are real and unequal all of the six values of v are unreal, unless two are equal as just explained.

54. With a single exception, all efforts to resolve the quintic algebraically, with which I have become acquainted, have belonged to what I have called the first class, and have had for their object to illustrate the general theory of equations by determining the quantities required under that theory which I have here called the elements. The object is theoretic rather than practical, and when it is attained we have merely the satisfaction of knowing that we have accomplished for the fifth degree all that is possible under the uniform theory which affords solutions for the inferior degrees. I presume that the determination of the elements will continue to be regarded as the object of greatest interest in the resolution of the quintic. Nevertheless, it is not mentioned as essential in the definition quoted in paragraph 1, wherein the problem is broadly stated as the expression of the roots of the quintic in terms of those of a resolvent sextic. It is possible to accomplish this without reference to the elements, by some method of what I have called the second class. This view of the problem has, in the exceptional case referred to, been acted upon by a writer of the highest authority, Professor Cayley. I quote his statement* in the paragraph now following.

55. "The roots of the given quintic equation are each of them rational functions of the roots $[\phi]$ of the auxiliary equation, so that the theory of the solution of an equation of the fifth order appears to be now carried to its extreme limit. We have in fact

$$\begin{aligned}
\phi_1\phi_6 + \phi_2\phi_4 + \phi_3\phi_5 &= (*\between x_1,\ 1)^4,\\
\phi_1\phi_2 + \phi_3\phi_4 + \phi_5\phi_6 &= (*\between x_2,\ 1)^4,\\
\phi_1\phi_5 + \phi_2\phi_3 + \phi_4\phi_6 &= (*\between x_3,\ 1)^4,\\
\phi_1\phi_3 + \phi_2\phi_6 + \phi_4\phi_5 &= (*\between x_4,\ 1)^4,\\
\phi_1\phi_4 + \phi_2\phi_5 + \phi_3\phi_6 &= (*\between x_5,\ 1)^4,
\end{aligned}$$

where $(*\between x_1,\ 1)^4$, etc., are the values, corresponding to the roots, x_1, etc., of the given equation, of a given quartic function. And combining these equations respectively with the quintic equations satisfied by the roots x_1, etc., respectively,

* "On a New Auxiliary Equation," Philosophical Transactions, Vol. 151, p. 264.

it follows that, conversely, the roots x_1, x_2, etc., are rational functions of the combinations $\phi_1\phi_6 + \phi_2\phi_4 + \phi_3\phi_5$, $\phi_1\phi_2 + \phi_3\phi_4 + \phi_5\phi_6$, etc., respectively, of the roots of the auxiliary equation. It is proper to notice that, combining together in every possible manner the six roots of the auxiliary equation, there are in all fifteen combinations of the form $\phi_1\phi_2 + \phi_3\phi_4 + \phi_5\phi_6$. But the combinations occurring in the above-mentioned equations are a completely determinate set of five combinations: the equation of the order 15, whereon depend the combinations $\phi_1\phi_2 + \phi_3\phi_4 + \phi_5\phi_6$, is not rationally decomposable into three quintic equations, but only into a quintic equation having for its roots the above-mentioned five combinations, and into an equation of the tenth order, having for its roots the other ten combinations, and being an irreducible equation. Suppose that the auxiliary equation and its roots are known; the method of ascertaining what combinations of roots correspond to the roots of the quintic equation would be to find the rational quintic factor of the equation of the fifteenth order, and observe what combinations of the roots of the auxiliary equation are also roots of this quintic factor." Here, as Mr. Cayley states elsewhere,

$$(*\between x_1, 1)^4 = 20(2a^2, 8ab, 22ac - 10b^2, 18ad - 10bc, 7ae - 10bd + 5c^2 \between x_1, 1)^4.$$

56. The method just described requires for determining each root the ascertainment of the common root of a quartic and a quintic, and the process suggested for selecting fit combinations involves considerable difficulty. Two other methods of the second class will now be presented as alternatives.

57. If the several values of ϕ as defined in paragraph 15 be added two and two, there results the following scheme:

$$\left.\begin{array}{ll}
\phi_1 + \phi_2 = 2(x_1 - x_3)(x_5 - x_4), & \phi_2 + \phi_6 = 2(x_1 - x_5)(x_3 - x_2), \\
\phi_1 + \phi_3 = 2(x_1 - x_2)(x_5 - x_3), & \phi_3 + \phi_4 = 2(x_1 - x_5)(x_4 - x_3), \\
\phi_1 + \phi_4 = 2(x_1 - x_4)(x_2 - x_3), & \phi_3 + \phi_5 = 2(x_2 - x_3)(x_4 - x_5), \\
\phi_1 + \phi_5 = 2(x_1 - x_5)(x_2 - x_4), & \phi_3 + \phi_6 = 2(x_1 - x_3)(x_4 - x_2), \\
\phi_1 + \phi_6 = 2(x_2 - x_5)(x_3 - x_4), & \phi_4 + \phi_5 = 2(x_1 - x_3)(x_2 - x_5), \\
\phi_2 + \phi_3 = 2(x_1 - x_4)(x_5 - x_2), & \phi_4 + \phi_6 = 2(x_1 - x_2)(x_4 - x_5), \\
\phi_2 + \phi_4 = 2(x_2 - x_4)(x_5 - x_3), & \phi_5 + \phi_6 = 2(x_1 - x_4)(x_3 - x_5). \\
\phi_2 + \phi_5 = 2(x_1 - x_2)(x_3 - x_4), &
\end{array}\right\} \quad (122)$$

In these products of differences each difference occurs three times, each time with different values of ϕ. If any pair of values of ϕ be taken at random, the remaining four values can be arranged as different pairs in three sets, two of which will exhibit common factors with the first pair, while the third will not. Let any three pairs having common factors be represented by $p_1 = 2(x_2 - x_3)(x_4 - x_5)$, $p_2 = 2(x_3 - x_1)(x_4 - x_5)$, and $p_3 = 2(x_1 - x_2)(x_4 - x_5)$. Then

$$p_1 x_1 + p_2 x_2 + p_3 x_3 = 0. \tag{123}$$

With the three quintics in x_1, x_2, and x_3 respectively, we have now four equations from which the values of these roots may be obtained by elimination. Once in three times, as has been seen, this process will fail, in which event the use of another combination will be successful. The method here given is simpler in appearance than that of Mr. Cayley, but it is open to the same objections, that it is only tentative, and that it is so laborious as to be unavailable in practice. The second method, now to be given, will be found free from both objections.

58. If we add two and two the several values of ψ as defined in (44), and compare the results with (122), we find that

$$\left.\begin{array}{ll}
\psi_1 + \psi_6 = x_1(\phi_1 + \phi_6), & \psi_4 + \psi_6 = x_3(\phi_4 + \phi_6), \\
\psi_2 + \psi_4 = x_1(\phi_2 + \phi_4), & \psi_1 + \psi_3 = x_4(\phi_1 + \phi_3), \\
\psi_3 + \psi_5 = x_1(\phi_3 + \phi_5), & \psi_2 + \psi_6 = x_4(\phi_2 + \phi_6), \\
\psi_1 + \psi_2 = x_2(\phi_1 + \phi_2), & \psi_4 + \psi_5 = x_4(\phi_4 + \phi_5), \\
\psi_3 + \psi_4 = x_2(\phi_3 + \phi_4), & \psi_1 + \psi_4 = x_5(\phi_1 + \phi_4), \\
\psi_5 + \psi_6 = x_2(\phi_5 + \phi_6), & \psi_2 + \psi_5 = x_5(\phi_2 + \phi_5), \\
\psi_1 + \psi_5 = x_3(\phi_1 + \phi_5), & \psi_3 + \psi_6 = x_5(\phi_3 + \phi_6). \\
\psi_2 + \psi_3 = x_3(\phi_2 + \phi_3), &
\end{array}\right\} \tag{124}$$

Hence any pair of values of ψ divided by the corresponding pair of values of ϕ gives a root of the quintic; and the five pairs formed by combining any value of ψ with the other five values respectively, each pair being similarly divided, give all the roots. It is to be remembered that $\psi = \tau\phi$, where $\tau = t - ba^{-1}$, that $\phi^2 = 500v$, and that $x = y - ba^{-1}$. The values of y are therefore to be found by dividing pairs of values of $t\phi$ or $tv^{\frac{1}{2}}$ by the corresponding pairs of values of ϕ or $v^{\frac{1}{2}}$ respectively. In practice, there are two modes of determining the needed values. We may determine the values of ϕ by the dexter resolvent, and from these the corresponding values of t by (106); or, in many cases preferably, we may ascertain the values of t by the central resolvent, and employ them in (5) to obtain those of v. The signs to be given to the several values of $v^{\frac{1}{2}}$ or ϕ in the latter case must be such as to agree with $\Sigma v^{\frac{1}{2}} = \Sigma\phi = 0$, and if more than one such arrangement is possible, with $\Sigma t v^{\frac{1}{2}} = \Sigma t\phi = 0$. In the highly improbable event that more than one arrangement fulfils both conditions, recourse may be had to (49) or (48).—As an illustration, let us take (12), for which the several values are

$$\left.\begin{array}{lrrrrrr} t\phi & 150 & -174 & 130 & -66 & -22 & -18 \\ \phi & 5 & -29 & 51 & -19 & -69 & 61 \end{array}\right\} \qquad (125)$$

Here the sum of any two values of $t\phi$, divided by the sum of the corresponding values of ϕ, gives a value of y, which increased by 1, the value of $-ba^{-1}$, gives a root of the quintic. If any one value of $t\phi$ be added successively to each of the others, and the sums be separately divided by the corresponding pairs of values of ϕ, we shall obtain all five values of y, and, by adding 1 to each, all five roots of the given quintic.

59. Various transcendental solutions of the quintic, by the aid of elliptic functions or of hypergeometric series, have been proposed in recent years. It would seem that no solution involving series can rival in simplicity the use of Lagrange's theorem. The employment of elliptic functions appears to have been suggested by De Moivre's transcendental solutions, by circular functions, of the cubic and of the solvable quintic known by his name. The use of tabulated functions has long been recognized as legitimate in analysis whenever necessary, with the implied understanding that no new function is to be introduced for tabulation so long as those already recognized will serve. Solutions by elliptic functions are circuitous and difficult, yet we are bound to prefer them to others which would require new tabulations. Were it not for this restriction, we might devise solutions of the quintic with great ease. Two such possible solutions will now be shown, in each of which it is assumed that the quintic has been brought by Jerrard's transformation to the form $x^5 + 5\varepsilon x + \zeta = 0$.

60. In the case just mentioned, if we put $v = \frac{1}{25}\varepsilon z$, the dexter resolvent (50) becomes, with slight changes,

$$(z-1)^4(z^2 - 6z + 25) = z\varepsilon^{-5}\zeta^4. \qquad (126)$$

"In this compact equation we have the unknown quantity expressed directly as a function of the parameter $\varepsilon^{-5}\zeta^4$. Were tables for such purposes worth calculating, a table of single entry might be constructed with great ease, showing a value of the parameter for each value of z, and therefore the converse, which would supply all that is needed to exhibit the roots of the trinomial in their normal form."*

61. A second possible solution involving special tabulation will be found to follow from consideration of De Moivre's solution of the cubic. If $x = \cos\theta$

* Vol. VI, p. 307.

$= \frac{1}{2}(\exp[i\theta] - \exp[-i\theta])$, where $i = \sqrt{(-1)}$,

$$x^3 - \tfrac{3}{4}x - \tfrac{1}{4}\cos 3\theta = 0. \quad (127)$$

To quote De Morgan's remark on the Irreducible Case,* "the best method of obtaining the roots is by having recourse to a registry of the roots of cubic equations, which is in the hands of every tyro, namely, the tables of sines and cosines." In other words, if a table of cosines (or sines) were needed for no other purpose, it would be useful at any rate as a registry of the roots of cubic equations. As a table of roots, however, it would be incomplete, since this method serves only when there are three real roots, and it would need to be supplemented, were no other solution known, by the values of $\exp\theta + \exp(-\theta)$. For quintics a similar "registry of roots" might be prepared, which would likewise serve only for roots all real, and would therefore need to be similarly supplemented. The function to be tabulated would be $\cos\theta + 5\cos\frac{3}{5}\theta$, which let us denote by $C\theta$. Then

$$x^5 - \tfrac{5}{8}x - \tfrac{1}{16}C5\theta = 0 \quad (128)$$

The trinomial to be solved having been first reduced to the form $x^5 - \frac{5}{8}x - k = 0$, the value of 5θ would be found from the new table, and, as before, $x = \cos\theta$. Or, we might have $x = \sin\theta$, and tabulate $S\theta = \sin\theta - 5\sin\frac{3}{5}\theta$, when the equation would be

$$x^5 - \tfrac{5}{8}x - \tfrac{1}{16}S5\theta = 0. \quad (129)$$

62. I close with the suggestion that the solution of the special Jacobian sextic discussed by MM. Hermite, Kronecker, Brioschi, Kiepert and others (and therefore, as shown by Signor Brioschi,† of the Malfattian or dexter resolvent), by means of elliptic functions, should be combined with the methods of exhibiting the roots now shown in paragraphs 2 and 58.

Milwaukee, August 5, 1885.

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* Penny Cyclopædia. † Mathematische Annalen, XIII, 154.

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61. Second suggestion.
62. Suggested combination of solution by elliptic functions with methods here given for exhibiting the roots.

On Linear Differential Equations whose Fundamental Integrals are the Successive Derivatives of the same Function.

By Thomas Craig.

I.

It is known that having given a linear differential equation of the n^{th} order, one of whose fundamental integrals is a function of another, the equation can be reduced to one of the order $n-1$. For example, having given the linear differential equation

$$1. \qquad \frac{d^n y}{dx^n} + p_1 \frac{d^{n-1}y}{dx^{n-1}} + p_2 \frac{d^{n-2}y}{dx^{n-2}} + \ldots + p_n y = 0,$$

of which the fundamental (*i. e.* linearly independent) integrals are

$$y_1, y_2, y_3 \ldots y_n;$$

then if $y_i = \phi(y_j)$, equation 1 can be reduced to the form

$$\frac{d^{n-1}y}{dx^{n-1}} + q_1 \frac{d^{n-2}y}{dx^{n-2}} + q_2 \frac{d^{n-3}y}{dx^{n-3}} + \ldots + q_{n-1} y = 0,$$

where $q_1, q_2, q_3 \ldots q_{n-1}$ are known functions of $p_1, p_2, p_3 \ldots p_n$.

Assuming now the equation

$$2. \qquad \frac{d^n y}{dx^n} + p_{11} \frac{d^{n-1}y}{dx^{n-1}} + p_{12} \frac{d^{n-2}y}{dx^{n-2}} + \ldots + p_{1n} y = 0,$$

having for integrals
$$\phi, \frac{d\varphi}{dx}, \frac{d^2\varphi}{dx^2} \ldots \frac{d^{n-1}\varphi}{dx^{n-1}},$$

it is required to find the form of the coefficients $p_{11}, p_{12} \ldots p_{1n}$. We have to start with the following system of equations

$$3. \qquad \begin{array}{llll}
1_0 & \dfrac{d^n\varphi}{dx^n} + p_{11}\dfrac{d^{n-1}\varphi}{dx^{n-1}} + \ldots + p_{1n}\phi & = 0 \\
1_1 & \dfrac{d^{n+1}\varphi}{dx^{n+1}} + p_{11}\dfrac{d^n\varphi}{dx^n} + \ldots + p_{1n}\dfrac{d\varphi}{dx} & = 0 \\
1_2 & \dfrac{d^{n+2}\varphi}{dx^{n+2}} + p_{11}\dfrac{d^{n+1}\varphi}{dx^{n+1}} + \ldots + p_{1n}\dfrac{d^2\varphi}{dx^2} & = 0 \\
\ldots & \ldots\ldots\ldots\ldots\ldots\ldots & \\
1_{n-1} & \dfrac{d^{2n-1}\varphi}{dx^{2n-1}} + p_{11}\dfrac{d^{2n-2}\varphi}{dx^{2n-2}} + \ldots + p_{1n}\dfrac{d^{n-1}\varphi}{dx^{n-1}} & = 0.
\end{array}$$

Differentiating the first of these and subtracting the result from the second we have, using accents to denote differentiation,

4. $$p'_{11}\phi^{(n-1)} + p'_{12}\phi^{(n-2)} + \dots + p'_{1n}\phi = 0,$$

one integral of which is of course the function ϕ. Differentiating now the second of equations 3 and subtracting the result from the third, then differentiating the third and subtracting from the fourth, and so on, we have the equations

$$\begin{aligned} p'_{11}\phi^{(n)} + p'_{12}\phi^{(n-1)} + \dots + p'_{1n}\phi' &= 0 \\ p'_{11}\phi^{(n+1)} + p'_{12}\phi^{(n)} + \dots + p'_{1n}\phi'' &= 0 \\ \dots\dots\dots\dots\dots\dots\dots\dots \\ p'_{11}\phi^{(2n-3)} + p'_{12}\phi^{(2n-4)} + \dots + p'_{1n}\phi^{(n-2)} &= 0, \end{aligned}$$

and these are what 4 becomes if we substitute for ϕ the derivatives

$$\phi',\ \phi'' \dots \phi^{(n-2)}.$$

Equation 4 therefore has all of its integrals derivatives of the same function, viz. the integrals are

$$\phi,\ \frac{d\varphi}{dx},\ \frac{d^2\varphi}{dx^2} \dots \frac{d^{n-2}\varphi}{dx^{n-2}}.$$

Dividing out 4 by p'_{11} and denoting the new coefficients by p'_{22}, p'_{23}, etc., gives the new differential equation

5. $$\frac{d^{n-1}y}{dx^{n-1}} + p'_{22}\frac{d^{n-2}y}{dx^{n-2}} + \dots + p'_{2n}y = 0.$$

The linearly independent integrals of this being

$$\phi,\ \phi' \dots \phi^{(n-2)},$$

we can, as before, form a differential equation of the order $n - 2$, viz.

6. $$\frac{d^{n-2}y}{dx^{n-2}} + p'_{33}\frac{d^{n-3}y}{dx^{n-3}} + \dots + p'_{3n}y = 0,$$

where $p'_{33} = p'_{23} \div p'_{22}$, etc., the integrals of which are found just as before to be

$$\phi,\ \frac{d\varphi}{dx},\ \frac{d^2\varphi}{dx^2} \dots \frac{d^{n-3}\varphi}{dx^{n-3}}.$$

Continuing this process we arrive at the equation of the second degree,

$$\frac{d^2y}{dx^2} + p'_{n-1,\,n-1}\frac{dy}{dx} + p'_{n-1,\,n}\,y = 0,$$

the integrals of which are ϕ and $\frac{d\varphi}{dx}$. Reducing this equation to one of the first order we have

$$\frac{dy}{dx} + \frac{p''_{n-1,\,n}}{p''_{n-1,\,n-1}}\,y = 0,$$

or for brevity,

$$\frac{dy}{dx} + qy = 0,$$

giving

$$y, = \phi, = e^{-\int q\,dx}.$$

We have thus determined the complete set of integrals of 2 and consequently also of the derived equations 5, 6, etc.

Writing the equation of the second order in the form

7.
$$\frac{d^2y}{dx^2} + p_1\frac{dy}{dx} + p_2 y = 0,$$

we have for its integrals

$$y_1 = e^{-\int q dx}, \quad y_2 = -qe^{-\int q dx},$$

where, as before, $q = \frac{p'_2}{p'_1}$. Substituting these integrals in the equation we obtain the identities

$$q^2 - q' + p_1 q + p_2 = 0,$$
$$-q^3 - q'' + 3qq' + p_1 q^2 - p_1 q' - p_2 q = 0.$$

Multiplying the first of these by $-q$ and subtracting from the second we have

8.
$$q'' + (p_1 - 2q)q' = 0,$$

as the condition to be satisfied by the coefficients of 7, in order that it may have the above integrals. The equation is of course only, theoretically, integrable when we know p_1: and in fact it is obvious that p_1 must be known, for even if we had a differential equation in q independent of p_1 it would still be necessary to know p_1 in order to find p_2 from q, $= \frac{p'_2}{p'_1}$.

The solution of 8 can be obtained in an infinite number of cases. First make $p_1 - 2q = F(x)$ where F is any functional symbol. We have then

$$\frac{d^2q}{dx^2} + F(x)\frac{dq}{dx} = 0,$$

giving
$$q = C\int e^{-\int F(x)dx}dx + C'$$

and
$$p_1 = 2q + F(x).$$
$$p_2 = \int p'_1 q dx.$$

Similarly we may write $p_1 - 2q = F(q)$ and obtain values of p_1 and p_2. The only practical difficulty in this second assumption is that x will be determined directly as a function of q instead of the converse.

EXAMPLES:

a.
$$q'' + (p_1 - 2q)q' = 0.$$

Make $p_1 = q$ and we have $q'' - qq' = 0$;

assuming $q' = Q$ this becomes $\frac{dQ}{dq} - q = 0$, giving, if we add no constant of integration,

$$Q = \frac{q^2}{2},$$

and finally

$$q = -\frac{2}{x}.$$

We have therefore

$$p_1 = -\frac{2}{x},$$

and

$$p_2 = \frac{2}{x^2}.$$

The differential equation is then

$$\frac{d^2y}{dx^2} - \frac{2}{x}\frac{dy}{dx} + \frac{2}{x^2}y = 0,$$

of which the integrals are x and x^2.

b. Making again $p_1 = q$ we have

$$q'' - qq' = 0$$

or, if q' be replaced by Q,

$$\frac{dQ}{dq} - q = 0.$$

Integrating and denoting by $\frac{C^2}{2}$ the constant of integration, we have

$$\frac{dy}{dx} + \frac{q^2 + C^2}{2},$$

from which follows

$$q = C \tan\frac{Cx}{2},$$

and the equation is readily found to be

$$\frac{d^2y}{dx^2} + C\tan\frac{Cx}{2}\frac{dy}{dx} + \frac{C^2}{2}\sec^2\frac{Cx}{2}y = 0,$$

the integrals of which are

$$y_1 = \left(\cos\frac{Cx}{2}\right)^{2C}$$

$$y_2 = \frac{dy_1}{dx} = -C\sin\frac{Cx}{2}\left(\cos\frac{Cx}{2}\right)^{2C-1}$$

c. Making $p_1 = 2q$ the equation is

$$\frac{d^2y}{dx^2} + 2C(1 - Cx)\frac{dy}{dx} + C^3x(Cx - 2)y = 0,$$

of which the integrals are

$$y_1 = e^{Cx\left(\frac{Cx}{2}-1\right)},$$

$$y_2 = \frac{dy_1}{dx} = C(Cx - 1)e^{Cx\left(\frac{Cx}{2}-1\right)}.$$

d. Making $p_1 - 2q = F(x)$, assume $F(x) = \frac{1}{x}$: then $q = C \log x + C'$, and the differential equation is

$$\frac{d^2y}{dx^2} + \left(2C \log x + \frac{1}{x}\right)\frac{dy}{dx} + \left((C \log x)^2 + \frac{C \log x}{x} + \frac{C}{x}\right) y = 0,$$

of which the integrals are

$$y_1 = e^{-(Cx \log x - Cx)}$$

$$y_2 = \frac{dy_1}{dx} = -C \log x \, e^{-(Cx \log x - Cx)}.$$

e. If we write $F(x) = -\frac{1}{x}$, we find

$$\frac{d^2y}{dx^2} + \left(Cx^2 - \frac{1}{x}\right)\frac{dy}{dx} + \left(\frac{C^2x^4}{4} + \frac{Cx}{2}\right) y = 0$$

$$y_1 = e^{-\frac{Cx^3}{6}}$$

$$y_2 = \frac{dy_1}{dx} = -\frac{Cx^2}{2} e^{-\frac{Cx^3}{6}}.$$

f. For $F(x) = \alpha$ (α a constant), we have

$$\frac{d^2y}{dx^2} + \left(\alpha - \frac{2C}{\alpha} e^{-\alpha x}\right)\frac{dy}{dx} + \frac{C^2}{\alpha^2} e^{-2\alpha x} y = 0$$

$$y_1 = e^{-\frac{C}{\alpha^2} e^{-\alpha x}}$$

$$y_2 = \frac{dy_1}{dx} = \frac{C}{\alpha} e^{-\alpha x} e^{-\frac{C}{\alpha^2} e^{-\alpha x}}.$$

Writing now as above $\frac{p''_{n-n,n}}{p''_{n-1,n-1}} = q$ we have $q'' + (p'_{n-1,n-1} - 2q) q' = 0$, which determines in the manner indicated q and $p'_{n-1,n-1}$. Substitute in the equations

$$\frac{d^2y}{dx^2} + p'_{n-1,n-1}\frac{dy}{dx} + p'_{n-1,n} \ y = 0$$

$$\frac{d^3y}{dx^3} + p'_{n-1,n-1}\frac{d^2y}{dx^2} + p'_{n-1,n}\frac{dy}{dx} = 0,$$

the value $y = e^{-\int q dx}$ and we have

$$q^2 - q' - p'_{n-1,n-1} q + p_{n-1,n} = 0,$$

$$-q^3 + 3qq' - q'' + p'_{n-1,n-1}(q^2 - q') - p'_{n-1,n} q = 0.$$

In these $p'_{n-1,n-1}$ is already known, so the first one suffices to determine $p'_{n-1,n}$. Going one step further back to the equations

$$\frac{d^3y}{dx^3} + p'_{n-2,n-2}\frac{d^2y}{dx^2} + p'_{n-2,n-1}\frac{dy}{dx} + p'_{n-2,n} \ y = 0,$$

$$\frac{d^4y}{dx^4} + p'_{n-2,n-2}\frac{d^3y}{dx^3} + p'_{n-2,n-1}\frac{d^2y}{dx^2} + p'_{n-2,n}\frac{dy}{dx} = 0,$$

$$\frac{d^5y}{dx^5} + p'_{n-2,n-2}\frac{d^4y}{dx^4} + p'_{n-2,n-1}\frac{d^3y}{dx^3} + p'_{n-2,n}\frac{d^2y}{dx^2} = 0,$$

we have for y the same value, viz., $y = e^{-\int q dx}, = \phi$, where q is known, therefore for $p'_{n-2,\,n-2},\ p'_{n-2,\,n-1},\ p'_{n-2,\,n}$ we have

$$p'_{n-2,\,n-2} = -\begin{vmatrix} \phi''' & \phi' & \phi \\ \phi^{iv} & \phi'' & \phi' \\ \phi^{v} & \phi''' & \phi'' \end{vmatrix} \div$$

$$p'_{n-2,\,n-1} = -\begin{vmatrix} \phi'' & \phi''' & \phi \\ \phi''' & \phi^{iv} & \phi' \\ \phi^{iv} & \phi^{v} & \phi'' \end{vmatrix} \div \text{ Denom.} = \begin{vmatrix} \phi'' & \phi' & \phi \\ \phi''' & \phi'' & \phi' \\ \phi^{iv} & \phi''' & \phi'' \end{vmatrix}.$$

$$p'_{n-2,\,n} = -\begin{vmatrix} \phi'' & \phi' & \phi''' \\ \phi''' & \phi'' & \phi^{iv} \\ \phi^{iv} & \phi''' & \phi^{v} \end{vmatrix} \div$$

And so the process is to be continued back to the equation of the n^{th} order, of which the general coefficient will be

$$p_{1i} = -\begin{vmatrix} \overset{1}{\phi^{(n-1)}} & \cdots & \overset{2}{\phi^{(n-2)}} & \cdots & \overset{i}{\phi^{(n)}} & \cdots & \phi \\ \phi^{(n)} & \cdots & \phi^{(n-1)} & \cdots & \phi^{(n+1)} & \cdots & \phi' \\ \vdots & & \vdots & & \vdots & & \vdots \\ \phi^{(2n-2)} & \cdots & \phi^{(2n-3)} & \cdots & \phi^{(2n-1)} & \cdots & \phi^{(n-1)} \end{vmatrix} \div$$

$$\text{Denom.} = \begin{vmatrix} \phi^{(n-1)} & \cdots & \phi^{(n-2)} & \cdots & \phi \\ \phi^{(n)} & \cdots & \phi^{(n-1)} & \cdots & \phi' \\ \vdots & & \vdots & & \vdots \\ \phi^{(2n-2)} & \cdots & \phi^{(2n-3)} & \cdots & \phi^{(n-1)} \end{vmatrix}.$$

The only thing at all arbitrary consists in the choice of a value for $p'_{n-1,\,n-1} - 2q$ in order that the equation

$$q'' + (p'_{n-1,\,n-1} - 2q)\,q'$$

may be integrated. Parting from the equation of the second order

$$\frac{d^2y}{dx^2} + p'_{n-1,\,n-1}\frac{dy}{dx} + p'_{n-1,\,n}\,y = 0,$$

all the coefficients p'_{ij} up to p_{1k} are determined by the solution of sets of linear equations, the coefficients of which are known functions of the known quantity q.

Assume now that the coefficients of the equation of the n^{th} order, viz.,

$$p_{11},\ p_{12},\ p_{13},\ \ldots\ p_{1n}$$

are simply periodic functions of the first kind, having ω for their common period.

It is known* that such an equation possesses at least one integral which is a periodic function of the second kind, and that the multiplier of this function is a root of the fundamental equation. If we choose, as we may,† y_1 for this integral, say $y_1 = \phi$, where $\phi(x+\omega) = \varepsilon\phi(x)$, then

$$\frac{d\varphi}{dx}, \frac{d^2\varphi}{dx^2} \cdots \frac{d^{n-1}\varphi}{dx^{n-1}}$$

will all have the same multiplier as ϕ, so that the roots of the fundamental equation will all be equal and be equal to ε.

It is easy to see that from the form of the integrals of our equation that $y_1, = \phi$, actually is the integral which is periodic of the second kind. If it were not such an integral it would have to be, as is well known, of the form

$$y_1 = \psi_{11} + x\psi_{12} + x^2\psi_{13} + \ldots + x^{n-1}\psi_{1n},$$

where ψ_{ij} is a periodic function of the second kind having ω for a period and ε for a multiplier. All the derivatives of y_1 would now be of the same form as y_1, and consequently no integral of the equation could be a periodic function of the second kind having ω for period and ε for multiplier. But there must be *one* such integral: therefore it follows obviously that y_1 is this integral.

We have now $$q'' + (p'_{n-1,\,n-1} - 2q)\,q' = 0$$

and $$y_1 = \phi, = e^{-\int q\,dx},$$

therefore $$\frac{1}{\varphi}\frac{d\varphi}{dx} = -q;$$

changing x into $x+\omega$, the left-hand side of this equation remains unaltered, and therefore q is a periodic function of the first kind having ω for period. Writing then

$$p'_{n-1,\,n-1} - 2q = F(x)$$

it follows that $F(x)$ must also be a periodic function of the first kind, having ω for period, since, from the manner of formation of $p'_{n-1,\,n-1}$ this quantity is such a function. Assuming then that

$$p_{11},\ p_{12} \cdots p_{1n}$$

are periodic functions of the first kind having ω for period, and forming by the method above indicated the equation

a. $$q'' + (p'_{n-1,\,n-1} - 2q)\,q' = 0$$

it is only necessary to assume for

$$p'_{n-1,\,n-1} - 2q$$

* G. Floquet: Théorie des équations différentielles linéaires à coefficients périodiques. Annales de l'École Normale Supérieure, 1883.

† *Vide* Floquet's memoir cited above, §8.

a periodic function of the first kind having the same period, and then determine q, and consequently $p'_{n-1,\,n-1}$ from α, and $p'_{n-1,\,n}$ from

$$q^2 - q' - p'_{n-1,\,n-1}\ q + p'_{n-1,\,n} = 0.$$

The remaining coefficients p'_{ij} and finally p_{1k} will then be determined as above. It is easy to see that the multiplier ε is equal to unity. We have in fact

$$\phi(x) = e^{-\int q(x)\,dx}$$

$$\phi(x+\omega) = \varepsilon\phi(x) = e^{-\int q(x)\,dx}$$

since $$q(x+\omega) = q(x);$$

the right-hand members of these two equations being equal, the left-hand members are also equal, and consequently $\varepsilon = 1$.

It is easy to see that a similar result holds in the case where the coefficients $p_{11}, p_{12} \ldots p_{1n}$ are doubly-periodic coefficients of the first kind having ω and ω' for periods.

If $\phi(x)$ is an integral, doubly-periodic of the second kind, and having ε and ε' for multipliers, *i. e.* if

$$\phi(x+\omega) = \varepsilon\phi(x),$$

$$\phi(x+\omega') = \varepsilon'\phi(x);$$

we have $$\phi(x) = e^{-\int q\,dx},$$

where $q(x+\omega) = q(x+\omega') = q(x)$. And so just as before it is seen that $\varepsilon = \varepsilon' = 1$.

Differentiating twice the first of equations 3, and subtracting the result from the third, we have, since

$$p'_{11}y^{(n)} + p'_{12}y^{(n-1)} + \ldots + p'_{1n}y' = 0,$$

9. $$p''_{11}y^{(n-1)} + p''_{12}y^{(n-2)} + \ldots + p''_{1n}y = 0.$$

This has for integrals $$\phi,\ \frac{d\varphi}{dx},\ \frac{d^2\varphi}{dx^2} \ldots \frac{d^{n-3}\varphi}{dx^{n-3}},\ z$$

where z has to be determined; this of course is easily done, since, knowing $n-2$ of the integrals of an equation of the $(n-1)^{\text{st}}$ order we can reduce the equation to one of the first order, and so determine the remaining integral.

Again, differentiating the first of equations 3 three times, and subtracting from the fourth, we have, after some simple reductions,

10. $$p'''_{11}y^{(n-1)} + p'''_{12}y^{(n-2)} + \ldots + p'''_{1n}y = 0,$$

of which we know $n-3$ integrals, viz.

$$\phi,\ \frac{d\varphi}{dx},\ \frac{d^2\varphi}{dx^2}, \ldots \frac{d^{n-4}\varphi}{dx^{n-4}},$$

and which can therefore be reduced to an equation of the second order, which on being integrated will enable us to find the remaining two integrals of the equation in question.

Similarly we find the equation

$$p_{11}^{\mathrm{iv}}y^{(n-1)}+p_{12}^{\mathrm{iv}}y^{(n-2)}\ldots+p_{1n}^{\mathrm{iv}}y=0$$

of which

$$\phi,\ \frac{d\varphi}{dx},\ \frac{d^2\varphi}{dx^2}\cdots\frac{d^{n-5}\varphi}{dx^{n-5}}$$

are integrals, and which can consequently be reduced to an equation of the third order; and

$$p_{11}^{\mathrm{v}}y^{(n-1)}+p_{12}^{\mathrm{v}}y^{(n-2)}+\ldots+p_{1n}^{\mathrm{v}}y=0$$

of which

$$\phi,\ \frac{d\varphi}{dx},\ \frac{d^2\varphi}{dx^2}\cdots\frac{d^{n-6}\varphi}{dx^{n-6}},$$

and which can consequently be reduced to an equation of the fourth order.

Finally we come to the equation

$$p_{11}^{(n-1)}y^{(n-1)}+p_{12}^{(n-1)}y^{(n-2)}+\ldots+p_{1n}^{(n-1)}y=0,$$

of which ϕ is the only known integral, and which can therefore be only reduced to an equation of the order $n-2$. The order of these equations may in turn be reduced by unity, and so new sets of equations will arise. The consideration of these equations will, however, be deferred until later.

II.

Equations whose integrals are

$$\begin{array}{cccc}
y_1 & y_2 & \cdots & y_a \\
\frac{dy_1}{dx} & \frac{dy_2}{dx} & \cdots & \frac{dy_a}{dx} \\
\frac{d^2y_1}{dx^2} & \frac{d^2y_2}{dx^2} & \cdots & \frac{d^2y_a}{dx^2} \\
\vdots & \vdots & & \vdots \\
\frac{d^{\lambda_1-1}y_1}{dx^{\lambda_1}} & \frac{d^{\lambda_2-1}y_2}{dx^{\lambda_2}} & \cdots & \frac{d^{\lambda_a-1}y_a}{dx^{\lambda_a}}
\end{array}$$

where $\lambda_1+\lambda_2+\ldots\lambda_a=n$.

We will suppose the quantities $\lambda_1,\lambda_2\ldots\lambda_a$ arranged in descending order of magnitude, or, at least, in case any of them are equal, in such an order that no one shall be greater than any preceding one. The differential equation is

1. $$\frac{d^ny}{dx^n}+p_{11}\frac{dy^{n-1}}{dx^{n-1}}+p_{12}\frac{d^{n-2}y}{dx^{n-2}}+\ldots+p_{1n}y=0.$$

Taking the integral y_1 we have the system

$$
\Lambda_1 \quad
\begin{aligned}
&\frac{d^n y_1}{dx^n} + p_{11}\frac{d^{n-1}y_1}{dx^{n-1}} + p_{12}\frac{d^{n-2}y_1}{dx^{n-1}} + \ldots + p_{1n}y_1 = 0,\\
&\frac{d^{n+1}y_1}{dx^{n+1}} + p_{11}\frac{d^n y_1}{dx^n} + p_{12}\frac{d^{n-1}y}{dx^{n-1}} + \ldots + p_{1n}\frac{dy_1}{dx} = 0,\\
&\ldots\ldots\ldots\ldots\ldots\ldots\ldots\ldots\\
&\frac{d^{n+\lambda_1-1}y_1}{dx^{n+\lambda_1-1}} + p_{11}\frac{d^{n+\lambda_1-2}y_1}{dx^{n+\lambda_1-2}} + p_{12}\frac{d^{n+\lambda_1-3}y_1}{dx^{n+\lambda_1-3}} + \ldots + p_{1n}\frac{d^{\lambda_1-1}y_1}{dx^{\lambda_1-1}} = 0.
\end{aligned}
$$

From the first and second of these we obtain as before

$$(1_1) \qquad p'_{11}\frac{d^{n-1}y_1}{dx^{n-1}} + p'_{12}\frac{d^{n-2}y_1}{dx^{n-2}} + \ldots + p'_{1n}y = 0,$$

from the second and third we have

$$(2_1) \qquad p'_{11}\frac{d^n y_1}{dx^n} + p'_{12}\frac{d^{n-1}y_1}{dx^{n-1}} + \ldots + p'_{1n}\frac{dy}{dx} = 0,$$

from the third and fourth

$$(3_1) \qquad p'_{11}\frac{d^{n+1}y_1}{dx^{n+1}} + p'_{12}\frac{d^n y}{dx^n} + \ldots + p'_{1n}\frac{d^2y}{dx^2} = 0,$$

and finally

$$p'_{11}\frac{d^{n+\lambda_1-2}y_1}{dx^{n+\lambda_1-2}} + p'_{12}\frac{d^{n+\lambda_1-3}y_1}{dx^{n+\lambda_1-3}} + \ldots + p'_{1n}\frac{d^{\lambda_1-2}y_1}{dx^{\lambda_1-2}} = 0.$$

These last equations are merely what the first becomes on replacing y_1 by $\frac{dy_1}{dx}, \frac{d^2y_1}{dx^2} \ldots \frac{d^{\lambda_1-2}y_1}{dx^{\lambda_1-2}}$. These quantities therefore are all integrals of the equation

$$(1'_1) \qquad p'_{11}y^{(n-1)} + p'_{12}y^{(n-2)} + \ldots + p'_{1n}y = 0.$$

A set of equations similar to Λ_1 are to be formed for each of the integrals $y_2, y_3 \ldots y_\alpha$; call the general set Λ_ι. this is

$$
\Lambda_\iota \quad
\begin{aligned}
&\frac{d^n y_\iota}{dx^n} + p_{11}\frac{d^{n-1}y_\iota}{dx^{n-1}} + p_{12}\frac{d^{n-2}y_\iota}{dx^{n-2}} + \ldots + p_{1n}y_\iota = 0.\\
&\frac{d^{n+1}y_\iota}{dx^{n+1}} + p_{11}\frac{d^n y_\iota}{dx^n} + p_{12}\frac{d^{n-1}y_\iota}{dx^{n-1}} + \ldots + p_{1n}\frac{dy_\iota}{dx} = 0.\\
&\ldots\ldots\ldots\ldots\ldots\ldots\ldots\ldots\\
&\frac{d^{n+\lambda_\iota-1}y_\iota}{dx^{n+\lambda_\iota-1}} + p_{11}\frac{d^{n+\lambda_\iota-2}y_\iota}{dx^{n+\lambda_\iota-2}} + p_{12}\frac{d^{n+\lambda_\iota-3}y_\iota}{dx^{n+\lambda_\iota-3}} + \ldots + p_{1n}\frac{d^{\lambda_\iota-1}y_\iota}{dx^{\lambda_\iota-1}} = 0.
\end{aligned}
$$

From these, as before, we form the system

$$
\begin{aligned}
&(1_\iota) \qquad p'_{11}y_\iota^{n-1} + p'_{12}y_\iota^{n-2} + \ldots + p'_{1n}y_\iota = 0,\\
&(2_\iota) \qquad p'_{11}y_\iota^{n} + p'_{12}y_\iota^{n-1} + \ldots + p'_{1n}y'_\iota = 0,\\
&\ldots\ldots\ldots\ldots\ldots\ldots\ldots\ldots\\
&\qquad p'_{11}y_\iota^{n+\lambda_\iota-2} + p'_{12}y_\iota^{n+\lambda_\iota-3} + \ldots + p'_{1n}y_\iota^{\lambda_\iota-2} = 0.
\end{aligned}
$$

Referring now to equation (1_1) it is clear from equation (1_ι) that $y_2, y_3 \ldots y_a$ are all integrals, and, since $\lambda_2, \lambda_3 \ldots \lambda_a$ are none of them greater than λ_1 and no element in this series is greater than any preceding one, it also follows that the derivatives of $y_2, y_3 \ldots y_a$ up to the orders $\lambda_2 - 2, \lambda_3 - 2 \ldots \lambda_a - 2$ are the remaining integrals. That is, the complete set of integrals of $(1_1')$ is

$$y_1, \frac{dy_1}{dx}, \frac{d^2y_1}{dx^2}, \ldots \frac{d^{\lambda_1-2}y_1}{dx^{\lambda_1-2}},$$
$$y_2, \frac{dy_2}{dx}, \frac{d^2y_2}{dx^2}, \ldots \frac{d^{\lambda_2-2}y_2}{dx^{\lambda_1-1}},$$
$$\cdots\cdots\cdots\cdots$$
$$y_a, \frac{dy_a}{dx}, \frac{d^2y_a}{dx^2}, \ldots \frac{d^{\lambda_a-2}y_a}{dx^{\lambda_a-1}},$$

or taking the general set of integrals,

$$(P_\iota)\quad \begin{array}{lllll} (1_\iota) & p_{11}'y_\iota^{n-1} & + p_{12}'y_\iota^{n-2} & + \ldots + p_{1n}'y_\iota & = 0, \\ (2_\iota) & p_{11}'y_\iota^{n} & + p_{12}'y_\iota^{n-1} & + \ldots + p_{1n}'y_\iota' & = 0, \\ \cdots & \cdots & \cdots & \cdots & \cdots \\ & p_{11}'y_\iota^{n+\lambda_\iota-2} & + p_{12}'y_\iota^{n+\lambda_\iota-3} & + \ldots + p_{1n}'y_\iota^{\lambda_\iota-2} & = 0. \end{array}$$

The first of these is also satisfied by

$$y_1, y_2 \ldots y_{\iota-1}, y_{\iota+1} \ldots y_a.$$

It is clear from what precedes that the following also satisfy (1_ι),

$$\frac{dy_1}{dx}, \frac{d^2y_1}{dx^2} \ldots \frac{d^{\lambda_1-2}y_1}{dx^{\lambda_1-2}},$$
$$\frac{dy_2}{dx}, \frac{d^2y_2}{dx^2} \ldots \frac{d^{\lambda_2-2}y_2}{dx^{\lambda_2-2}},$$
$$\cdots\cdots\cdots\cdots$$
$$\frac{dy_{\iota-1}}{dx}, \frac{d^2y_{\iota-1}}{dx^2} \ldots \frac{d^{\lambda_{\iota-1}-2}y}{dx^{\lambda_{\iota-1}-2}}.$$

The quantities

$$\frac{dy_{\iota+1}}{dx}, \frac{d^2y_{\iota+1}}{dx^2} \ldots \frac{d^{\lambda_{\iota+1}-2}y_{\iota+1}}{dx^{\lambda_{\iota+1}-1}},$$
$$\cdots\cdots\cdots\cdots$$
$$\frac{dy_a}{dx}, \frac{d^2y_a}{dx^2} \ldots \frac{d^{\lambda_a-2}y_a}{dx^{\lambda_a-1}},$$

will obviously also satisfy (1_ι). We have thus the same complete set of integrals as before obtained, viz. the known integrals of the derived equation

$$\frac{d^{n-1}y}{dx^{n-1}} + p_{22}^{(1)}\frac{d^{n-2}y}{dx^{n-2}} + \ldots + p_{2n}^{(1)}y = 0,$$

where $p^{(1)}_{2i} = p'_{1i} \div p'_{11}$, are

$$y_1,\ \frac{dy_1}{dx},\ \frac{d^2y_1}{dx^2},\ \ldots\ \frac{d^{\lambda_1-2}y_1}{dx^{\lambda_1-2}},$$
$$y_2,\ \frac{dy_2}{dx},\ \frac{d^2y_2}{dx^2},\ \ldots\ \frac{d^{\lambda_2-2}y_2}{dx^{\lambda_2-2}},$$
$$y_a,\ \frac{dy_a}{dx},\ \frac{d^2y_a}{dx},\ \ldots\ \frac{d^{\lambda_a-2}y_a}{dx^{\lambda_a-2}}.$$

We have in all these $\lambda_1 + \lambda_2 + \ldots + \lambda_a - a,\ = n - a$, known integrals, *i. e.* there remain $a - 1$ integrals to be determined. It will of course be noticed that if for any value of i $\lambda_i = 1$ the corresponding function y_i is not an integral of the derived equation. Referring now to the system (P_i): writing as above

$$\frac{p'_{12}}{p'_{11}} = p^{(1)}_{22},\ \frac{p'_{13}}{p'_{11}} = p^{(1)}_{23},\ \text{etc.}$$

this is

$$(P_i)_1 \qquad \begin{array}{llll} y_i^{(n-1)} & + p^{(1)}_{22} y_i^{(n-2)} & + \ldots + p^{(1)}_{2n} y_i & = 0, \\ y_i^{(n)} & + p^{(1)}_{22} y_i^{(n-1)} & + \ldots + p^{(1)}_{2n} y'_i & = 0, \\ \ldots & \ldots & \ldots & \ldots \\ y_i^{(n+\lambda_i-2)} & + p^{(1)}_{22} y_i^{(n+\lambda_i-3)} & + \ldots + p^{(1)}_{2n} y_i^{(\lambda_i-2)} & = 0. \end{array}$$

We have now as is easily seen, for the integrals of the equation

$$P^{(1)} \qquad p^{(1)\prime}_{22} y^{(n-2)} + p^{(1)\prime}_{23} y^{(n-3)} + \ldots + p^{(1)\prime}_{2n} y = 0,$$

the following system,

$$y_1\ \frac{dy_1}{dx}\ \frac{d^2y_1}{dx^2}\ \ldots\ \frac{d^{\lambda_1-3}y_1}{dx^{\lambda_1-3}},$$
$$y_2\ \frac{dy_2}{dx}\ \frac{d^2y_2}{dx^2}\ \ldots\ \frac{d^{\lambda_2-3}y_2}{dx^{\lambda_2-3}},$$
$$\ldots\ldots\ldots\ldots\ldots$$
$$y_i\ \frac{dy_i}{dx}\ \frac{d^2y_i}{dx^2}\ \ldots\ \frac{d^{\lambda_i-3}y_i}{dx^{\lambda_i-3}},$$
$$\ldots\ldots\ldots\ldots\ldots$$
$$y_a\ \frac{dy_a}{dx}\ \frac{d^2y_a}{dx^2}\ \ldots\ \frac{d^{\lambda_a-3}y_a}{dx^{\lambda_a-3}}.$$

We have thus $n - 2a$ integrals of the equation of order $n - 2$, leaving $2(a - 1)$ integrals to be determined. Again we can form a new equation of the degree $n - 3$ of which we know $n - 3a$ integrals and $3(a - 1)$ to be determined. This process will be continued until $n - ka$ becomes either equal to or less than a. If $\lambda_1 = \lambda_2 \ldots \lambda_a$ then we shall have $n - ka = a$ as the number of integrals of the equation which cannot again be reduced. These will be $y_1, y_2 \ldots y_a$. If

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the λ's are all different, then $n-k\alpha$ will be equal to unity, and y_1 will be the only known integral. The number of known integrals will of course depend upon the number of λ's which are equal to λ_1. It is easy to find the order of the equation which can be no further reduced. We know that λ_1 is the greatest of the λ's, at least it is not less than any succeeding λ. By continuing the process of reduction long enough we must come to an equation of which

$$y_1 \text{ and } \frac{dy_1}{dx}$$

are integrals; other known integrals such as y_2, $\frac{dy_2}{dx}$, y_3, $\frac{dy_3}{dx}$, etc., may or may not exist according to the relative values of the λ's. The equation having y_1 and $\frac{dy_1}{dx}$ for integrals is readily found to be

$$\frac{d^{n-\lambda_1+2}y}{dx^{n-\lambda_1+2}} + q_1\frac{d^{n-\lambda_1+1}y}{dx^{n-\lambda_1+1}} + \ldots + q_{n-\lambda_1+2}\,y = 0.$$

The coefficients $q_1, q_2 \ldots q_{n-\lambda_1+2}$ are of course derived by known processes from $p_{11}, p_{12}, \ldots p_{1n}$. As we know at least two integrals of this equation, viz. y_1 and $\frac{dy_1}{dx}$, we can reduce it one degree more, giving

$$\frac{d^{n-\lambda_1+1}y}{dx^{n-\lambda_1+1}} + r_1\frac{d^{n-\lambda_1}y}{dx^{n-\lambda_1}} + \ldots + r_{n-\lambda_1+1}\,y = 0.$$

The degree of this equation can no longer be reduced, at least in the sense in which the word 'reduction' has been heretofore employed. We have then $n-k=n-\lambda_1+1$, or $k=\lambda_1-1$, *i. e.* $n-\lambda_1+1$ is the degree of the equation of the lowest order which can be obtained by the above process of reduction. Of course if we make the substitution

$$y = y_1\int z\,dx$$

we find an equation in z of order $n-\lambda_1$, *none* of whose integrals are known.

Suppose now that the coefficients

$$p_{11}, p_{12}, \ldots p_{1n}$$

of the original equation are uniform simply periodic functions of the first kind having ω for period. Then one or more of the integrals will be uniform periodic functions of the second kind having for multipliers the different roots of the fundamental equation. If the roots of the fundamental equation are all simple, then we have as fundamental integrals of the given differential equation n uniform periodic functions of the second kind each possessing its own multiplier, which is different from all the others.

Suppose $\Delta = 0$ to be the fundamental equation, and $\varepsilon_1, \varepsilon_2, \ldots \varepsilon_t$ its distinct roots, and $\nu_1, \nu_2, \ldots \nu_t$ the orders parting from which the minors of Δ cease to be all zero for $\varepsilon = \varepsilon_i (i = 1, 2, \ldots t)$; then it is known* that the differential equation admits $\nu_1 + \nu_2 + \ldots + \nu_t$ distinct integrals, which are uniform periodic functions of the second kind. Again, it is known that, in order that the differential equation may admit of n integrals which are periodic functions of the second kind, it is necessary and sufficient that each root of the fundamental equation $\Delta = 0$ shall annul all the minors of Δ up to the order equal to the degree of multiplicity of the root. Finally, those integrals of the equation which are not periodic functions of the second kind are of the form of polynomials in x, the coefficients of the different powers of x being periodic functions of the second kind, all having the same multiplier. In the case in hand we have the differential equation

$$\frac{d^n y}{dx^n} + p_{11}\frac{d^{n-1}y}{dx^{n-1}} + p_{12}\frac{d^{n-2}y}{dx^{n-2}} + \ldots + p_{1n}y = 0,$$

the coefficients of which are uniform periodic functions of the first kind, having ω for period. The integrals of the equation are

$$\begin{array}{l} y_1, \frac{dy_1}{dx}, \frac{d^2y_1}{dx^2}, \ldots \frac{d^{\lambda_1-1}y_1}{dx^{\lambda_1-1}}, \\ y_2, \frac{dy_2}{dx}, \frac{d^2y_2}{dx^2}, \ldots \frac{d^{\lambda_2-1}y_2}{dx^{\lambda_2-1}}, \\ \ldots\ldots\ldots\ldots\ldots \\ y_\alpha, \frac{dy_\alpha}{dx}, \frac{d^2y_\alpha}{dx^2}, \ldots \frac{d^{\lambda_\alpha-1}y_\alpha}{dx^{\lambda_\alpha-1}}. \end{array}$$

Since the derivatives of $y_1, y_2, \ldots y_\alpha$ must be functions of the same kind as the y's themselves, it is clear that the fundamental equation cannot have more than α distinct roots. If the equation have α distinct roots $\varepsilon_1, \varepsilon_2, \ldots \varepsilon_\alpha$, these will of course be the multipliers of $y_1, y_2, \ldots y_\alpha$, and consequently of their derivatives; *i. e.* in this case the equation has n distinct integrals which are periodic functions of the second kind. The degrees of multiplicity of the roots are respectively $\lambda_1, \lambda_2, \ldots \lambda_\alpha$,

and obvious, ε_1 makes each minor of Δ up to the order λ_1 vanish, ε_2 makes each minor vanish up to the order λ_2, and so on. The case when the fundamental equation has fewer than α roots is included in the above general theorem; as an illustration, however, suppose that the equation has roots

$$\varepsilon_1, \varepsilon_2, \ldots \varepsilon_\beta, \qquad \beta < \alpha$$

---

* Floquet: Annales de l'École Normale, February, 1883, page 48.

belonging to the functions

$$y_1,\ y_2,\ \dots\ y_\beta.$$

Then it is obvious that we have

$$\lambda_1 + \lambda_2 + \dots + \lambda_\beta$$

integrals, which are periodic functions of the second kind.

Similar remarks apply of course in the case of coefficients which are doubly periodic functions of the first kind, except that in this case we have two fundamental equations corresponding to the two sets of multipliers.

Resume now the original system of equations of the n^{th} order, satisfied by y_ι and its $\lambda_\iota - 1$ derivatives, viz.

$$\begin{aligned}
&\frac{d^n y_\iota}{dx^n} + p_{11}\frac{d^{n-1}y_\iota}{dx^{n-1}} + \dots + p_{1n}y = 0,\\
&\frac{d^{n+1}y_\iota}{dx^{n+1}} + p_{11}\frac{d^n y_\iota}{dx^n} + \dots + p_{1n}\frac{dy}{dx} = 0,\\
&\dots\dots\dots\dots\dots\dots\dots\dots\\
&\frac{d^{n+\lambda_\iota-1}y_\iota}{dx^{n+\lambda_\iota-1}} + p_{11}\frac{d^{n+\lambda_\iota-2}y_\iota}{dx^{n+\lambda_\iota-2}} + \dots + p_{1n}\frac{d^{\lambda_\iota-1}y}{dx^{\lambda_\iota-1}} = 0.
\end{aligned}$$

From the first and third of these we derive after easy reductions the equation

$$p''_{11}y_\iota^{(n-1)} + p''_{12}y_\iota^{(n-2)} + \dots + p''_{1n}y_\iota = 0,$$

or, writing

$$\frac{p''_{12}}{p''_{11}} = p^{(2)}_{22},\ \frac{p''_{13}}{p''_{11}} = p^{(2)}_{23}\dots,$$

$$y_\iota^{(n-1)} + p^{(2)}_{22}y_\iota^{(n-2)} + \dots + p^{(2)}_{2n}y = 0;$$

similarly

$$(P_\iota),\qquad \begin{aligned}
&y_\iota^{(n)} + p^{(2)}_{22}y_\iota^{(n+1)} + \dots + p^{(2)}_{2n}y' = 0,\\
&y_\iota^{(n+1)} + p^{(2)}_{22}y_\iota^{(n)} + \dots + p^{(2)}_{2n}y_\iota'' = 0,\\
&\dots\dots\dots\dots\dots\dots\dots\dots\\
&y_\iota^{(n+\lambda_\iota-3)} + p^{(2)}_{22}y_\iota^{(n+\lambda_\iota-4)} + \dots + p^{(2)}_{2n}y^{(\lambda_\iota-3)} = 0.
\end{aligned}$$

The integrals of the equation

$$P^{(2)}\qquad \frac{d^{n-1}y}{dx^{n-1}} + p^{(2)}_{22}\frac{d^{n-2}y}{dx^{n-2}} + \dots + p^{(2)}_{2n}y = 0$$

are easily seen to be

$$\begin{aligned}
&y_1,\ \frac{dy_1}{dx},\ \frac{d^2y_1}{dx^2}\dots\frac{d^{\lambda_1-2}y_1}{dx^{\lambda_1-2}},\\
&y_2,\ \frac{dy_2}{dx},\ \frac{d^2y_2}{dx^2}\dots\frac{d^{\lambda_2-2}y_2}{dx^{\lambda_2-2}},\\
&\dots\dots\dots\dots\\
&y_\alpha,\ \frac{dy_\alpha}{dx},\ \frac{d^2y_\alpha}{dx^2}\dots\frac{d^{\lambda_\alpha-2}y_\alpha}{dx^{\lambda_\alpha-2}}.
\end{aligned}$$

We have thus $n-2\alpha$ integrals of the equation of the order $n-1$, and these are the same as the known $n-2\alpha$ integrals of the equation

$$P^{(1)'} \qquad p_{22}^{(1)'}y^{(n-2)}+p_{23}^{(1)'}y^{(n-3)}+\ldots+p_{2n}^{(1)'}y=0,$$

or say

$$y^{(n-2)}+p_{23}^{(1)}y^{(n-3)}+\ldots+p_{2n}^{(1)}y=0.$$

There remain now $2\alpha-1$ unknown integrals of $P^{(2)}$ and $2(\alpha-1)$ unknown integrals of $P^{(1)'}$.

From $(P_\iota)_2$ we can form again the equation

$$p_{22}^{(2)'}y^{(n-2)}+p_{23}^{(2)'}y^{(n-3)}+\ldots+p_{2n}^{(2)'}y=0,$$

or say

$$y^{(n-2)}+p_{23}^{(2)}y^{(n-3)}+\ldots+p_{2n}^{(2)}y=0,$$

the integrals which are easily seen to be

$$\begin{array}{l} y_1,\ \frac{dy_1}{dx},\ \frac{d^2y_1}{dx^2}\ldots\frac{d^{\lambda_1-4}y_1}{dx^{\lambda_1-4}}, \\ y_2,\ \frac{dy_2}{dx},\ \frac{d^2y_2}{dx^2}\ldots\frac{d^{\lambda_2-4}y_2}{dx^{\lambda_2-4}}, \\ \ldots\ldots\ldots\ldots\ldots \\ y_\alpha,\ \frac{dy_\alpha}{dx},\ \frac{d^2y_\alpha}{dx^2}\ldots\frac{d^{\lambda_\alpha-4}y_\alpha}{dx^{\lambda_\alpha-4}}. \end{array}$$

These $n-3\alpha$ integrals are the known integrals of

$$y^{(n-3)}+p_{44}^{(1)}y^{(n-4)}+\ldots+p_{4n}^{(1)}y=0$$

mentioned above. There remain now $3\alpha-2$ unknown integrals of the new equation of order $n-2$ and $3(\alpha-1)$ integrals of the equation of order $n-3$.

This process can of course be continued just as in the earlier case, where the coefficients were $p_{ij}^{(1)}$; it is hardly worth while, however, saying anything more about it.

We may go back, however, to the original system once more, and by combination of the first and fourth, the second and fifth, etc., and making some easy reductions, obtain

$$\begin{array}{ll} y_\iota^{(n-1)}+p_{22}^{(3)}y^{(n-2)}+\ldots+p_{2n}^{(3)}y &=0, \\ y_\iota^{(n-1)}+p_{22}^{(3)}y^{(n-2)}+\ldots+p_{2n}^{(3)}y' &=0, \\ \ldots\ldots\ldots\ldots\ldots & \\ y_\iota^{(n+\lambda_\iota-4)}+p_{22}^{(3)}y^{(n+\lambda_\iota-5)}+\ldots+p_{2n}^{(3)}y^{(\lambda_\iota-4)} &=0, \end{array}$$

where

$$p_{22}^{(3)}=p_{12}'''\div p_{11}''',\ p_{23}^{(3)}=p_{13}'''\div p_{11}''',\ \text{etc.}$$

The integrals of

$$\frac{d^{n-1}y}{dx^{n-1}}+p_{22}^{(3)}\frac{d^{n-2}y}{dx^{n-2}}+\ldots+p_{2n}^{(3)}y=0$$

are obviously

$$
\begin{array}{l}
y_1, \ \dfrac{dy_1}{dx}, \ \dfrac{d^2y_1}{dx^2}, \ldots \dfrac{d^{\lambda_1-4}y_1}{dx^{\lambda_1-4}}, \\
y_2, \ \dfrac{dy_2}{dx}, \ \dfrac{d^2y_2}{dx^2}, \ldots \dfrac{d^{\lambda_2-4}y_2}{dx^{\lambda_2-4}}, \\
\cdots\cdots\cdots\cdots\cdots \\
y_a, \ \dfrac{dy_a}{dx}, \ \dfrac{d^2y_a}{dx^2}, \ldots \dfrac{d^{\lambda_a-4}y_a}{dx^{\lambda_a-4}}.
\end{array}
$$

We know thus $n - 3a$ integrals of

$$y^{(n-1)} + p_{22}^{(3)} y^{(n-2)} + \ldots + p_{2n}^{(3)} y = 0,$$

and these are the known integrals of the equations

$$y^{(n-2)} + p_{33}^{(2)} y^{(n-3)} + \ldots + p_{3n}^{(2)} y = 0,$$
$$y^{(n-3)} + p_{44}^{(1)} y^{(n-4)} + \ldots + p_{4n}^{(1)} y = 0.$$

There remain then $3a - 3$ integrals of the equation of order $n - 3$, $3a - 2$ unknown integrals of the equation of order $n - 2$, and $3a - 1$ unknown integrals of the equation of order $n - 1$.

Taking the general case, combine the first of the original set with the $i + 1$st, the second with the $i + 2$nd, etc.; we have as the result

$$y^{(n-1)} + p_{22}^{(i)} y^{(n-2)} + \ldots + p_{2n}^{(i)} y = 0,$$

where

$$p_{22}^{(i)} = \frac{d^ip_{12}}{dx^i} \div \frac{d^ip_{11}}{dx^i}, \quad p_{23}^{(i)} = \frac{d^ip_{13}}{dx^i} \div \frac{d^ip_{11}}{dx^i}, \text{ etc.}$$

The known integrals of this are

$$
\begin{array}{l}
y_1, \ \dfrac{dy_1}{dx}, \ \dfrac{d^2y_1}{dx^2}, \ldots \dfrac{d^{\lambda_1-i-1}y_1}{dx^{\lambda_1-i-1}}, \\
y_2, \ \dfrac{dy_2}{dx}, \ \dfrac{d^2y_2}{dx^2}, \ldots \dfrac{d^{\lambda_2-i-1}y_2}{dx^{\lambda_2-i-1}}, \\
\cdots\cdots\cdots\cdots\cdots \\
y_a, \ \dfrac{dy_a}{dx}, \ \dfrac{d^2y_a}{dx^2}, \ldots \dfrac{d^{\lambda_a-i-1}y_a}{dx^{\lambda_a-i-1}}.
\end{array}
$$

These are the known integrals now of the set of equations, including the $p^{(i)}$ equation,

$$
\begin{array}{llll}
y^{(n-1)} + p_{22}^{(i)} y^{(n-2)} & + \ldots + p_{2n}^{(i)} y & = 0, \\
y^{(n-2)} + p_{33}^{(i-1)} y^{(n-3)} & + \ldots + p_{3n}^{(i-1)} y & = 0, \\
y^{(n-3)} + p_{44}^{(i-2)} y^{(n-4)} & + \ldots + p_{4n}^{(i-2)} y & = 0, \\
\cdots\cdots\cdots\cdots & \cdots\cdots\cdots & \\
y^{(n-i)} + p_{i+1,\,i+1}^{(1)} y^{(n-i-1)} & + \ldots + p_{i+1,\,n}^{(1)} y & = 0.
\end{array}
$$

The numbers of unknown integrals of these equations are respectively

$$ia - 1, \ ia - 2, \ ia - 3, \ldots ia - i.$$

In résumé: we have the two equations

$$(P) = \frac{d^n y}{dx^n} + p_{11}\frac{d^{n-1}y}{dx^{n-1}} + p_{12}\frac{d^{n-2}y}{dx^{n-2}} + \ldots + p_{1n}y = 0,$$

$$\frac{d^{n-1}y}{dx^{n-1}} + p_{22}^{(1)}\frac{d^{n-2}y}{dx^{n-2}} + p_{23}^{(1)}\frac{d^{n-3}y}{dx^{n-3}} + \ldots + p_{2n}y = 0,$$

having the common $n - \alpha$ integrals

$$y_1, \frac{dy_1}{dx} \ldots \frac{d^{\lambda_1 - 2}y_1}{dx^{\lambda_1 - 2}},$$

$$\ldots\ldots\ldots$$

$$y_\alpha, \frac{dy_\alpha}{dx} \ldots \frac{d^{\lambda_\alpha - 2}y_\alpha}{dx^{\lambda_\alpha - 2}};$$

the three equations

$$(P) = 0,$$

$$\frac{d^{n-1}y}{dx^{n-1}} + p_{22}^{(2)}\frac{d^{n-2}y}{dx^{n-2}} + \ldots + p_{2n}^{(2)}y = 0,$$

$$\frac{d^{n-2}y}{dx^{n-2}} + p_{33}^{(1)}\frac{d^{n-3}y}{dx^{n-3}} + \ldots + p_{3n}^{(1)}y = 0,$$

having the $n - 2\alpha$ common integrals

$$y_1, \frac{dy_1}{dx} \ldots \frac{d^{\lambda_1 - 3}y_1}{dx^{\lambda_1 - 3}},$$

$$\ldots\ldots\ldots$$

$$y_\alpha, \frac{dy_\alpha}{dx} \ldots \frac{d^{\lambda_\alpha - 3}y_\alpha}{dx^{\lambda_\alpha - 3}},$$

and the $i + 1$ equations

$$(P) = 0$$

$$\frac{d^{n-1}y}{dx^{n-1}} + p_{22}^{(i)}\frac{d^{n-2}y}{dx^{n-2}} + \ldots + p_{2n}^{(i)}y = 0,$$

$$\frac{d^{n-2}y}{dx^{n-2}} + p_{33}^{(i-1)}\frac{d^{n-3}y}{dx^{n-3}} + \ldots + p_{3n}^{(i-1)}y = 0,$$

$$\ldots\ldots\ldots\ldots\ldots\ldots\ldots\ldots\ldots$$

$$\frac{d^{n-i}y}{dx^{n-i}} + p_{i+1,i+1}^{(1)}\frac{d^{n-i-1}y}{dx^{n-i-1}} + \ldots + p_{i+1,n}^{(1)}y = 0,$$

having the $n - i\alpha$ common integrals

$$y_1, \frac{dy_1}{dx} \ldots \frac{d^{\lambda_1 - i - 1}y_1}{dx^{\lambda_1 - i - 1}},$$

$$\ldots\ldots\ldots$$

$$y_\alpha, \frac{dy_\alpha}{dx} \ldots \frac{d^{\lambda_\alpha - i - 1}y_\alpha}{dx^{\lambda_\alpha - i - 1}}.$$

From each of these systems we may deduce an equation satisfied only by the common integrals, *i. e.* we would have, including the given equation, a new system of equations of the respective degrees $n, n-\alpha, n-2\alpha, \ldots n-i\alpha \ldots$ satisfied only by the corresponding group of common integrals. The properties of each of the new equations will of course be exactly the same as those of the original equation.

In all that precedes it has been tacitly assumed, unless statement is made to the contrary, that all the functions $y_1, y_2, \ldots y_\alpha$, and consequently their derivatives, are uniform functions; further, if a_ι denote a singular point of the function, the function will remain finite at the point a_ι on being multiplied by a determinate finite power of $(x-a_\iota)$. The coefficients $p_{11}, p_{12}, \ldots p_{1n}$ are therefore of the form

$$\frac{P_{11}}{x-a_\iota}, \frac{P_{12}}{(x-a_\iota)^2}, \ldots \frac{P_{1n}}{(x-a_\iota)^n}$$

where

$$P_{1k} = \frac{F_{k(\rho-1)}(x)(x-a_\iota)^k}{\psi^k}$$

ρ denoting the total number of singular points; $F_m(x)$ an integral function of x at most of the degree m; and $\psi = (x-a_1)(x-a_2)\ldots(x-a_\rho)$.*

It is of course perfectly obvious that the coefficients of the various derived equations are of the same form as those of the given equation, formed as they are merely by differentiation, multiplication and division. This being the case it follows that the unknown integrals of these derived equations are uniform functions of the same character as the known integrals

$$\begin{array}{l} y_1, \frac{dy_1}{dx}, \frac{d^2y_1}{dx^2}, \ldots \frac{d^{k_1}y_1}{dx^{k_1}}, \\ y_2, \frac{dy_2}{dx}, \frac{d^2y_2}{dx^2}, \ldots \frac{d^{k_2}y_2}{dx^{k_2}}, \\ \ldots\ldots\ldots\ldots \\ y_\alpha, \frac{dy_\alpha}{dx}, \frac{d^2y_\alpha}{dx^2}, \ldots \frac{d^{k_\alpha}y_\alpha}{dx^{k_\alpha}}. \end{array}$$

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* Crelle, Vol. LXVI. Fuchs: Zur Theorie der linearen Differentialgleichungen mit veränderlichen Coefficienten.

On Perpetuants, with Applications to the Theory of Finite Quantics.

By J. Hammond.

1. The following formulæ are taken from two memoirs, both published in the *American Journal of Mathematics*, viz.:

Sylvester: On Subinvariants, *i. e.* Semi-Invariants to Binary Quantics of an Unlimited Order. (§4. Perpetuants, Vol. V, pp. 105–118.)

Cayley: A Memoir on Seminvariants (Vol. VII, pp. 1–25).

$$(2)=\frac{x^2}{\mathbf{2}},\qquad (3)=\frac{x^3}{\mathbf{2}\cdot\mathbf{3}},\qquad (4)=\frac{x^7}{\mathbf{2}\cdot\mathbf{3}\cdot\mathbf{4}},\quad (5)=\frac{x^{15}}{\mathbf{2}\cdot\mathbf{3}\cdot\mathbf{4}\cdot\mathbf{5}},$$

$$(2,\,2)=\frac{x^4}{\mathbf{2}\cdot\mathbf{4}},\ (3,\,2)=\frac{x^5}{\mathbf{2}\cdot\mathbf{2}\cdot\mathbf{3}},\ (4,\,2)=\frac{x^9}{\mathbf{2}\cdot\mathbf{2}\cdot\mathbf{3}\cdot\mathbf{4}}$$

$$(3,\,3)=\frac{x^6+x^{11}}{\mathbf{2}\cdot\mathbf{3}\cdot\mathbf{4}\cdot\mathbf{6}},\ (2,\,2,\,2)=\frac{x^6}{\mathbf{2}\cdot\mathbf{4}\cdot\mathbf{6}}$$

$$(4)+(2,\,2)=\frac{x^4}{\mathbf{2}\cdot\mathbf{3}\cdot\mathbf{4}}$$

$$(5)+(3,\,2)=\frac{x^5}{\mathbf{2}\cdot\mathbf{3}\cdot\mathbf{4}\cdot\mathbf{5}}+S_5$$

$$(6)+(4,\,2)+(3,\,3)+(2,\,2,\,2)=\frac{x^6}{\mathbf{2}\cdot\mathbf{3}\cdot\mathbf{4}\cdot\mathbf{5}\cdot\mathbf{6}}+S_6$$

The word "Perpetuant" with its corresponding idea, and most of the formulæ are due to Prof. Sylvester. The notation, in which the numbers $\mathbf{2},\,\mathbf{3},\ldots,$ printed a little larger and thicker than usual, stand for $1-x^2,\ 1-x^3,\ldots$; and the corrections S_5, S_6, corresponding to Quintic and Sextic Syzygies, belong to Prof. Cayley. Notice that, when m and n are unequal, we have $(m,\,n)=(m)\times(n)$; thus

$$(3,\,2)=(3)\times(2).\quad \text{But, if } (n)=\phi x,\ (n,\,n)=\frac{\varphi x^2+(\varphi x)^2}{2},\ \text{thus}$$

$$(3,\,3)=\tfrac{1}{2}\{\phi x^2+(\phi x)^2\}\ \text{where}\ \phi x=\frac{x^3}{\mathbf{2}\cdot\mathbf{3}}$$

$$=\frac{1}{2}\left(\frac{x^6}{\mathbf{4}\cdot\mathbf{6}}+\frac{x^6}{\mathbf{2}\cdot\mathbf{2}\cdot\mathbf{3}\cdot\mathbf{3}}\right),$$ which, when reduced, gives the value written above.

The value of (3, 3) has, by a mere clerical error, been incorrectly given on p. 14 of Prof. Cayley's Memoir.

2. The syzygies denoted by S_5 and S_6 are irreducible, or ground-syzygies, and will be written $S_5^{(1)}$, $S_6^{(1)}$, for the sake of uniformity in what follows. But among the syzygies of degree seven are found the compounds $(2)\,S_5^{(1)}$, formed by multiplying the Quintic Syzygies by the Quadric Perpetuants, and these compounds are not all of them independent, but are connected with certain syzygies of the sixth degree by $S_7^{(2)}$ distinct relations, or syzygies of the second grade. And similar reasoning applies to the eighth degree. Hence

$$S_7 = S_7^{(1)} + (2)\,S_5^{(1)} - S_7^{(2)}$$
$$S_8 = S_8^{(1)} + (2)\,S_6^{(1)} + (3)\,S_5^{(1)} - S_8^{(2)}.$$

Moreover, in the formula for S_9 there will be compound syzygies of the second grade, in number $(2)\,S_7^{(2)}$, connected with certain of the $S_8^{(2)}$ by means of $S_9^{(3)}$ distinct relations, or syzygies of the third grade; and so on for higher degrees than the ninth.

We are not as yet provided with the means of obtaining generating functions for Perpetuants and Syzygies of degree n, or even of making the attempt with any chance of success; although Capt. MacMahon's recent researches (MacMahon: On Perpetuants, *American Journal of Mathematics*, Vol. VII, pp. 26–46 and pp. 256–263) have rendered it extremely probable that the generating function for Perpetuants of the n^{th} degree is

$$(n) = \frac{x^{2^{n-1}-1}}{2\cdot3\cdot4\ldots n} \text{ when } n > 2.$$

This result will be confirmed for the cases $n = 6$ and $n = 7$, and the case $n = 5$ will be reconsidered as the simplest example of the method employed.

It seems that in general $S_n^{(\lambda)}$ (syzygies of degree n and grade λ) are not capable of being represented by any simple form of generating function, unless $n = 2\lambda + 3$, when by an extension of the method used, in the Johns Hopkins Circular for April, 1883 (p. 85), to find $S_5^{(1)}$, generating functions of a simple form may be obtained for the series of syzygies $S_5^{(1)}$, $S_7^{(2)}$, $S_9^{(3)}$, . . .

It should, however, be noticed that Prof. Cayley's Method of Capitation and Decapitation (see his memoir referred to) gives the simplest possible proof that the complete number of Quintic Syzygies has been found.

This will be sufficiently obvious if we remark that each congruence of the form $32^{\alpha-1}.2^{\beta} \equiv 32^{\beta-1}.2^{\alpha}$, when completed into a syzygy, contains a distinct irreducible Quartic Form; so that no repetition of the capitation process is possible.

Observe also that, if $2\alpha + 2\beta + 1 = w = 2n + 1$, the number of either the Quintic Syzygies, or of the special class of included Quartic Forms, is the same as the number of partitions of n into two unequal parts, zeros excluded. By means of these partitions I first arrived at the expression $\frac{x^7}{2.4}$ for the generating function of $S_5^{(1)}$, (see the J. H. U. Circular referred to).

3. Instead of $32^{\alpha-1}\ldots$ writing $C_{2\alpha+1}\ldots$, and $Q_{2\alpha}\ldots$ instead of $2^{\alpha}\ldots$, the general formula for Syzygies of the $S_5^{(1)}$ type is

$$C_{2\alpha+1}Q_{2\beta} - C_{2\beta+1}Q_{2\alpha} - a\Phi_{\alpha,\beta} = 0$$

where $\Phi_{\alpha,\beta}$ denotes the Quartic Form just mentioned, and the parent Quantic may be either $a + b\frac{x}{1} + c\frac{x^2}{1.2} + d\frac{x^3}{1.2.3} + \ldots$, or $(a, b, c, \ldots \between x, y,)^i$; but in the latter case i must not be less than either $2\alpha + 1$ or $2\beta + 1$. Hence the applications to the Theory of Finite Quantics.

It is easily seen that all Syzygies of the $S_7^{(2)}$ type (*i. e.* those of the seventh degree and second grade) are included in the formula

$$\begin{aligned} &Q_{2\alpha}(C_{2\beta+1}Q_{2\gamma} - C_{2\gamma+1}Q_{2\beta} - a\Phi_{\beta,\gamma}) \\ &- Q_{2\beta}(C_{2\alpha+1}Q_{2\gamma} - C_{2\gamma+1}Q_{2\alpha} - a\Phi_{\alpha,\gamma}) \\ &+ Q_{2\gamma}(C_{2\alpha+1}Q_{2\beta} - C_{2\beta+1}Q_{2\alpha} - a\Phi_{\alpha,\beta}) \\ &+ \ a(Q_{2\alpha}\Phi_{\beta,\gamma} \ - Q_{2\beta}\Phi_{\alpha,\gamma} \ + Q_{2\gamma}\Phi_{\alpha,\beta}) = 0 \end{aligned}$$

connecting compounds of the type $(2)S_5^{(1)}$ with a special class of Sextic Syzygies, and that, if $2\alpha + 2\beta + 2\gamma + 1 = w = 2n + 1$, their number is the same as that of the partitions of n into three unequal parts, zeros excluded.

This may by a convenient abbreviation be written

$$Q_{2\alpha}\overline{\beta\gamma} - Q_{2\beta}\overline{\alpha\gamma} + Q_{2\gamma}\overline{\alpha\beta} + a.\overline{\alpha\beta\gamma} = 0$$

and there is a similar formula for some of the syzygies of the type $S_8^{(2)}$, viz.

$$C_{2\alpha+1}\overline{\beta\gamma} - C_{2\beta+1}\overline{\alpha\gamma} + C_{2\gamma+1}\overline{\alpha\beta} + a(C_{2\alpha+1}\Phi_{\beta,\gamma} - C_{2\beta+1}\Phi_{\alpha,\gamma} + C_{2\gamma+1}\Phi_{\alpha,\beta}) = 0;$$

but this, unlike the former, will not give the complete set of $S_8^{(2)}$ syzygies, because for the eighth degree the compounds are of two types $(2)S_6^{(1)}$ and $(3)S_5^{(1)}$ (*vide* Art. 2). In like manner all the $S_9^{(3)}$ syzygies (and a second special class of $S_8^{(3)}$ syzygies of odd weight) are given by the formula,

$$\begin{aligned} &Q_{2\alpha}(Q_{2\beta}\overline{\gamma\delta} \ - Q_{2\gamma}\overline{\beta\delta} \ + Q_{2\delta}\overline{\beta\gamma} \ + a\overline{\beta\gamma\delta}) \\ &- Q_{2\beta}(Q_{2\alpha}\overline{\gamma\delta} \ - Q_{2\gamma}\overline{\alpha\delta} \ + Q_{2\delta}\overline{\alpha\gamma} \ + a\overline{\alpha\gamma\delta}) \\ &+ Q_{2\gamma}(Q_{2\alpha}\overline{\beta\delta} \ - Q_{2\beta}\overline{\alpha\delta} \ + Q_{2\delta}\overline{\alpha\beta} \ + a\overline{\alpha\beta\delta}) \\ &- Q_{2\delta}(Q_{2\alpha}\overline{\beta\gamma} \ - Q_{2\beta}\overline{\alpha\gamma} \ + Q_{2\gamma}\overline{\alpha\beta} \ + a\overline{\alpha\beta\gamma}) \\ &- \ a(Q_{2\alpha}\overline{\beta\gamma\delta} - Q_{2\beta}\overline{\alpha\gamma\delta} + Q_{2\gamma}\overline{\alpha\beta\delta} - Q_{2\delta}\overline{\alpha\beta\gamma}) = 0 \end{aligned}$$

connecting compounds of the type $(2)S_7^{(2)}$ with the special class of $S_8^{(3)}$ syzygies, and if $2\alpha+2\beta+2\gamma+2\delta+1=w=2n+1$, their number is that of the partitions of n into four unequal parts, zeros excluded.

This formula also may be abbreviated, by writing a single symbol for each syzygy of the second grade contained in it, *i. e.* for each expression in ( ); and when this is done, it will be easy to write down the corresponding formula for all the $S_{11}^{(4)}$ syzygies, which will be found to connect five compounds of the type $(2)S_9^{(3)}$ with one of the type $aS_{10}^{(3)}$ by a linear relation. And just as the $S_8^{(3)}$ syzygies were shown to form a special class only, and not the complete set, because of the existence of compounds of the two types $(2)S_6^{(1)}$ and $(3)S_5^{(1)}$; so the $S_{10}^{(3)}$ will form a special class because of the existence of the two types of compounds $(2)S_8^{(2)}$ and $(3)S_7^{(2)}$. The $S_{11}^{(4)}$ on the other hand will be completely determined (just as $S_9^{(3)}$ and $S_7^{(2)}$ were), their number being that of the partitions of $n=\frac{1}{2}(w-1)$ into five unequal parts, zeros excluded.

Precisely similar reasoning holds in every case, and leads to the following THEOREM:—*All the $S_{2\lambda+3}^{(\lambda)}$ syzygies are of odd weight* $(w=2n+1)$, *and connect compounds of the type* $(2)\,S_{2\lambda+1}^{(\lambda-1)}$ *with a special class of $S_{2\lambda+2}^{(\lambda-1)}$ syzygies, their number being that of the partitions of n into $\lambda+1$ unequal parts, zeros excluded. But for a finite Quintic of order i, none of these parts must exceed $\frac{i-1}{2}$ if i be odd, or $\frac{i}{2}-1$ if i be even.*

4. Let $\binom{n}{k}$ denote the number of partitions of n into k unequal parts, zeros excluded. By the subtraction of unity from each part we obtain the partitions of $n-k$ into k unequal parts, zeros included; and these are the same as the partitions of $n-k$ into k unequal parts, zeros excluded, + those of $n-k$ into $k-1$ unequal parts, zeros also excluded. This is the meaning of the formula

$$\binom{n}{k}=\binom{n-k}{k}+\binom{n-k}{k-1}.$$

Now the theorem is that

$$S_{2\lambda+3}^{(\lambda)}=\Sigma\binom{n}{\lambda+1}x^{2n+1}.$$

Whence $$(1-x^{2\lambda+2})\,S_{2\lambda+3}^{(\lambda)}=\Sigma\left\{\binom{n}{\lambda-1}-\binom{n-\lambda-1}{\lambda+1}\right\}x^{2n+1}.$$

The above partition formula reduces this to $\Sigma\binom{n-\lambda-1}{\lambda}x^{2n+1}$, or to $\Sigma\binom{m}{\lambda}x^{2m+1+2\lambda+2}$ if we write $n=m+\lambda+1$. So that finally we obtain

$$S^{(\lambda)}_{2\lambda+3}=\frac{x^{2\lambda+2}}{1-x^{2\lambda+2}}\,S^{(\lambda-1)}_{2\lambda+1}=\frac{x^{\lambda^2+3\lambda+3}}{2.4.6.\ldots(\lambda+1)\text{ factors}}.$$

Particular cases are

$$S^{(1)}_5=\frac{x^7}{2.4}$$
$$S^{(2)}_7=\frac{x^{13}}{2.4.6}$$
$$S^{(3)}_9=\frac{x^{21}}{2.4.6.8}$$
$$S^{(4)}_{11}=\frac{x^{31}}{2.4.6.8.10}$$
. . . .

Postponing, for the present, the application to Finite Quantics, I remark that Syzygies of each of these four types are recognizable in the Numerator of the Rep. Generating Function for the Binary 12^ic, the type $S^{(4)}_{11}$ being represented by a single term. (*Vide* pp. 44 and 45 of Sylvester's Tables of the Generating Functions and Groundforms of the Binary Duodecimic, with remarks, &c., *Am. Journ.*, Vol. IV, pp. 41 *et seq.*)

and it should always be borne in mind that each of these generating functions will also serve to express the number of syzygies of a special class whose degree and grade are each lower by unity than those indicated. Thus the same generating function serves to express the total number of $S^{(2)}_7$ syzygies, and that of a special class of $S^{(1)}_6$ syzygies.

5. A simple generating function may be defined as one of the form $\frac{x^w}{(1-x^\lambda)(1-x^\mu)(1-x^\nu)\ldots}$, where $\lambda, \mu, \nu, \ldots$ are all different.

It corresponds to all the partitions obtainable from a single partition of w, denoted by x^w, by superadding any number of each of the parts $\lambda, \mu, \nu, \ldots$ This partition of w may be perfectly determinate or only partially so; but in the case of perpetuants, here considered, both its parts and the superadded ones $\lambda, \mu, \nu, \ldots$ must be non-unitary. A good example is the generating function for Quintic Perpetuants where $w=15$, the superadded parts being 2, 3, 4, and 5, and the partition of w may be any one of the three 555, 5532, or 5433.

A first reduction of the formulæ for perpetuants is obtained by using only those simple generating functions in which $\lambda, \mu, \nu, \ldots$ are all even and the partition of w contains a known number of odd parts; the other generating functions used being expanded in a series of these.

Thus in the table, which gives the first and only reduction for Quintic Perpetuants; all the simple generating functions in the n^{th} row correspond to

	I	II	III
1	$\frac{x^5}{2.4}$	$\frac{x^5+x^7}{2.4}$	$-\frac{x^7}{2.4}$
2	$\frac{x^8+x^{10}}{2.4}$	$\frac{x^8+x^{10}}{2.4}$	0
3	$\frac{x^{11}+x^{13}+x^{15}}{2.4}$	$\frac{x^{11}+x^{13}}{2.4}$	$\frac{x^{15}}{2.4}$
4	$\frac{x^{14}+x^{16}+x^{18}+x^{20}}{2.4}$	$\frac{x^{14}+x^{16}}{2.4}$	$\frac{x^{18}+x^{20}}{2.4}$
etc.	etc.	etc.	etc.

partitions containing n odd parts. Columns I and II are obtained by decomposition, as explained below, from $\frac{x^5}{2.3.4.5}$ and $\frac{x^5}{2.2.3}$; and III by the row from row subtraction of II from I. The addition of the positive terms of III gives $\frac{x^{15}}{2.3.4.5}$, or the generating function of Quintic Perpetuants; the negative term is the generating function of Quintic Syzygies.

A system of partitions of the type $5^{A+1}4^B3^C2^D$ can be decomposed into partitions of the types

$$\begin{array}{lllll} 54^B2^D, & 54^B32^D, & 54^B3^22^D, & 54^B3^32^D, & \text{etc.} \\ & 5^24^B2^D, & 5^24^B32^D, & 5^24^B3^22^D, & \text{etc.} \\ & & 5^34^B2^D, & 5^34^B32^D, & \text{etc.} \\ & & & 5^44^B2^D, & \text{etc.} \end{array}$$

and to each of these corresponds a term of the decomposition of the generating function of $5^{A+1}4^B3^C2^D$, *i. e.* a term of

$$\frac{x^5}{2.3.4.5} = \frac{x^5}{2.4} + \frac{x^8+x^{10}}{2.4} + \frac{x^{11}+x^{13}+x^{15}}{2.4} + \frac{x^{14}+x^{16}+x^{18}+x^{20}}{2.4} + \text{etc.}$$

where the simple generating functions of partitions with the same number of odd parts have been grouped together, as in Col. I of the table.

To obtain Col. II notice that products of the type (3, 2) or $3^{\alpha+1}2^{\beta}.2^{\gamma+1}$ can be decomposed into the products $32^{\beta}.2^{\gamma+1}$, $3^2 2^{\beta}.2^{\gamma+1}$, $3^3 2^{\beta}.2^{\gamma+1}$, $3^4 2^{\beta}.2^{\lambda+1}$, etc. Or passing to their corresponding generating functions

$$\frac{x^5}{2\cdot 2\cdot 3}=\frac{x^5}{2\cdot 2}+\frac{x^8}{2\cdot 2}+\frac{x^{11}}{2\cdot 2}+\frac{x^{14}}{2\cdot 2}+\ldots$$

The transition from this to Col. II of the table, preparatory to its ultimate subtraction from Col. I, though simple and even trifling in appearance, deserves some notice. It signifies that each of the products spoken of has to be transformed, by means of Prof. Cayley's Multiplication Theory, into a series of *simple partitions;* and when this has been done, we have a system of equations for the reduction of the partitions of Col. I. The unreduced partitions now correspond to the positive terms of Col. III, which, when collected, constitute the Generating Function for Quintic Perpetuants; the reduced partitions are made to depend on these and on the products (3, 2). To see the exact meaning of the negative term in row 1, or the Generating Function for Quintic Syzygies, observe that the products $32^{\beta}.2^{\gamma+1}$ may be divided into three sets, viz.

(*a*) those for which $\beta=\gamma$, generating function $\frac{x^5}{4}$

(*b*) " " $\beta<\gamma$, " " $\frac{x^7}{2\cdot 4}$

(*c*) " " $\beta>\gamma$, " " $\frac{x^7}{2\cdot 4}$

and that either of the sets (*b*) or (*c*) taken with (*a*) being capable of reducing all the partitions of one odd part in Col. I, the other set is incapable of giving any fresh reductions and therefore yields syzygies.

6. Reasoning precisely similar to that employed in the preceding article gives the decomposition

$$\frac{x^6}{2\cdot 3\cdot 4\cdot 5\cdot 6}=\frac{x^6}{2\cdot 4\cdot 6}+\frac{x^9+x^{11}}{2\cdot 4\cdot 6}+\frac{x^{12}+x^{14}+x^{16}}{2\cdot 4\cdot 6}$$
$$+\frac{x^{15}+x^{17}+x^{19}+x^{21}}{2\cdot 4\cdot 6}+\frac{x^{18}+x^{20}+x^{22}+x^{24}+x^{26}}{2\cdot 4\cdot 6}$$
$$+\frac{x^{21}+x^{23}+x^{25}+x^{27}+x^{29}+x^{31}}{2\cdot 4\cdot 6}+\text{etc.}$$

where we may at once dismiss the first term with the remark that it corresponds to partitions with no odd part, and is therefore exactly reducible by means of the products (2, 2, 2). The remaining terms, with their common denominator omitted for the sake of brevity, are arranged in the first column of the table.

Den^r. $2\cdot4\cdot6$

	I	II	III
1	x^9+x^{11}	$x^9+x^{11}+x^{13}$	$-x^{13}$
2	$x^{12}+x^{14}+x^{16}$	$x^{12}+x^{14}+x^{16}$	0
3	$x^{15}+x^{17}+x^{19}+x^{21}$	$x^{15}+x^{17}+x^{19}$	x^{21}
4	$x^{18}+x^{20}+x^{22}+x^{24}+x^{26}$	$x^{18}+x^{20}+x^{22}$	$x^{24}+x^{26}$
5	$x^{21}+x^{23}+x^{25}+x^{27}+x^{29}+x^{31}$	$x^{21}+x^{23}+x^{25}$	$x^{27}+x^{29}+x^{31}$
etc.	etc.	etc.	etc.

The products (4, 2) yield the second column, in which the successive rows are the terms of the expansion

$$\frac{x^9}{2\cdot2\cdot3\cdot4}=\frac{x^9}{2\cdot2\cdot4}+\frac{x^{12}}{2\cdot2\cdot4}+\frac{x^{15}}{2\cdot2\cdot4}+\text{etc.}$$

after reduction to the common denominator $2\cdot4\cdot6$. The reasoning of the preceding article applies here, *mutatis mutandis*, and need not be repeated. The third column is found, as before, by subtraction and gives the following reduced formula for Sextic Perpetuants.

(A) $$(6)+(3,\ 3)=\frac{x^{21}}{2\cdot3\cdot4\cdot5\cdot6}+\left(S_6-\frac{x^{13}}{2\cdot4\cdot6}\right).$$

The syzygies denoted by the last term are easily shown to be identical with the special class of sextic syzygies before mentioned, given by the general formula

$$Q_{2\alpha}\Phi_{\beta,\gamma}-Q_{2\beta}\Phi_{\alpha,\gamma}+Q_{2\lambda}\Phi_{\alpha,\beta}=0,$$

and having the same generating function (see art. 4).

For, from the way in which the reduction has been effected, it is evident that the syzygies in question contain no products of either of the types (2, 2, 2) or (3, 3), but simply connect those of the (4, 2) products in which only one odd part occurs with one another and with forms of lower degree. In the general formula these forms of lower degree are contained in the forms Φ; thus $\Phi_{3,1} = 43 + 322$, and so for others.

Moreover, if we call a (4, 2) syzygy one which does not contain any (3, 3) product, all the (4, 2) syzygies are contained in the above general formula. For it is readily seen that the capitation of an identity, or of a syzygy which contains a (3, 3) product, can never give rise to a (4, 2) syzygy; so that the complete set of these is obtained by first capitating all the Quintic Syzygies with 4.2 throughout, and then capitating the syzygies thus formed with 4.2 throughout, and repeating the process indefinitely. We shall thus find, for the generating function of (4, 2) syzygies, the generating function for Quintic Syzygies multiplied by $\frac{x^6}{6}$; which is precisely the result obtained previously, proving that all the (4, 2) syzygies have been found.

7. The removal of the (3, 3) products from the reduced formula alone remains to be performed, and when this is done we shall obtain at the same time the generating function for Sextic Perpetuants and that for Sextic Syzygies.

The result of removing a single (3, 3) product, symbolized by x^n, is either

$$\left(\frac{x^{21}}{2\cdot3\cdot4\cdot5\cdot6} - x^n\right) + \left(S_6 - \frac{x^{18}}{2\cdot4\cdot6}\right),$$

or

$$\frac{x^{21}}{2\cdot3\cdot4\cdot5\cdot6} + \left(S_6 - x^n - \frac{x^{18}}{2\cdot4\cdot6}\right),$$

where in the former $-x^n$ denotes a reduction of the partitions, and in the latter a syzygy. Hence we see that any (3, 3) product must either give a reduction of the partitions contained in $\frac{x^{21}}{2\cdot3\cdot4\cdot5\cdot6}$ or else a syzygy.

Now the generating function of (3, 3) is $\frac{x^6 + x^{11}}{2\cdot3\cdot4\cdot6}$, and the effect of the reduction of Article 6 on the partitions denoted by

$$\frac{x^6}{2\cdot3\cdot4\cdot5\cdot6} = \frac{x^6 + x^{11} + x^{16} + x^{21} + x^{26} + x^{31} + x^{36} + x^{41} + \text{etc.}}{2\cdot3\cdot4\cdot6}$$

has been to remove the terms $x^6 + x^{11} + x^{16}$ from the numerator of this last fraction; so that it would appear at first sight that no fresh reduction is given by any of the (3, 3) products, in which case all of them would correspond to syzygies, this however is known to be untrue. But if we write the generating function of (3, 3) in the equivalent form

$$\frac{x^6 + x^9 + x^{11} + x^{12} + x^{14} + x^{15} + x^{17} + x^{18} + x^{20} + x^{23}}{2\cdot4\cdot6} + \frac{x^{21} + x^{26}}{2\cdot3\cdot4\cdot6}$$

we see that the next two terms $x^{31}+x^{36}$ may also be removed from the numerator, leaving $x^{31}=x^{36}+x^{41}+$ etc. The unreduced partitions are now

$$(6)=\frac{x^{31}}{2.3.4.5.6}.$$

That portion of the generating function for (3, 3) which does not give reductions will correspond to syzygies; and hence we have, introducing the term x^{13} in the numerator to represent the (4, 2) syzygies,

$$S_6=\frac{x^6+x^9+x^{11}+x^{12}+x^{13}+x^{14}+x^{15}+x^{17}+x^{18}+x^{20}+x^{23}}{2.4.6}.$$

To complete the proof it would be necessary to identify this with the complete set of Sextic Syzygies, which would involve the calculation of such of them as have not already been found by Capt. MacMahon. The above value of S_6 may, by introducing common factors in its numerator and denominator, be written in either of the equivalent, but more concise, forms

$$S_6=\frac{x^6+x^{11}+x^{13}-x^{16}-x^{21}-x^{26}}{2.3.4.6}=\frac{x^6+x^{13}-2x^{16}-x^{18}+x^{31}}{2.3.4.5.6},$$

of which the second has been given by Prof. Cayley.

8. The formulæ for the seventh degree are

$$(5,2)=\frac{x^{17}}{2.2.3.4.5},\ (4,3)=\frac{x^{10}}{2.2.3.3.4},\ (3,2,2)=\frac{x^{7}}{2.2.3.4},$$

$$(7)+(5,2)+(4,3)+(3,2,2)=\frac{x^7}{2.3.4.5.6.7}+S_7,$$

by means of which we may verify the relation

$$(7)+(4,3)=\frac{x^{28}}{2.3.4.5.6.7}+\left(S_7-\frac{x^9+x^{11}}{2.4.6}\right),$$

but remembering that $S_7=S_7^{(1)}+(2)\,S_5^{(1)}-S_7^{(2)}$ (see Article 2), and substituting for $S_5^{(1)}$ and $S_7^{(2)}$ their values given in Article 4; this reduces to

(B) $$(7)+(4,3)=\frac{x^{28}}{2.3.4.5.6.7}+S_7^{(1)},$$

indicating that every ground-syzygy of the seventh degree contains at least one term of the (4, 3) type. This theorem was originally obtained by the method employed in Articles 5 and 6, and there is every reason to believe that a tolerably simple independent proof of it is possible; but my present object is briefly to indicate the formulæ for Perpetuants of the seventh degree and to consider in the remaining articles the case of Finite Quantics. We have

$$(4, 3) = \frac{x^{10} + x^{12} + x^{13} + x^{14} + x^{15} + x^{16} + x^{17} + \ldots}{2 \cdot 3 \cdot 4 \cdot 6}$$

$$= \frac{N}{2 \cdot 3 \cdot 4 \cdot 6} + \frac{x^{28} + x^{35} + x^{42} + x^{49} + x^{56}}{2 \cdot 3 \cdot 4 \cdot 5 \cdot 6}$$

where $N = x^{10} + x^{12} + x^{13} + x^{14} + x^{15} + x^{16} + x^{17} + x^{18} + x^{19} + x^{20} + x^{21} + x^{22} + x^{23}$
$+ x^{24} + x^{25} + x^{26} + x^{27} + x^{28} + x^{30} + x^{31} + x^{32} + x^{34} + x^{36} + x^{37} + x^{39}$
$+ x^{41} + x^{44} + x^{46} + x^{51}$.

And if we write

$$\frac{x^{28}}{2 \cdot 3 \cdot 4 \cdot 5 \cdot 6 \cdot 7} = \frac{x^{28} + x^{35} + x^{42} + x^{49} + x^{56} + x^{63} + x^{70} + x^{77} + \text{etc.}}{2 \cdot 3 \cdot 4 \cdot 5 \cdot 6}$$

the method of Article 7 will give

$$(7) = \frac{x^{63}}{2 \cdot 3 \cdot 4 \cdot 5 \cdot 6 \cdot 7},$$

and $$S_7^{(1)} = \frac{N}{2 \cdot 3 \cdot 4 \cdot 6} = \frac{x^{10} + x^{12} + x^{13} + x^{14} + x^{16} - x^{28} - x^{35} - x^{42} - x^{49} + x^{56}}{2 \cdot 3 \cdot 4 \cdot 5 \cdot 6};$$

where the second value of $S_7^{(1)}$ is obtained from the first by introducing the common factor $1 - x^5$ in both numerator and denominator. The first value would be more convenient than the second if we had in our possession a complete list of the ground-syzygies and wished to compare them with the indicated value of $S_7^{(1)}$. Such a comparison would be in itself a most complete and satisfactory proof of the results given in this article, but would require some very laborious calculations. The terms $x^{10} + x^{13} + x^{14}$ of N are easily seen to correspond to a set of syzygies of the form $4^{\alpha+1} 3^{\beta+1} 2^{\gamma}. 32^{\delta} - 54^{\alpha} 3^{\beta+1} 2^{\gamma}. 2^{\delta+1} \equiv 0$; but I have not attempted to compare any of the remaining twenty-six terms with syzygies.

Two different reductions have been made use of in these articles; the first of these corresponds to the reduction of the generating function of a Finite Quantic to its Representative form, and consists in removing all terms which contain Quadric Factors; the second reduction removes all terms with Cubic Factors. For the eighth degree the formula after the first reduction is

$$\text{(C)} \quad (8) + (5, 3) + (4, 4) = \frac{x^{36}}{2 \cdot 3 \cdot 4 \cdot 5 \cdot 6 \cdot 7 \cdot 8} + S_8^{(1)} + (3) S_5^{(1)} + \left(\frac{x^{21}}{2 \cdot 4 \cdot 6 \cdot 8} - S_8^{(2)} \right),$$

the terms (6, 2), (4, 2, 2), (3, 3, 2), (2, 2, 2, 2), and $(2) S_6^{(1)}$, which contain Quadric Factors, having been removed. The term whose numerator is x^{21} corresponds to the "special class of $S_8^{(2)}$ syzygies" mentioned in Article 3. This reduction does not assume the knowledge of either (6) or $S_6^{(1)}$, but only of their difference; the result is therefore unaffected by any error in the determination of (6), should any such error exist.

9. To the formula $(5) = \frac{x^{15}}{2.3.4.5}$ and to (A), (B), and (C) of the preceding articles may be added one for the ninth degree, viz.

$$\text{(D)}\quad (9)+(6,3)+(5,4)+(3,3,3)=\frac{x^{45}}{2\cdot 3\cdot 4\cdot 5\cdot 6\cdot 7\cdot 8\cdot 9}+S_9^{(1)}+(3)\,S_6^{(1)}+(4)\,S_5^{(1)}-S_9^{(3)}$$

but the proof of this formula, as well as that of (C), has been omitted for the sake of brevity.

Observe (1) that in each of them all terms containing a Quadric Factor have been removed from both sides of the equation.

(2) That the expressions found in Article 4 for $S_5^{(1)}$, $S_7^{(3)}$, and $S_9^{(3)}$ have been substituted for them in the formulæ for degrees 5, 7, and 9 respectively.

(3) That each of them contains a term of the form

$$\frac{x^{\frac{n(n+1)}{2}}}{2\cdot 3\ldots n}, \text{ where } n=5,\ 6,\ 7,\ 8,\ 9.$$

This term alone expresses the unreduced partitions, viz. if the partition expressing any Quintic Perpetuant be $5^{\delta+1}4^{\gamma+1}3^{\beta+1}2^{\alpha}$, the unreduced partitions in (A), (B), (C) and (D) may be found by the successive superaddition of $\varepsilon+1$ parts each $=6$, $\zeta+1$ parts each $=7$, $\eta+1$ parts each $=8$, and $\theta+1$ parts each $=9$; so that the unreduced partitions in (D) will be of the form $9^{\theta+1}8^{\eta+1}7^{\zeta+1}6^{\varepsilon+1}5^{\delta+1}4^{\gamma+1}3^{\beta+1}2^{\alpha}$.

Applications to the Theory of Finite Quantics.

10. In a recent number of the *American Journal of Mathematics* (Vol. VII, p. 337) I found for the Generating Function of any Binary Quantic the expression

$$\frac{1-\Sigma_1+\Sigma_2-\Sigma_3+\ldots.}{\Pi(1-a^r x^s)}$$

where $(r.s)$ is the deg. order of any groundform, and the product Π contains a factor corresponding to each groundform. If there be two groundforms of the same deg. order Π will of course contain a corresponding square factor, if three a cubic factor, and so on. In the numerator Σ_n is the aggregate of terms corresponding to the irreducible syzygies of the n^{th} grade, or say the irreducible n^{th} syzygies; and the portion of the generating function corresponding to the portion Σ_n of the numerator will, when expanded, give a series of terms corresponding to all the n^{th} syzygies reducible or irreducible. The generating function when written in this form serves to express the fact that, for any given deg. order, the number of asyzygetic covariants is equal to the total number of covariants minus the total number of first syzygies, of that deg. order, plus the total number of second syzygies minus the total number of third syzygies, and so on.

It may now be easily shown that the numerator of any Representative form of Generating Function for a Binary Quantic will consist of blocks of alternately positive and negative terms; the terms in the first block representing covariants, which may be either irreducible or compound, but none of the compounds are divisible by any covariant represented in the denominator; the terms in the second block, in like manner, representing first syzygies not divisible by any of the denominator covariants; those in the third block second syzygies, and so on. To see this it is only necessary to replace all the factors $(1-a^r x^s, 1-a^{r'}x^{s'} \ldots)$ of Π which do not occur in the denominator of the representative form under consideration by geometrical series multiplying its numerator, which will therefore be $(1-\Sigma_1+\Sigma_2-\Sigma_3+\ldots)\times(1+a^r x^s+x^{2r}x^{2s}+\ldots)(1+a^{r'}x^{s'}+a^{2r'}x^{2s'}+\ldots)\ldots$; viz. the compound covariants of the first block will simply be powers and products of $(r.s)$, $(r'.s')$, ...; and the compound syzygies of the subsequent blocks will be simply divisible by these powers and products. If after multiplying by these geometrical series, we allow terms such as $+a^m x^n$ and $-a^m x^n$ to cancel each other,* there will result a Representative Generating Function of the form

$$\frac{C-S_1+S_2-S_3+\ldots}{D};$$

in which D is a representative divisor of Π, *i. e.* it is the product of factors $(1-a^p x^q)(1-a^{p'}x^{q'})\ldots$ representing groundforms whose deg. orders are $(p.q)$, $(p'.q')$, ...; and of the covariants in C, or the n^{th} syzygies in S_n, those which are reducible will be divisible by powers and products of the groundforms $(r.s)$, $(r's')$ ..., but not divisible by any of the groundforms $(p.q)$, $(p'q')$, ... represented by the factors of D.

11. Dr. F. Franklin, in a paper "On the Calculation of the Generating Functions and Tables of Groundforms for Binary Quantics" (*American Journal*

*Positive and negative terms may in most cases be freely allowed to destroy one another, for a $(n+1)^{th}$ syzygy may be considered as a relation between the *compound* n^{th} syzygies diminishing their number. But subsequent corrections will have to be made, not only for cases of the coexistence of groundform and syzygy, but also wherever $(n+1)^{th}$ syzygies exist simultaneously with *irreducible* n^{th} syzygies of the same deg. order. Examples of this do in fact occur even for the Binary Quintic, which has irreducible first and second syzygies of the same deg. order.

of Mathematics, Vol. III, p. 137), has given the numerator of the representative generating function whose denominator is

$$D = (1 - a^p x^q)(1 - a^{p'} x^{q'})(1 - a^{p''} x^{q''}) \ldots$$

in the form

$$L(1 - \Sigma a^p x^q + \Sigma a^{p+p'} x^{q+q'} - \Sigma a^{p+p'+p''} x^{q+q'+q''} + \ldots),$$

where L is the expansion of the generating function in an infinite series, and is multiplied by the finite expanded value of D which forms the series in brackets. He also proves that the covariants represented by terms of the numerator are not divisible by any of the denominator forms $(p.q)$, $(p'.q')$, $(p''.q'')$, ... This form of numerator will of course reduce to that previously given, viz. $C - S_1 + S_2 - S_3 + \ldots$, if we freely allow positive and negative terms to destroy one another; but among the terms of $L\Sigma a^p x^q$ some will correspond to compound covariants of the type $(p.q)\Theta$, where Θ is one of the asyzygetic forms represented by the terms of L, and will properly be subtracted from L in order to reduce it to C; others, *which cannot thus be subtracted*, will correspond to syzygies, each of which will give an expression for a compound of the type $(p.q)\Theta$ in terms of the asyzygetic compounds. So also, some of the terms of $L\Sigma a^{p+p'} x^{q+q'}$ will correspond to compound syzygies, reducing the syzygies of the preceding block to S_1, and some of them to "second syzygies"; and reductions will take place, in like manner, in each of the alternately positive and negative blocks of the numerator.

But since positive and negative terms have been freely allowed to cancel each other,* no term in S_1 is of the same deg. order as a term in C; thus the syzygy giving $(p.q)\Theta$ is not of the same deg. order as any term of C, and since the asyzygetic compounds not contained in C are all of them divisible by one or more of the denominator forms $(p.q)$, $(p'.q')$, $(p''.q'')$, ..., each of the syzygies S_1 will connect only compound covariants which are divisible thus.

This reasoning may, without difficulty, be extended so as to give the meaning of terms in the remaining blocks S_2, S_3, ..., when it will be seen that each term of the block S_n denotes a n^{th} syzygy not itself divisible by any denominator form but connecting only such compound $(n-1)^{\text{th}}$ syzygies as are so divisible. If we call a, Q_{2a}, $Q_{2\beta}$, $Q_{2\gamma}$, ... denominator forms, the first, second, and third syzygies whose expressions have been given in Article 3 fulfil these conditions.

* Subsequent corrections will of course have to be made for such cases of coexistence as are mentioned in the preceding note.

12. Referring to p. 135 of Dr. Franklin's paper, just quoted, it will be seen that the representative function will be expressed in a finite form whenever its denominator D is divisible by the denominator of the reduced form, but in no other case. For a given quantic there is sometimes a large number of different finite representative generating functions, even when the number of factors in the denominator is restricted. As an example, for the Quintic there is no finite representative generating function with less than six factors in its denominator, but as many as 18 can be found with six factors. Here the denominator of the reduced form is

$$(1-a^4)(1-a^6)(1-a^8)(1-ax)(1-ax^3)(1-ax^5),$$

the denominator of the simplest representative form being

$$(1-a^4)(1-a^8)(1-a^{12})(1-a^2x^2)(1-a^3x^6)(1-ax^5);$$

but the three invariant factors of the denominator may be chosen in 3 different ways, viz.

$$(1-a^4)(1-a^8)(1-a^{12})$$
$$(1-a^4)(1-a^8)(1-a^{18})$$
$$(1-a^8)(1-a^{12})(1-a^{18}),$$

and similarly the three covariant factors may be chosen in 6 different ways; or there are in all 18 different ways. For one of the covariant factors must be $(1-ax^5)$, and denoting the other two by $(1-a^px^q)(1-a^rx^s)$, $(p.q)$ may be either (2.2), (3.3), or (4.4), and $(r.s)$ may be either (2.6) or (3.9).

In Prof. Sylvester's very extensive and extremely valuable Tables of Generating Functions and Groundforms, published in various numbers of the *American Journal of Mathematics*, the representative form given is in each case the simplest that can be found by multiplying both numerator and denominator of the reduced form, given in the same table, by such a factor as will make the denominator, and therefore the numerator, representative. But since no two representative forms give us exactly the same information, it may in some cases be convenient, instead of those given in the tables, to use other forms; and any one of these may be obtained from the reduced form by the same process.

Everything in this and in the two preceding articles may be either directly applied or easily extended to Systems of Binary Quantics, but in what follows the quantic $(a, b, c, \ldots \between x, y)^n$, and its simplest representative function, will alone be considered when $n = 5, 6, 7, 8, 9, 10, 12$.

13. The *source* of any covariant is what remains of it after putting $x = 1$ and $y = 0$, and is a *perpetuant* when not expressible as a rational integral function of the sources of covariants of lower deg. order, however great the order of the parent quantic may be.

A syzygy containing non-perpetuant sources may be a non-perpetuant syzygy in two distinct cases: (1) after substituting for each non-perpetuant source its value in terms of perpetuants, the syzygy may be expressible as a linear function of the compound syzygies to the quantic of infinite order; (2) this substitution may cause the syzygy to vanish. Irreducible syzygies to a quantic of finite order, when not included in either of these classes, are also *perpetuant syzygies.*

Since all groundforms of degrees 1, 2, and 3, have perpetuant sources, and since there are no perpetuant syzygies of lower degree than 5; no single binary quantic has a syzygy of lower degree than 5, and all its syzygies of degree 5 are perpetuant syzygies. For, all compound sources of degree 4 are made up of perpetuant sources of degrees 1, 2, and 3; so that no syzygy of degree 4 can exist which contains a non-perpetuant source, *i. e.* there is no non-perpetuant syzygy.

Syzygies of degree 5 to any binary quantic, besides terms divisible by the source of the quantic itself, or a, contain terms formed by multiplying a source of degree 2 by one of degree 3; or denoting these by $(2)(3)$, $(2)'(3)'$, ... we may write any syzygy of degree 5 in the form of a congruence, thus

$$\lambda(2)(3) + \mu(2)'(3)' + \ldots \equiv 0 \pmod{a},$$

where λ, μ, ... are numerical multipliers to be determined.

But this is a perpetuant congruence, and corresponds to a perpetuant syzygy. Again, there are no second syzygies of lower degree than 7; for there are no compound syzygies of degree 5, and therefore no second syzygies of that degree, and denoting the syzygies of degree 5 by $(\overline{5})$, $(\overline{5})'$, ..., the only compound syzygies of degree 6 are $a(\overline{5})$, $a(\overline{5})'$, ... which are clearly not connected by a second syzygy. For degree 7 the compound syzygies not divisible by a are $(2)(\overline{5})$, $(2)'(\overline{5})'$, ..., so that any second syzygy of degree 7 may be written as a congruence, thus $\lambda(2)(\overline{5}) + \mu(2)'(\overline{5})' + \ldots \equiv 0 \pmod{a}$, corresponding to a perpetuant second syzygy, of which the general value has been given in Article 3.

By extending this reasoning to the 3d, 4th, 5th, ... syzygies in succession, we may prove that there are no n^{th} syzygies of lower degree than $2n + 3$, and that the n^{th} syzygies of this degree are perpetuant n^{th} syzygies, so that their number is given by the partition theorem of Article 3. And it should be remembered that every n^{th} syzygy thus found contains a $(n-1)^{\text{th}}$ syzygy whose weight is the same, its degree being less by unity than that of the n^{th} syzygy.

14. The number of partitions of n into $\lambda + 1$ unequal parts, zeros excluded and no part greater than μ, is the coefficient of $c^{\lambda+1}z^n$ in

$$(1 + cz)(1 + cz^2)(1 + cz^3)\ldots(1 + cz^\mu) - (1 + cz + cz^2 + cz^3 + \ldots + cz^\mu)$$

when λ has any positive integral value from 1 to μ inclusive.

But by the theorem of Article 3 this is the number of λ^{th} syzygies of degree $2\lambda + 3$ and weight $2n + 1$, to the quantic of order i; where $i - 1 = 2\mu$ or $i - 2 = 2\mu$, according as i is odd or even.

Hence we may, if $w = 2n + 1$, instead of the above expression, use

$$\frac{ax}{g}(1 + a^2gx^2)(1 + a^2gx^4)\ldots(1 + a^2gx^{2\mu}) - \frac{ax}{g}(1 + a^2gx^2 + a^2gx^4 + \ldots + a^2gx^{2\mu}),$$

in which the coefficient of $a^{2\lambda+3}g^\lambda x^w$ is the number of λ^{th} syzygies of degree $2\lambda + 3$ and weight w, to either the quantic of order $2\mu + 1$ or that of order $2\mu + 2$. Thus, for the Quintic or Sextic,

$$\frac{ax}{g}(1 + a^2gx^2)(1 + a^2gx^4) - \frac{ax}{g}(1 + a^2gx^2 + a^2gx^4) = a^5gx^7$$

represents the well-known syzygy of degree 5 and weight 7, which is the same for both quantics.

For the 7^{ic} or 8^{ic} we have

$$\frac{ax}{g}(1 + a^2gx^2)(1 + a^2gx^4)(1 + a^2gx^6) - \frac{ax}{g}(1 + a^2gx^2 + a^2gx^4 + a^2gx^6)$$
$$= a^5g(x^7 + x^9 + x^{11}) + a^7g^2x^{13},$$

so that these two quantics have one syzygy of degree 5 for each of the weights 7, 9 and 11, and a second syzygy of degree 7 and weight 13. It is to be remembered that this second syzygy contains a syzygy of deg. weight 6.12 the only representative of the special class mentioned in Article 3.

For the 9^{ic} or 10^{ic} we have in like manner

$$\frac{ax}{g}(1+a^2gx^2)(1+a^2gx^4)(1+a^2gx^6)(1+a^2gx^8) - \frac{ax}{g}(1+a^2gx^2+a^2gx^4+a^2gx^6+a^2gx^8)$$
$$= a^5g(x^7 + x^9 + 2x^{11} + x^{13} + x^{15}) + a^7g^2(x^{13} + x^{15} + x^{17} + x^{19}) + a^9g^3x^{21},$$

which must be interpreted as before.

And for the 11^{ic} or 12^{ic},

$$\frac{ax}{g}(1 + a^2gx^2)(1 + a^2gx^4)(1 + a^2gx^6)(1 + a^2gx^8)(1 + a^2gx^{10})$$
$$- \frac{ax}{g}(1 + a^2gx^2 + a^2gx^4 + a^2gx^6 + a^2gx^8 + a^2gx^{10})$$
$$= a^5g(x^7 + x^9 + 2x^{11} + 2x^{13} + 2x^{15} + x^{17} + x^{19})$$
$$+ a^7g^2(x^{13} + x^{15} + 2x^{17} + 2x^{19} + 2x^{21} + x^{23} + x^{25})$$
$$+ a^9g^3(x^{21} + x^{23} + x^{25} + x^{27} + x^{29})$$
$$+ a^{11}g^4x^{31}.$$

The above is a complete determination of the syzygies of degree 5, second syzygies of degree 7, third syzygies of degree 9, and fourth syzygies of degree 11, for the quantics considered. A partial determination of the syzygies of degree 6, second syzygies of degree 8, and third syzygies of degree 10, for these quantics, will be obtained by simply dividing those terms that contain any power of g higher than the first by ag, thus for the 9^{ic} or 10^{ic} we have

$$a^6 g\,(x^{13} + x^{15} + x^{17} + x^{19}) + a^8 g^3 x^{21}.$$

As was shown in Article 11, the successive positive and negative blocks in the numerator of any representative generating function, for which the denominator forms are a, Q_3, Q_4, ... ought to contain terms corresponding to all of these syzygies, and when this is not so, the corrections spoken of in the foot-notes of Articles 10 and 11 will have to be made.

15. To illustrate the nature of these corrections, take the case of the 12^{ic}. Here writing $g = -1$ and replacing deg. weight by deg. order, the odd-degree formula becomes

$$\begin{aligned}
&- a^5 (x^{22} + x^{26} + 2x^{30} + 2x^{34} + 2x^{38} + x^{42} + x^{46})\\
&+ a^7 (x^{34} + x^{38} + 2x^{42} + 2x^{46} + 2x^{50} + x^{54} + x^{58})\\
&- a^9 (x^{50} + x^{54} + x^{58} + x^{62} + x^{66})\\
&+ a^{11} x^{70};
\end{aligned}$$

while the even-degree formula will be found to be

$$\begin{aligned}
&- a^6 (x^{22} + x^{26} + 2x^{30} + 2x^{34} + 2x^{38} + x^{42} + x^{46})\\
&+ a^8 (x^{38} + x^{42} + x^{46} + x^{50} + x^{54})\\
&- a^{10} x^{58};
\end{aligned}$$

viz. the latter is in this case obtained from the former by dividing the terms of degrees 7, 9, and 11 by $-ax^{12}$ instead of by ag.

These are to be compared with the following terms selected from the numerator of Prof. Sylvester's representative generating function for the 12^{ic} (*American Journal of Mathematics*, Vol. IV, pp. 44, 45).

$$\begin{aligned}
&a^5 (4x^{22} + x^{24} + x^{26} - x^{30} - 2x^{34} - 2x^{38} - x^{42} - x^{46})\\
&a^6 (6x^{22} + 2x^{24} + x^{26} - x^{30} - 3x^{34} - 2x^{38} - x^{42} - x^{46})\\
&a^7 (- 4x^{34} - x^{38} + x^{42} + 2x^{46} + 2x^{50} + x^{54} + x^{58})\\
&a^8 (- 2x^{38} - x^{40} + 2x^{46} + x^{50} + x^{54})\\
&a^9 (x^{50} - x^{58} - x^{62} - x^{66})\\
&- a^{10} x^{58}\\
&+ a^{11} x^{70}
\end{aligned}$$

where the terms of the 5th and 6th degrees in a have been taken partly from the first positive, and partly from the first negative block; those of the 7th and 8th degrees partly from the first negative, and partly from the second positive block; those of the 9th degree partly from the second positive, and partly from the second negative, $-a^{10}x^{58}$ from the second negative, and $a^{11}x^{70}$ from the third positive block.

Now writing, as in Articles 10 and 11, the numerator in the form

$$C - S_1 + S_2 - S_3 + S_4,$$

we have

$$\begin{aligned}
C &= \ldots + a^5(\ldots + 5x^{22} + x^{24} + 2x^{26} + x^{30}) \\
&\qquad + a^6(\ldots + 7x^{22} + 2x^{24} + 2x^{26} + x^{30}) + \ldots \\
-S_1 &= -a^5(x^{22} + x^{26} + 2x^{30} + 2x^{34} + 2x^{38} + x^{42} + x^{46}) \\
&\quad - a^6(\ldots + x^{22} + x^{26} + 2x^{30} + 3x^{34} + 2x^{38} + x^{42} + x^{46}) \\
&\quad - a^7(\ldots + 5x^{34} + 2x^{38} + x^{42}) \\
&\quad - a^8(\ldots + 3x^{38} + x^{40} + x^{42}) - \ldots \\
S_2 &= a^7(x^{34} + x^{38} + 2x^{42} + 2x^{46} + 2x^{50} + x^{54} + x^{58}) \\
&\quad + a^8(\ldots + x^{38} + x^{42} + 2x^{46} + x^{50} + x^{54}) \\
&\quad + a^9(\ldots + 2x^{50} + x^{54}) + \ldots \\
-S_3 &= -a^9(x^{50} + x^{54} + x^{58} + x^{62} + x^{66}) \\
&\quad - a^{10}(\ldots + x^{58}) - \ldots \\
S_4 &= a^{11}x^{70} + \ldots
\end{aligned}$$

which are expressions for the separated blocks. It will be seen, if we recombine these, and thus return to the original form of the numerator, that contiguous blocks have a tendency to overlap one another; what I have called, for want of a better word, *corrections* separate them again and restore each term to its proper block. The restorations depending on the odd-degree formulæ of the last article may be relied on as exact, those depending on the even-degree formulæ can only be considered as approximations to an exact restoration.

16. Passing on to the consideration of the first block C in the numerator of any representative function which contains, among its denominator forms, the quantic itself and all its quadricovariants, including invariants; it will be seen at once that since none of the covariants represented by terms of C are divisible by any of the denominator forms, C cannot contain any terms of the first or second degree, and all its terms of the third, fourth, and fifth degrees will represent groundforms. It may contain terms of the sixth degree representing compounds obtained by multiplying together a pair of cubic groundforms; and the terms of higher degree that it may contain may be found from a table of the partitions of

any number n from which the parts 1 and 2 have been removed;* thus

n	Partitions
1	None
2	None
3	3
4	4
5	5
6	6, 3.3
7	7, 4.3
8	8, 5.3, 4.4
9	9, 6.3, 5.4, 3.3.3
etc.	. . . .

Since there are no syzygies of any kind of lower degree than the fifth, the groundforms of the first, second, third, and fourth degrees may be found by simply adding the denominator forms of these degrees to those represented by terms in C, and will always be identical with those obtained by Tamisage.†

Since no terms of the 5th degree occur in S_2, S_3, . . . , the numerator N may be written $N = C - S_1$ if we reject all terms of higher and all of lower degree than the fifth; and then the theory is that all the groundforms, except those in the denominator, and no compounds are represented by the terms of C, and that all the syzygies are perpetuant syzygies, which are all represented by terms of S_1. To obtain the complete list of groundforms of the fifth degree to the quantic of order i, we take the value of S_1 from the following table:

i	S_1
5	a^5x^{11}
6	a^5x^{16}
7	$a^5(x^{13} + x^{17} + x^{21})$
8	$a^5(x^{18} + x^{22} + x^{26})$
9	$a^5(x^{15} + x^{19} + 2x^{23} + x^{27} + x^{31})$
10	$a^5(x^{20} + x^{24} + 2x^{28} + x^{32} + x^{36})$
11	$a^5(x^{17} + x^{21} + 2x^{25} + 2x^{29} + 2x^{33} + x^{37} + x^{41})$
12	$a^5(x^{22} + x^{26} + 2x^{30} + 2x^{34} + 2x^{38} + x^{42} + x^{46})$
etc.	. . . .

* It may possibly be useful at this point to recall the corresponding formulæ in the theory of Perpetuants, viz. (A), (B), (C), and (D) of Articles 6, 8, and 9.

† In the denominator for the Nonic (*American Journal of Mathematics*, Vol. II, p. 236) the printer has omitted the factor $(1 - a^2x^2)$.

the value of C is then found by adding S_1 to the value of N taken from Prof. Sylvester's Tables of Generating Functions and Groundforms, in the pages of this Journal, to which reference is made; and finally, the denominator forms of degree 5, if there are any, are added.

Thus when $i = 7$ (Vol. II, p. 228)

$$N = a^5(x + 2x^3 + 2x^5 + 2x^7 + 2x^9 - x^{17} - x^{21})$$
$$C = a^5(x + 2x^3 + 2x^5 + 2x^7 + 2x^9 + x^{13})$$

there is no denominator form of degree 5, and all the groundforms are represented by terms of C; these are the same as given in the table p. 230, with the additional groundform of deg. order (5.13).

When $i = 8$ (Vol. II, p. 232)

$$N = a^5(x^2 + 2x^4 + 2x^6 + x^8 + 3x^{10} + x^{14} - x^{18} - x^{22} - x^{26})$$

the addition of S_1 to this simply destroys the negative terms and there is no additional groundform; but there is a denominator form (the Quintinvariant) a^5, and this added to C gives the expression

$$a^5(1 + x^2 + 2x^4 + 2x^6 + x^8 + 3x^{10} + x^{14})$$

corresponding to the complete set of quintic groundforms given in the table (p. 233). These two examples will sufficiently explain the way in which the following table is calculated

	i	Additional Groundforms
Vol. II, p. 224	5	None
" 225	6	None
" 230	7	a^5x^{13}
" 233	8	None
" 242	9	$a^5(x^{15} + x^{19})$
" 247	10	a^5x^{20}
Not calculated	11	—
Vol. IV, p. 48	12	$a^5(x^{22} + x^{26} + x^{30})$

Some of these were found by a different method as long ago as the beginning of 1883; there are also additional Groundforms for the 11^{ic} which are not given here, but may be found without much difficulty (See the J. H. U. Circulars, No. 22, April, 1883).

17. The success of the method of the last article was dependent on the combination of three accidents, viz.

(1) All the Numerator Covariants were Groundforms.

(2) All the Numerator Syzygies were Perpetuant Syzygies.

(3) All the Perpetuant Syzygies were Numerator Syzygies.

These will not happen for the sixth or higher degrees.

Example 1. The Quintic has six perpetuant syzygies of degree 6, but not one of them is represented by a term in the Numerator of the Representative Function, which (see Vol. II, p. 224) has no negative term of the sixth degree.

Example 2. In the Numerator of the Representative Function for the Sextic (Vol. II, p. 225) the term $+a^6x^4$ represents the square of the covariant of deg. order (3.2). The products $(2.0)^2(2.4)$, $(2.0)(4.4)$, $(2.4)(4.0)$ are not Numerator Covariants because each of them is divisible by a denominator form.

Example 3. The syzygy $an - b^2e + 6bl + 2ck + fg = 0$ of deg. order (7.9) for the Quintic (see Prof. Cayley's 10th Memoir on Quantics), is a Perpetuant Syzygy, and a Numerator Syzygy for the Quintic because, though the syzygy itself is not, each of its terms is divisible by one of the denominator forms (a, b, c, g, q, u); but it is not a Numerator Syzygy for the Sextic, because, though a, b, c remain denominator forms for the Sextic, neither f nor g is a denominator form. In fact f is the source of the covariant (3, 12) and g of the covariant (4.4), both of which are numerator forms for the Sextic.

Example 4. Of the two negative terms $-a^6(x^{14} + x^{16})$ in the Numerator for the Septimic (Vol. II, p. 228), one represents a Perpetuant Syzygy, the other a non-perpetuant Syzygy.

For the syzygies of the fifth degree we may write

Deg. Order.	
(5.13)	$(3.7)(2.6) - (3.11)(2.2) = (1.7)(4.6)$
(5.17)	$(3.7)(2.10) - (3.15)(2.2) = (1.7)(4.10)$
(5.21)	$(3.11)(2.10) - (3.15)(2.6) = (1.7)(4.14)$.

Multiplying the first of these by (2.10), the second by $-(2.6)$, the third by (2.2), and adding, we deduce

(6.16) $$(2.10)(4.6) - (2.6)(4.10) + (2.2)(4.14 = 0,$$

which is one of the perpetuant syzygies in question.

It should be noticed that the four are so connected that when any three of them are known the fourth can be found by means of the second syzygy.

(7.23)
$$\begin{aligned}&(2.10)\{(3.7)(2.6) - (3.11)(2.2) - (1.7)(4.6)\}\\ -&(2.6)\{(3.7)(2.10) - (3.15)(2.2) - (1.7)(4.10)\}\\ +&(2.2)\{(3.11)(2.10) - (3.15)(2.6) - (1.7)(4.14)\}\\ +&(1.7)\{(2.10)(4.6) - (2.6)(4.10) + (2.2)(4.14)\} = 0,\end{aligned}$$

which could not exist if one of them were missing. Thus the existence of the term $+a^7x^{23}$ in the numerator, besides indicating the second syzygy of deg. order (7.23) implies the existence of the syzygy of deg. order (5.13).

From the following three perpetuants,

$$\Theta = ai - 8bh + \dots$$
$$\Phi = i(ac - b^2) - 4adh + \dots$$
$$\Psi = i(ae - 4bd + 3c^2) - 4afh + \dots,$$

we may obtain, by eliminating i, three sources for the 7^{ic} connected by a syzygy.

Using the letters P, Q, R, to denote these three sources, we have

$$P = (ac - b^2)\Theta - a\Phi$$
$$Q = (ae - 4bd + 3c^2)\Theta - a\Psi$$
$$R = (ae - 4bd + 3c^2)\Phi - (ac - b^2)\Psi$$

connected by the syzygy

$$aR + (ae - 4bd + 3c^2)P - (ac - b^2)Q = 0.$$

It might easily be proved that P, Q, R, when considered as sources for the 7^{ic}, are irreducible; they may therefore be taken for the sources of the three non-perpetuant groundforms of deg. order (4.8), (4.4), and (5.7) respectively, and the above syzygy may be written

(6.14) $$(1.7)(5.7) + (2.6)(4.8) - (2.10)(4.4) = 0,$$

which is the syzygy represented by $-a^6x^{14}$.

Referring to the definitions of Article 13 it is clear that this syzygy belongs to the second kind of non-perpetuant syzygies.

18. I will conclude by stating a theorem which I believe to be true, but have in vain tried to prove.

Every irreducible syzygy must contain among its terms at least one binary combination of the groundforms.

The converse of this, viz.

Every syzygy containing at least one binary combination of the groundforms among its terms is irreducible, is obviously true.

Note on Space Divisions.

BY E. H. MOORE, JR., *Denver, Col., and* C. N. LITTLE, *Lincoln, Neb.*

The Division of a Plane into Polygons by n unlimited straight lines.

1. Since the lines are unlimited the finite polygons are *convex;* a division of the plane stretching to infinity between two lines reappears at the opposite side of the plane between the same two lines, and the two infinite divisions make one polygon. Thus all the conclusions given hold after projection, and in obtaining them it is sufficient to consider the simplest form into which the plane can be projected.

2. The total number of polygons is $\frac{1}{2}n(n-1)+1$.* This may be shown by mathematical induction, noticing that an additional line adds one polygon for every segment of it.

3. When n lines form an n-gon, the resulting figure may be called an *n-wheel.* The n-gon may be taken finite and convex; there is a bordering *chain* of n 3-gons; in the exterior angular space between two adjacent sides, there are two of these 3-gons and $n-3$ 4-gons. Thus the distribution of polygons in an n-wheel is

$$1\ n\text{-gon},\ \tfrac{1}{2}n(n-3)\ \text{4-gons and } n\ \text{3-gons}.$$

4. There is a chain of bordering 3-gons; and the remaining 4-gons are also in chains. Chain d consists of n 4-gons; each 4-gon is bordered on its two interior sides by two 4-gons of chain $d-1$, and angles at their intersection on

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*Cf. Pilgrim; *Zeitschrift für Math. u. Physik;* 24, 188; 1879: "Ueber die Anzahl der Theile, in welche ein Gebiet k^{ter} Stufe durch n Gebiete $(k-1)^{\text{ter}}$ Stufe getheilt werden kann."

a 4-gon of chain $d - 2$, and its $\left.\begin{matrix}\text{opposite}\\ \text{adjacent}\end{matrix}\right\}$ sides are $\left.\begin{matrix}\text{adjacent}\\ \text{non-adjacent}\end{matrix}\right\}$ in the n-gon. If n be even, the outer chain contains $\frac{1}{2}n$ 4-gons.

5. In the distribution with n lines, if there be an $(n-m)$-gon where $n-m > m+4$, there is only one $(n-m)$-gon and no one of the remaining polygons has more than $m+4$ sides. This follows by considering first the $(n-m)$-wheel formed by the $n-m$ sides of the $(n-m)$-gon. Thus n lines may give an n-wheel, or a distribution with an $(n-1)$-gon, an $(n-2)$-gon ... or (n even) an $\frac{n+6}{2}$gon, (n odd) an $\frac{n+5}{2}$gon, and there are no other polygons of higher order than 4, 5, 6 ... or (n even) $\frac{1}{2}(n+2)$, (n odd) $\frac{1}{2}(n+3)$, respectively.

6. Let p_r denote the number of r-gons in the distribution.

$$\sum_{r=3}^{r=n} p_r = \frac{n(n-1)}{2} + 1 \qquad \text{(No. 2)}$$

$$\sum_{r=3}^{r=n} rp_r = 2n(n-1);$$

i. e. the number of sides of polygons is double the number of segments of lines. Also,

$$n \text{ even}, \sum_{r=\frac{1}{2}(n+6)}^{r=n} p_r \leqq 1;\ n \text{ odd}, \sum_{r=\frac{1}{2}(n+5)}^{r=n} p_r \leqq 1. \qquad \text{(No. 5)}$$

7. The distribution with one $(n-1)$-gon is completely specified by the position of the n^{th} line, *i. e.* by the order in which it crosses the chains of the $(n-1)$-wheel; for example, a line 121 cuts from a 4-gon of chain 2 a 3-gon vertical to the $(n-1)$-gon.

The distributions for three simple cases are as follows:

$(n-1)$-wheel and	$(n-1)$-gon.	5-gons.	4-gons.	3-gons.
line 121	1	1	$\frac{n^2-3n-2}{2}$	n
line 212	1	2	$\frac{n^2-3n-6}{2}$	$n+1$
line 232	1	5	$\frac{n^2-3n-18}{2}$	$n+4$

8. A table of all possible distributions for $n = 3, \ldots 7$:

n			7-gon	6-gons	5-gons	4-gons	3-gons	Total $\frac{n(n-1)}{2}+1$	
3	3-wheel						4	4	
4	4-wheel					3	4	7	
5	5-wheel				1	5	5	11	
6	6-wheel			1	0	9	6	16	(No. 3)
	5-wheel and line	121			2	8	6		(No. 7)
		212			3	6	7		(No. 7)
		232			6	0	10		(No. 7)
7	7-wheel		1	0	0	14	7	22	(No. 3)
	6-wheel and line	121		1	1	13	7		(No. 7)
		212		1	2	11	8		(No. 7)
		232		1	5	5	11		(No. 7)
		323		1	3	9	9		
	$p_7 = 0,\ p_6 = 0$				6	6	10		
					5	8	9		
					4	10	8		
					3	12	7		

9. Five lines ($n = 5$) always make one pentagon. A conic may be described to touch the five lines; the conic may be projected into an ellipse, and the five tangent lines then evidently form a convex pentagon, which is the pentagon here otherwise noticed.

Simple Cases of Division of Space by Planes; and Flat Space of k Dimensions by Flat Spaces of $k - 1$ Dimensions.

10. Since the dividing flats are unlimited the resulting k-fold polyhedroids are convex. A division of the k-flat stretching to infinity between q $(k - 1)$-flats reappears in the opposite direction between the same flats, and the two infinite divisions constitute one k-fold polyhedroid.

11. Since the n^{th} dividing $(k-1)$-flat adds one k-fold division for every $(k-1)$-fold division of itself, made by its $n-1$ $(k-2)$-flat intersections with the other dividing flats, the following table* of the total numbers of divisions is readily made.

Divisions of	$n=1$,	2,	3,	4,	5,	6,	7,	8,	9,	10,	11.
Plane	1	2	4	7	11	16	22	29	37	46	56
Space	1	2	4	8	15	26	42	64	93	130	176
Four-flat	1	2	4	8	16	31	57	99	163	256	386

A general expression for the number of divisions in terms of n or of figurate numbers is also readily found. For example, n planes divide space into $\frac{(n-2)(n-1)n}{|\underline{3}} + n$ polyhedra, or the $(n-2)^{th}$ figurate number of the fourth order plus the n^{th} figurate number of the second order.

12. Four planes divide space into eight tetrahedra, and $n+1$ $(n-1)$-flats divide an n-flat into 2^n n-fold $(n+1)$-hedroids.

The only 3-fold pentahedron has as faces two 3-gons and three 4-gons. The distribution with five planes is ten pentahedra and five tetrahedra. Each tetrahedron has a tetrahedron vertically opposite at each of its four solid angles. This distribution appears from the consideration that ten pentahedra are required to use the twice fifteen 4-gons furnished by the five planes.

There are two 3-fold hexahedra: the A 6-hedron with two 5-gons, two 4-gons and two 3-gons as faces; and the B 6-hedron having six 4-gons as faces. The unique distribution for six planes is obtained from the distribution for five planes upon the passage of a sixth plane, as follows:

	A 6-hedra	B 6-hedra	5-hedra	4-hedra
A central pentahedron cut with a 5-gon section gives	2			
Two adjacent tetrahedra " 3-gon "			2	2
Two " pentahedra " 3-gon "	2			2
One " pentahedron " 3-gon "			2	
One cornering tetrahedron " 4-gon "			2	
Two " pentahedra " 4-gon "	2		2	
Two " pentahedra " 4-gon "		2	2	
Uncut			2	2
Distribution	6	2	12	6

* Cf. Pilgrim, *loc. cit.*

13. In a 4-flat any four 3-flats meet in a point and divide the 4-flat into eight parts. A fifth 3-flat makes sixteen 4-fold pentahedroids.

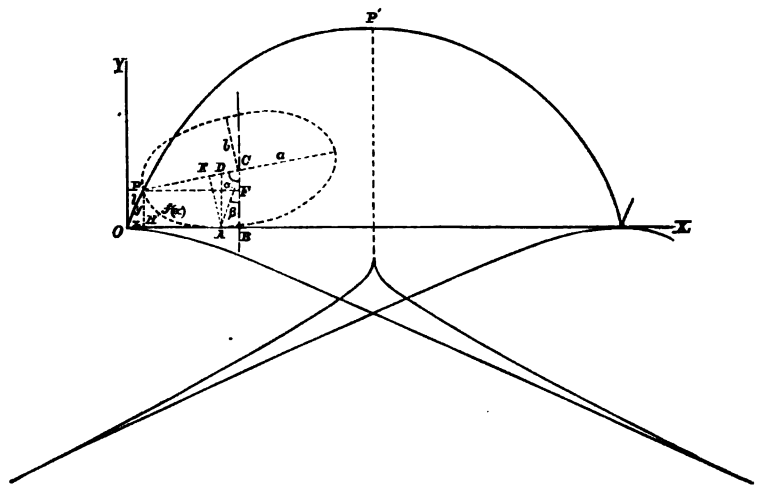
There are but two 4-fold hexahedroids; the A 6_4-hedroid has as faces two 3-fold tetrahedra and four 3-fold pentahedra; the B 6_4-hedroid has as faces six 3-fold pentahedra.

Six 3-flats divide a 4-flat into six 4-fold pentahedroids, ten B 6_4-hedroids and fifteen A 6_4-hedroids, an unique distribution.

NEW HAVEN, CONN., Dec. 1884.

Note on a Roulette.

By A. V. Lane, Ph. D.



Let us imagine an ellipse of semi-axes a and b to roll on a right line, taking one extremity of its major axis as the generating point.

With the notation of the diagram, we have

$$x = OH = OA + AB - PF = \text{arc } PA + r \sin\beta - a \sin\alpha. \quad (1)$$

$$y = HP = CB - CF = r \cos\beta - a \cos\alpha. \quad (2)$$

Now the sub-normal in the ellipse equals $\frac{b^2}{a^2}$ multiplied by the abscissa of the point of contact. $\therefore DE = \frac{b^2}{a^2} CE$. But $\frac{EA}{CE} = \tan(\alpha - \beta)$ and $\frac{EA}{DE} = \tan\alpha$.

$\therefore \frac{\tan(\alpha - \beta)}{\tan\alpha} = \frac{b^2}{a^2}$. Whence

$$\sin\beta = \frac{(a^2 - b^2)\sin\alpha\cos\alpha}{(a^4\cos^2\alpha + b^4\sin^2\alpha)^{\frac{1}{2}}} \quad (3)$$

$$\sin(\alpha - \beta) = \frac{b^2 \sin\alpha}{(a^4\cos^2\alpha + b^4\sin^2\alpha)^{\frac{1}{2}}}. \quad (4)$$

From the equation of the ellipse, we find

$$r=\frac{ab}{[a^2\sin^2(\alpha-\beta)+b^2\cos^2(\alpha-\beta)]^{\frac{1}{2}}}=\frac{(a^4\cos^2\alpha+b^4\sin^2\alpha)^{\frac{1}{2}}}{(a^2\cos^2\alpha+b^2\sin^2\alpha)^{\frac{1}{2}}}. \tag{5}$$

$$\text{The arc } PA=\int_0^{\alpha}\frac{a^2b^2d\alpha}{(a^2\cos^2\alpha+b^2\sin^2\alpha)^{\frac{3}{2}}}=f(\alpha). \tag{6}$$

Making the necessary substitutions from (3)–(6) in (1) and (2) and letting $c^2=a^2-b^2$, we have, as the equation of the roulette,

$$\begin{cases} x=f(\alpha)+\dfrac{c^2\sin\alpha\cos\alpha}{(a^2\cos^2\alpha+b^2\sin^2\alpha)^{\frac{1}{2}}}-a\sin\alpha & (7)\\ y=(a^2\cos^2\alpha+b^2\sin^2\alpha)^{\frac{1}{2}}-a\cos\alpha. & (8)\end{cases}$$

For brevity let ( ) represent $a^2\cos^2\alpha+b^2\sin^2\alpha$. Then

$$\frac{dx}{d\alpha}=(\ )^{\frac{1}{2}}-a\cos\alpha=y, \tag{9}$$

$$\frac{dy}{d\alpha}=-\frac{c^2\sin\alpha\cos\alpha}{(\ )^{\frac{1}{2}}}+a\sin\alpha=f(\alpha)-x, \tag{10}$$

$$\frac{dy}{dx}=\frac{a+\cos\alpha(\ )^{\frac{1}{2}}}{\sin\alpha(\ )^{\frac{1}{2}}}=\frac{f(\alpha)-x}{y}, \tag{11}$$

$$\frac{d^2y}{dx^2}=-\frac{(\ )^{\frac{3}{2}}-a\cos\alpha[a^2-2(\ )]}{\sin^2\alpha(\ )^{\frac{3}{2}}[(\ )^{\frac{1}{2}}-a\cos\alpha]}. \tag{12}$$

$$\text{The radius of curvature } \rho=\pm\frac{[a^2+(\ )+2a\cos\alpha(\ )^{\frac{1}{2}}]^{\frac{3}{2}}[(\ )^{\frac{1}{2}}-a\cos\alpha]}{\sin\alpha\{(\ )^{\frac{3}{2}}-a\cos\alpha[a^2-2(\ )]\}}. \tag{13}$$

$$\rho_0=\rho_{360}=0.\ \rho_{180}=\frac{4a^3}{2a^2-b^2}.\ \rho_{180}-2a=\frac{2ab^2}{2a^2-b^2}.$$

With the same axes and origin, but calling x' and y' the coordinates of the evolute, we obtain its equation by substituting from (7), (8), (11) and (12) in the expressions

$$x'=x-\frac{1+\left(\frac{dy}{dx}\right)^2}{\frac{d^2y}{dx^2}}\cdot\frac{dy}{dx}, \tag{14}$$

$$y'=y+\frac{1+\left(\frac{dy}{dx}\right)^2}{\frac{d^2y}{dx^2}}. \tag{15}$$

Whence the equation of the evolute is

$$\begin{cases} x' = f(\alpha) + \dfrac{a^2b^2 \sin\alpha\,[a + \cos\alpha\,(\ )^{\frac{1}{2}}]}{(\ )^{\frac{1}{2}}\{(\ )^{\frac{3}{2}} - a\cos\alpha\,[a^2 - 2(\ )]\}}, & (16) \\[2ex] y' = -\dfrac{a^2b^2\sin^2\alpha}{(\ )^{\frac{3}{2}} - a\cos\alpha\,[a^2 - 2(\ )]} & (17) \end{cases}$$

$$y'_0 = y'_{360} = 0 . y'_{90} = y'_{270} = -\frac{a^2}{b} \cdot y'_{180} = -\frac{2ab^2}{2a^2 - b^2}.$$

Let [ ] represent $(\ )^{\frac{3}{2}} - a\cos\alpha\,[a^2 - 2(\ )]$.

Let { } represent $\cos\alpha\,(\ )^{\frac{1}{2}}[2a^2 + c^2\sin^2\alpha] - a^3 + 3a^3\cos^2\alpha + 2a\sin^2\alpha\,(\ )$.

Then
$$\frac{dx'}{d\alpha} = a^2b^2\,\frac{[a + \cos\alpha\,(\ )^{\frac{1}{2}}]}{(\ )^{\frac{1}{2}}} \cdot \frac{\{\ \}}{[\ ]^2}, \tag{18}$$

$$\frac{dy'}{d\alpha} = -a^2b^2\sin\frac{\{\ \}}{[\ ]^2}, \tag{19}$$

$$\frac{d^2y'}{d\alpha^2} = -a^2b^2\,\frac{\cos\alpha\,[\ ]\{\ \} + \sin\alpha\,[\ ]\dfrac{d\{\ \}}{d\alpha} - 2\sin\alpha\,\{\ \}\dfrac{d[\ ]}{d\alpha}}{[\ ]^3}, \tag{20}$$

$$\frac{dy'}{dx'} = -\frac{1}{\dfrac{dy}{dx}} = -\frac{\sin\alpha\,(\ )^{\frac{1}{2}}}{a + \cos\alpha\,(\ )^{\frac{1}{2}}}, \tag{21}$$

$$\frac{d^2y'}{dx'^2} = -\frac{[\ ]^3}{a^2b^2[a + \cos\alpha\,(\ )^{\frac{1}{2}}]^3\{\ \}}. \tag{22}$$

To determine maximum and minimum values of y', we put $(19) = 0$. Whence $\sin\alpha = 0$, $\cos\alpha = \pm 1$; also $\{\ \} = 0$. This last gives (letting $w = \cos^2\alpha$) a biquadratic in w, two of whose roots are unity. Dividing by $(w - 1)^2$, we have the quadratic

$$w^2 - \frac{8a^4 - 3a^2b^2 - b^4}{c^2(3a^2 + b^2)}\,w + \frac{a - 4a^4b^2 + 4a^2b^4}{c^4(3a^2 + b^2)} = 0,$$

whose roots are

$$\frac{1}{2c^2(3a^2 + b^2)}\{8a^4 - 3a^2b^2 - b^4 \pm \sqrt{52a^8 - 4a^6b^2 - 39a^4b^4 - 10a^2b^6 + b^8}\} = \frac{1}{P}\{M \pm \sqrt{N}\}.$$

$\frac{M}{P} > 1$, and the plus sign cannot therefore be used before the radical as it would make $\cos\alpha > 1$. $M > 0$, $M - \sqrt{N}$ is always real, equals 0 when $\frac{a}{b} = \sqrt{2}$, and, for all other values of this ratio, is positive. Let $\frac{a}{b} = k$ and $+\frac{1}{2}\sqrt{1 + \sqrt{13}}$ $= +1.073027+ = \mu$. Then $w < 1$ for values of $k > \mu$, equals 1 when k equals μ and equals 0 when $k = \sqrt{2}$; giving in these cases possible values to $\cos\alpha$. But when $k < \mu$, $w > 1$ and $\cos\alpha$ impossible. So the values of y' to be investigated are those corresponding to

$$\cos\alpha = \pm 1 \text{ (from } \sin\alpha = 0\text{)},$$

$\cos\alpha = -1$ (derived from $(w-1)^2 = 0$ or from $w = \frac{1}{P}\{M - \sqrt{N}\}$ when $k = \mu$; $\cos\alpha = +1$ will not reduce $\{\ \}$ to 0),

$$\cos\alpha = -\sqrt{\frac{M-\sqrt{N}}{P}} \text{ when } k > \mu \text{ and } < \sqrt{2},$$

$$\cos\alpha = 0 \text{ when } k = \sqrt{2},$$

$$\cos\alpha = +\sqrt{\frac{M-\sqrt{N}}{P}} \text{ when } k > \sqrt{2},$$

approaching the limit $+\sqrt{\frac{4-\sqrt{13}}{3}} = +0.362606$—as k approaches ∞.

When $\cos\alpha = +1$, $\frac{d^2y'}{da^2} = -\frac{b^2}{a}$ and $y' = 0$ is a maximum; or, all other values of y' being negative, this is *numerically* a minimum.

When $\cos\alpha = -1$, $\frac{d^2y'}{da^2} = -\frac{b}{k}\,\frac{4k^4-2k^2-3}{(2k^2-1)^2}$ and $y' = -\frac{2ab^2}{2a^2-b^2}$ is numerically a maximum when $k < \mu$ and a minimum when $k > \mu$. When $k = \mu$, $\frac{d^2y'}{da^2} = 0$. But an investigation of the sign of $\frac{d^3y'}{da^3}$ shows it to be always negative and therefore the evolute always convex to the axis of X, for values of $k =$ or $< \mu$. This, combined with the fact that (21) shows the tangent to be perpendicular to the axis of X when $\cos\alpha = -1$, whatever the value of k, proves y' numerically a maximum when $\cos\alpha = -1$, $k = \mu$.

Omitting from (20) terms containing $\{\ \}$ as a factor, we have

$$\left[\frac{d^2y'}{da^2}\right]_{\{\ \}=0} = a^2b^3\sin\alpha^2.\ \frac{\cos\alpha(\ )^{\frac{1}{2}}[2a^3+8ac^2\sin^2\alpha]+b^2(3a^2-b^2)+(5c^2+a^2)c^2\cos^2\alpha-4c^4\cos^4\alpha}{(\ )^{\frac{1}{2}}[\ ]^2}.$$

This is clearly positive for positive values of $\cos\alpha$ and for $\cos\alpha = 0$, $k = \sqrt{2}$. It is also positive when $\cos\alpha$ is negative, k lying between $\sqrt{2}$ and μ.

Hence, for all values of $\cos\alpha$ derived from $\{\ \} = 0$, y' is numerically a maximum ($\cos\alpha < 0$, $k = \mu$ has already been shown to give such a value to y'). This maximum disappears when $k =$ or $< \mu$, because the value derived from $\{\ \} = 0$ for $\cos\alpha$ is then > 1.

The length of the half branch of the evolute may be found by substituting in (13) that value of α which makes y' a numerical maximum, doubling and then subtracting ρ_{180}.

The limiting forms of the roulette are the cycloid ($k = 1$) and the semicircle ($k = \infty$); of the evolute, a cycloid (equal to the involute) and a point (the centre of the involute).

To rectify the roulette—

Substituting in the formula $dl = dx\sqrt{1+\left(\frac{dy}{dx}\right)^2}$, we find

$$dl = \left[\frac{b^2(a^2+b^2)-(b^4+a^2b^2-2a^4)\cos^2\alpha}{(\ )} - 2a^3\cos\alpha\,(\ )^{-\frac{1}{2}}\right]^{\frac{1}{2}} d\alpha.$$

Let $\cos\alpha = u$, $d\alpha = -\frac{du}{\sqrt{1-u^2}}$, $f = b^2(a^2+b^2)$, $g = b^4 + a^2b^2 - 2a^4$, $h = a^2 - 2b^2$; then

$$dl = -\left[\frac{f-gu^2}{b^2+hu^2-c^2u^4} - 2a^3.\frac{u}{1-u^2}(b^2+c^2u^2)^{-\frac{1}{2}}\right]^{\frac{1}{2}} du.$$

Now $$\frac{f-gu^2}{b^2+hu^2-c^2u^4} = A + Cu^2 + Eu^4 + \text{etc.}$$

where $$A = \frac{f}{b^2},\ C = -\frac{1}{b^2}\left(g + \frac{fh}{b^2}\right),$$

$$E = \frac{1}{b^4}\left(c^2f + gh + \frac{fh^2}{b^2}\right), \text{ etc.}$$

The n^{th} coefficient of this series may be formed thus—first considering the minus signs as plus, add together $\frac{b^2c^2}{b^4}$ times the $(n-2)^{\text{th}}$ coefficient and $\frac{h}{b^2}$ times the $(n-1)^{\text{th}}$ coefficient and multiply this sum by $(-1)^{n-1}$. Again, $(b^2+c^2u^2)^{-\frac{1}{2}} = b^{-1} - \frac{1}{2}b^{-3}c^2u^2 + \frac{3}{8}b^{-5}c^4u^4 - \text{etc.}$, and $-2a^3.\frac{u}{1-u^2}(b^2+c^2u^2)^{-\frac{1}{2}} = Bu + Du^3 + Fu^5 + \text{etc.}$, where $B = -2a^3b^{-1}$, $D = -2a^3(-\frac{1}{2}b^{-3}c^2 + b^{-1})$, $F = -2a^3(\frac{3}{8}b^{-5}c^4 - \frac{1}{2}b^{-3}c^2 + b^{-1})$, etc.

The n^{th} coefficient of this series may be formed by multiplying $-2a^3$ by the sum of n coefficients of the preceding series.

We may now write $$dl = -(A+R)^{\frac{1}{2}}du,$$
where $$R = Bu + Cu^2 + Du^3 + \text{etc.}$$
Expanding and substituting for R its value, we have

$$dl = A'du + B'udu + C'u^2du + D'u^3du + E'u^4du + \text{etc.}$$

where $A' = -A^{\frac{1}{2}}$, $B' = -\frac{1}{2}A^{-\frac{1}{2}}B$, $C' = -\frac{1}{2}A^{-\frac{1}{2}}C + \frac{1}{8}A^{-\frac{3}{2}}B^2$,

$$D' = -\frac{1}{2}A^{-\frac{1}{2}}D + \frac{1}{8}A^{-\frac{3}{2}}(2BC) - \frac{1}{16}A^{-\frac{5}{2}}B^3,$$

$$E' = -\frac{1}{2}A^{-\frac{1}{2}}E + \frac{1}{8}A^{-\frac{3}{2}}(2BD+C^2) - \frac{1}{16}A^{-\frac{5}{2}}(3B^2C) + \frac{5}{128}A^{-\frac{7}{2}}B^4, \text{ etc.}$$

Integrating and determining the constant from $\alpha = 0$, $u = +1$, $l = 0$, we have

$$l = A'u + \frac{B'u^2}{2} + \frac{C'u^3}{3} + \frac{D'u^4}{4} + \frac{E'u^5}{5} + \text{etc.} - A' - \frac{B'}{2} - \frac{C'}{3} - \frac{D'}{4} - \frac{E'}{5} - \text{etc.}$$

For $\alpha = \Pi$, $u = -1$, $l = \text{arc } OP' = \frac{1}{2}$ branch. Then arc $OP' = -2\left[A' + \frac{C'}{3} + \frac{E'}{5} + \text{etc.}\right]$.

Calling S the area included between a portion of the curve, beginning at the origin of coordinates, the X axis and the ordinate, we have

$$S=\int ydx=\int y^2da$$
$$=2a^2\int\cos^2ada+b^2\int\sin^2ada-2a\int(\;)^{\frac{1}{2}}\cos ada.$$

The first integral equals $\frac{1}{2}\alpha+\frac{1}{4}\sin 2\alpha$, the second equals $\frac{1}{2}\alpha-\frac{1}{4}\sin 2\alpha$.

To integrate the last term—expanding we have

$$(\;)^{\frac{1}{2}}\cos ada=b\cos ada+\frac{c^2\cos^3ada}{2b}-\frac{c^4\cos^5ada}{8b^3}+\text{etc.}$$
$$=b\cos ada+\frac{c^2}{2^3b}(\cos 3ada+3\cos ada)$$
$$-\frac{c^4}{2^7b^3}(\cos 5ada+5\cos 3ada+10\cos ada)+\text{etc.}$$

Integrating this and substituting, we have $S=\frac{1}{2}(2a^2+b^2)\alpha+\frac{1}{4}(2a^2-b^2)$ $\sin 2\alpha$

$$-2a\left[b\sin\alpha+\frac{c^2}{2^3b}(\tfrac{1}{3}\sin 3\alpha+3\sin\alpha)-\frac{c^4}{2^7b^3}(\tfrac{1}{5}\sin 5\alpha+\tfrac{5}{3}\sin 3\alpha+10\sin\alpha)+\text{etc.}\right]$$

Making $\alpha=2\Pi$, we have the area for one branch equal to $\Pi(2a^2+b^2)$.

Calling V the volume generated by the revolution of S about the X-axis, we have

$$V=\Pi\int y^2dx=\Pi\int y^3da$$
$$=\Pi\int[(\;)^{\frac{3}{2}}da-3ab^2\cos ada-a(3c^2+a^2)\cos^3ada+3a^2(\;)^{\frac{1}{2}}\cos^2ada].$$

Expanding the irrational expressions and transforming the higher powers of $\cos\alpha$ to cosines of multiples of α as before, we find

$$\frac{V}{\Pi}=\frac{b}{4}(3c^2+4b^2)\alpha+\frac{3bc^2\sin 2\alpha}{2^3}+\frac{3c^4}{2^6b}(\tfrac{1}{4}\sin 4\alpha+\tfrac{4}{2}\sin 2\alpha+3\alpha)-\text{etc.}$$
$$-3ab^2\sin\alpha-\frac{3ac^2+a^3}{2^2}(\tfrac{1}{3}\sin 3\alpha+3\sin\alpha)$$
$$+3a^2\left[\tfrac{1}{2}b\alpha+\tfrac{1}{4}b\sin 2\alpha+\frac{c^2}{2^4b}(\tfrac{1}{4}\sin 4\alpha+\tfrac{4}{2}\sin 2\alpha+3\alpha)\right.$$
$$\left.-\frac{c^4}{2^8b^3}(\tfrac{1}{6}\sin 6\alpha+\tfrac{6}{4}\sin 4\alpha+\tfrac{15}{2}\sin 2\alpha+10\alpha)+\text{etc.}\right].$$

For $\alpha=2\Pi$, we have the volume thus generated by the area of one branch equal to

$$\Pi^2\left[\tfrac{1}{2}b(3a^2+b^2)+\frac{18c^4}{2^6b}-\text{etc.}+3a^2\left(b+\frac{bc^2}{2^4b}-\frac{20c^4}{2^8b^3}+\text{etc.}\right)\right].$$

UNIVERSITY OF TEXAS, AUSTIN, July, 1885.

The Cubi-quadric System.

"A familiar demonstration of the working."
(*Love's Labor Lost*, Act I, Scene 2.)

BY J. HAMMOND.

THE CUBI-QUADRIC SYSTEM.

1. The fifteen covariants of this system are taken from Salmon's Higher Algebra (3d ed. p. 178), but the cubic is here reduced to its canonical form by writing zero for both b and c in Salmon's formulae. These are arranged according to their degree in the coefficients of the Quadric instead of their order in the variables, a uniform notation is introduced, and what must be considered (for the present purpose) simpler forms are used instead of L_4 and R.

The two notations are given connectively below.

Salmon's	The Present Notation.
u	$A_{13} = ax^3 + dy^3$
$H =$ Hessian of u	$A_{22} = adxy$
Q	$A_{33} = ad(ax^3 - dy^3)$
D	$A_{40} = a^2d^2$
v	$B_{02} = Ax^2 + 2Bxy + Cy^2$
L_1	$B_{11} = aCx + dAy$
Jacobian of u, v	$B_{13} = -B(ax^3 - dy^3) - xy(aCx - dAy)$
I	$B_{20} = -adB$
Jacobian of v, H	$B_{22} = ad(Ax^2 - Cy^2)$
L_3	$B_{31} = ad(aCx - dAy)$
Δ	$C_{00} = AC - B^2$
L_2	$C_{11} = B(aCx - dAy) - (dA^2x - aC^2y)$
$L_4 + IL_1$	$C_{31} = ad(dA^2x + aC^2y)$
$*L = R + 8\Delta I$	$D_{20} = a^2C^3 + d^2A^3 - 2adABC$
M	$D_{40} = ad(a^2C^3 - d^2A^3)$

* R is the resultant of the cubic u and quadric v, L the resultant of the two linears L_1 and L_2.

Here the suffixed letters show the degree in the coefficients of the Quadric, and their suffixes denote the degree in the coefficients of the Cubic and order in the variables; thus C_{31} is of the second degree in the coefficients of the Quadric, of the third degree in those of the Cubic, and linear in the variables.

2. The following syzygies may, by the use of the canonical form, be easily verified.*

No.	deg. deg. order.	
		No. (1), not involving groundforms in which the coefficients of the quadric appear.
(1)	0.6.6	$A_{13}^2 A_{40} - 4A_{22}^3 - A_{33}^2 = 0.$
		Nos. (2) to (7), of degree 1 in the coefficients of the quadric.
(2)	1.3.5	$A_{13}B_{22} - A_{33}B_{02} - 2A_{22}B_{13} = 0$
(3)	1.4.4	$A_{13}B_{31} - A_{33}B_{11} + 2A_{22}B_{22} = 0$
(4)	1.4.6	$A_{13}^2 B_{20} + 2A_{22}^2 B_{02} - A_{13}A_{22}B_{11} - A_{33}B_{13} = 0$
(5)	1.5.3	$A_{40}B_{13} + A_{22}B_{31} - A_{33}B_{20} = 0$
(6)	1.5.5	$A_{13}A_{40}B_{02} + 2A_{13}A_{22}B_{20} - 2A_{22}^2 B_{11} - A_{33}B_{22} = 0$
(7)	1.6.4	$A_{13}A_{40}B_{11} - 2A_{22}A_{40}B_{02} - 4A_{22}^2 B_{20} - A_{33}B_{31} = 0$
		Nos. (8) to (20), of degree 2 in the coefficients of the quadric.
(8)	2.2.4	$A_{13}C_{11} + B_{02}B_{22} + B_{11}B_{13} = 0$
(9)	2.2.6	$A_{13}^2 C_{00} - A_{13}B_{02}B_{11} + B_{02}^2 A_{22} + B_{13}^2 = 0$
(10)	2.3.3	$A_{33}C_{00} - B_{02}B_{31} + B_{13}B_{20} + A_{22}C_{11} = 0$
(11)	2.3.3	$B_{11}B_{22} - B_{02}B_{31} + 2A_{22}C_{11} = 0$
(12)	2.3.5	$A_{13}B_{02}B_{20} - 2A_{13}A_{22}C_{00} + A_{22}B_{02}B_{11} - B_{13}B_{22} = 0$
(13)	2.4.4	$A_{13}C_{31} - A_{22}B_{11}^2 - B_{22}^2 = 0$
(14)	2.4.4	$A_{33}C_{11} + B_{13}B_{31} + B_{22}^2 = 0$
(15)	2.4.4	$A_{13}C_{31} - A_{13}B_{11}B_{20} - 2A_{22}B_{02}B_{20} - A_{40}B_{02}^2 + B_{13}B_{31} = 0$
(16)	2.4.4	$A_{40}B_{02}^2 - 4A_{22}^2 C_{00} + 4A_{22}B_{02}B_{20} - B_{22}^2 = 0$
(17)	2.5.3	$2A_{22}B_{11}B_{20} - 2A_{22}C_{31} + A_{40}B_{02}B_{11} - B_{22}B_{31} = 0$

* Only nine of these forty-four syzygies are fundamental, in the sense in which the word is used on p. 342, Vol. VII of the *American Journal of Mathematics*. For we know that any other covariant can be expressed in the form $\phi(A_{13}, A_{22}, A_{33}, B_{02}, B_{13}, C_{00})$, see Sylvester, Vol. I, p. 119, in this Journal. The nine fundamental syzygies are therefore those in which the remaining nine covariants appear for the first time. Or it may be simpler to say that Nos. (1) to (5) inclusive are fundamental, since from them it is possible to obtain each of the five covariants A_{40}, B_{11}, B_{22}, B_{20}, and B_{31}, in the form $\phi(A_{13}, A_{22}, A_{33}, B_{02}, B_{13}, C_{00})$, and that Nos. (8), (13), (23), and (27) are also fundamental.

All the non-fundamental can be deduced from the fundamental, when these are known, without any direct reference to the actual values of the covariants, *e. g.* from the three fundamental syzygies Nos. (2), (3) and (8) we have

$$B_{11}(2) - B_{02}(3) + 2A_{22}(8) = B_{11}(A_{13}B_{22} - A_{33}B_{02} - 2A_{22}B_{13}) - B_{02}(A_{13}B_{31} - A_{33}B_{11} + 2A_{22}B_{22}) + 2A_{22}(A_{13}C_{11} + B_{02}B_{22} + B_{11}B_{13}) = A_{13}(B_{11}B_{22} - B_{02}B_{31} + 2A_{22}C_{11})$$

from which the non-fundamental No. (11) is obtained.

(18)	2.5.3	$A_{13}A_{40}C_{00}+A_{13}B_{20}^2-A_{22}C_{31}-B_{22}B_{31}=0$
(19)	2.6.2	$A_{40}B_{11}^2-4A_{22}A_{40}C_{00}-4A_{22}B_{20}^2-B_{31}^2=0$
(20)	2.6.4	$A_{22}B_{11}B_{31}+2A_{22}B_{20}B_{22}+A_{40}B_{02}B_{22}-A_{33}C_{31}=0.$

No.	deg. deg. order.	Nos. (21) to (32), of degree 3 in the coefficients of the quadric.
(21)	3.2.4	$A_{13}B_{11}C_{00}+2A_{22}B_{02}C_{00}-B_{02}^2B_{20}-B_{02}B_{11}^2-B_{13}C_{11}=0$
(22)	3.3.3	$2A_{22}B_{11}C_{00}-B_{02}B_{11}B_{20}-B_{02}C_{31}-B_{22}C_{11}=0$
(23)	3.3.3	$A_{13}D_{20}-B_{11}^3+2B_{22}C_{11}+B_{02}C_{31}=0$
(24)	3.4.2	$A_{22}D_{20}+A_{40}B_{02}C_{00}+B_{02}B_{20}^2-B_{11}C_{31}=0$
(25)	3.4.2	$2A_{22}D_{20}-B_{11}^2B_{20}-B_{11}C_{31}-B_{31}C_{11}=0$
(26)	3.4.4	$A_{22}B_{11}C_{11}+2A_{22}B_{22}C_{00}-B_{02}B_{20}B_{22}+B_{13}C_{31}=0$
(27)	3.5.3	$A_{13}D_{40}-B_{11}^2B_{31}+B_{22}C_{31}=0$
(28)	3.5.3	$4A_{22}B_{20}C_{11}-2A_{22}B_{31}C_{00}+A_{40}B_{02}C_{11}+B_{11}B_{20}B_{22}+B_{22}C_{31}=0$
(29)	3.5.3	$A_{33}D_{20}-2A_{40}B_{13}C_{00}-B_{11}^2B_{31}-2B_{20}^2B_{13}-B_{22}C_{31}=0$
(30)	3.6.2	$A_{22}D_{40}+A_{40}B_{22}C_{00}+B_{20}^2B_{22}-B_{31}C_{31}=0$
(31)	3.6.2	$2A_{22}D_{40}-A_{40}B_{11}C_{11}-B_{11}B_{20}B_{31}-B_{31}C_{31}$
(32)	3.7.3	$2A_{22}A_{40}B_{11}C_{00}+2A_{22}B_{20}C_{31}+2A_{22}B_{11}B_{20}^2-A_{40}B_{11}^3+A_{40}B_{02}C_{31}+A_{33}D_{40}=0.$
		Nos. (33) to (41), of degree 4 in the coefficients of the quadric.
(33)	4.2.2	$B_{02}D_{20}-B_{11}^2C_{00}-C_{11}^2=0$
(34)	4.3.3	$2A_{22}C_{00}C_{11}-B_{02}B_{20}C_{11}-B_{11}^2C_{11}-B_{11}B_{22}C_{00}-B_{13}D_{20}=0$
(35)	4.4.2	$B_{02}D_{40}+B_{11}B_{20}C_{11}-B_{11}B_{31}C_{00}-C_{11}C_{31}=0$
(36)	4.4.2	$B_{02}D_{40}-B_{22}D_{20}-2C_{11}C_{31}=0$
(37)	4.5.1	$2A_{40}C_{00}C_{11}-B_{11}D_{40}+2B_{20}^2C_{11}+B_{31}D_{20}=0$
(38)	4.5.3	$B_{11}B_{31}C_{11}-B_{20}B_{22}C_{11}+B_{13}D_{40}+B_{22}B_{31}C_{00}=0$
(39)	4.6.2	$A_{40}B_{11}^2C_{00}+B_{11}^2B_{20}^2-B_{22}D_{40}-C_{31}^2=0$
(40)	4.6.2	$A_{40}C_{11}^2+2B_{20}B_{31}C_{11}+B_{22}D_{40}-B_{31}^2C_{00}=0$
(41)	4.7.1	$2A_{40}B_{11}B_{20}C_{00}-A_{40}B_{11}D_{20}+2A_{40}C_{00}C_{31}+2B_{20}^3B_{11}+2B_{20}^2C_{31}+B_{31}D_{40}=0.$
		Nos. (42) and (43), of degree 5 in the coefficients of the quadric, and linear in the variables.
(42)	5.5.1	$2A_{40}B_{11}C_{00}^2+2B_{11}B_{20}^2C_{00}+B_{11}B_{20}D_{20}+C_{11}D_{40}-C_{31}D_{20}=0$
(43)	5.7.1	$2A_{40}B_{31}C_{00}^2+A_{40}C_{11}D_{20}-B_{11}B_{20}D_{40}+2B_{20}^2B_{31}C_{00}+2B_{20}B_{31}D_{20}-C_{31}D_{40}=0.$
		No. (44), invariant, of degree 6 in the coefficients of the quadric.
(44)	6.8.0	$4A_{40}^2C_{00}^3+8A_{40}B_{20}^2C_{00}^2+4A_{40}B_{20}C_{00}D_{20}-A_{40}D_{20}^2+4B_{20}^4C_{00}+4B_{20}^3D_{20}+D_{40}^2=0.$

3. The Numerical Generating Function for the Cubi-quadric System, given by Professor Sylvester (*American Journal of Mathematics*, Vol. II, p. 295), is the fraction $\frac{N}{D}$; where

$$D=(1-b^2)(1-c^4)(1-bc^2)(1-b^3c^2)(1-bx^2)(1-c^2x^2)(1-cx^2)$$

$$\begin{aligned}N=1&+c^3x^3+bcx+bcx^3+bc^2x^2+bc^3x+b^2cx+b^2c^2x^2+b^2c^3x+b^2c^4x^2+b^3c^2x^2+b^3c^4\\&-bc^3x^5-bc^5x^3-b^2c^3x^3-b^2c^4x^4-b^2c^5x^3-b^2c^6x^4-b^3c^4x^4-b^3c^5x^3-b^3c^6x^4\\&-b^3c^6x^2-b^4c^4x^2-b^4c^7x^5.\end{aligned}$$

It is not difficult to find a Real Generating Function similar to that obtained for the Quintic by Prof. Cayley in his Tenth Memoir on Quantics; viz. we have

$$\text{(I)}\ldots\quad\frac{\begin{array}{l}1-A_{22}^2D_{20}\\+A_{33}(1-D_{20})\\+B_{11}(1-A_{22}^2D_{20})\\+B_{13}(1-A_{22})\\+B_{23}(1-D_{20})\\+B_{31}(1-A_{22})\\+C_{11}(1-A_{22})\\+B_{11}^2(1-A_{22})\\+C_{31}(1-A_{22})\\+B_{11}B_{31}(1-A_{22})\\+B_{11}C_{11}(1-A_{22})\\+D_{40}(1-A_{22})\end{array}}{(1-C_{00})(1-A_{40})(1-B_{20})(1-D_{20})(1-B_{02})(1-A_{22})(1-A_{13})}\quad\text{corresponding to}\quad\frac{\begin{array}{l}1-b^3c^6x^4\\+c^3x^3-b^3c^5x^3\\+bcx-b^4c^7x^5\\+bcx^3-bc^3x^5\\+bc^2x^2-b^4c^4x^2\\+bc^3x-bc^5x^3\\+b^2cx-b^2c^3x^3\\+b^2c^2x^2-b^2c^4x^4\\+b^2c^3x-b^2c^5x^3\\+b^2c^4x^2-b^2c^6x^4\\+b^3c^2x^2-b^3c^4x^4\\+b^3c^4-b^3c^6x^2\end{array}}{\text{Den}^{\text{r}}.\ D\text{ as above.}}$$

Reasoning precisely similar to that employed by Prof. Cayley in the memoir just mentioned will show that each of the 12 negative terms in the numerator corresponds to a syzygy connecting terms such as $P\phi(C_{00}, A_{40}\ldots.A_{13})$, where P is any one of the twelve positive terms of the numerator and C_{00}, A_{40}, etc. are the seven denominator forms; and that there are besides these 12 syzygies 36 others. We should therefore expect to find 48 irreducible syzygies in all; but it will be found that 4 of the 12 syzygies, represented in the numerator, are in fact reducible, so that the true number is 44, as given in the preceding article. The four reducible syzygies are those corresponding to the terms

$-A_{22}^2D_{20}$	deg.	deg.	order	3.6.4
$-A_{22}^2B_{11}D_{20}$	"	"	"	4.7.5
$-A_{22}B_{11}B_{31}$	"	"	"	2.6.4
$-A_{22}B_{11}C_{11}$	"	"	"	3.4.4

The first of these may be calculated as follows: The only compounds of deg. deg. order 3.6.4 which contain the twelve positive terms of the numerator in the first degree are found to be these

$$A_{22}^2B_{20}C_{00},\ A_{22}^2D_{20},\ A_{22}A_{40}B_{02}C_{00},\ A_{22}B_{02}B_{20}^2,\ A_{40}B_{02}^2B_{20}$$
$$B_{11}\,.\,A_{13}A_{40}C_{00},\ B_{11}\,.\,A_{13}B_{20}^2$$
$$B_{11}^2\,.\,A_{22}B_{20},\ B_{11}^2\,.\,A_{40}B_{02}$$
$$C_{31}\,.\,A_{13}B_{20}.$$

Then in

$$\alpha A_{22}^2B_{20}C_{00}+\beta A_{22}^2D_{20}+\gamma A_{22}A_{40}B_{02}C_{00}+\ldots=0,$$

the values of the covariants, given in Art. 1, are substituted, and the numerical factors α, β, γ, . . . are determined by elementary algebra.

In this way we find the syzygy

$$(a)\qquad A_{22}^2D_{20}+A_{22}A_{40}B_{02}C_{00}+A_{22}B_{02}B_{20}^2+A_{13}A_{40}B_{11}C_{00}+A_{13}B_{11}B_{20}^2 \\ -2A_{22}B_{11}^2B_{20}-A_{40}B_{02}B_{11}^2=0.$$

But from Art. 2 we obtain

$$\begin{aligned} A_{22}(24)-B_{11}(17)+B_{11}(18) &= A_{22}(A_{22}D_{20}+A_{40}B_{02}C_{00}+B_{02}B_{20}^2-B_{11}C_{31}) \\ &\quad -B_{11}(2A_{22}B_{11}B_{20}-2A_{22}C_{31}+A_{40}B_{02}B_{11}-B_{22}B_{31}) \\ &\quad +B_{11}(A_{13}A_{40}C_{00}+A_{13}B_{20}^2-A_{22}C_{31}-B_{22}B_{31}) \\ &= (a). \end{aligned}$$

The remaining three syzygies are

$$(b)\quad 2A_{13}A_{22}B_{20}D_{20}+A_{13}A_{40}B_{02}D_{20}+8A_{22}^2B_{11}B_{20}C_{00}-A_{22}^2B_{11}D_{20}+3A_{22}A_{40}B_{02}B_{11}C_{00} \\ -5A_{22}B_{02}B_{11}B_{20}^2-2A_{40}B_{02}^2B_{11}B_{20}-A_{13}A_{40}B_{11}^2C_{00}-A_{13}B_{11}^2B_{20}^2 \\ -2A_{22}B_{02}B_{20}C_{31}-A_{40}B_{02}^2C_{31}=0,$$

$$(c)\quad 2A_{22}B_{20}B_{22}+A_{40}B_{02}B_{22}+A_{13}B_{20}B_{31}+A_{13}A_{40}C_{11}-A_{22}B_{11}B_{31}=0,$$

$$(d)\quad 2A_{22}B_{22}C_{00}-B_{02}B_{20}B_{22}+A_{13}B_{31}C_{00}-A_{13}B_{20}C_{11}-B_{02}B_{11}B_{31}+A_{22}B_{11}C_{11}=0.$$

And the four reducing equations are

$$(a)=A_{22}(24)-B_{11}(17)+B_{11}(18)$$
$$(b)=2A_{22}B_{20}(23)+4A_{22}B_{20}(22)+A_{40}B_{02}(23)+2A_{40}B_{02}(22)-B_{11}(a)$$
$$(c)=A_{40}(8)-B_{11}(5)+B_{20}(3)$$
$$(d)=B_{11}(10)-B_{20}(8)+C_{00}(3),$$

where the first of these has been given above and is here repeated.

4. The twelve negative terms in the numerator of the Real Generating Function have necessarily been selected in a very arbitrary manner; for it is possible to write most of the syzygies in a great many different forms, and, after we have decided on one of these forms, any term of the syzygy might be selected to represent it. Thus, leaving the other eight as they are, we may take for the four reducible syzygies

$$(a)' = B_{11}(17) - B_{11}(18)$$
$$(b)' = B_{11}(a)'$$
$$(c)' = B_{11}(5)$$
$$(d)' = B_{11}(10)$$

which only differ from them in this, that those portions of them which are simply divisible by any of the denominator covariants have been rejected.

There is now no term $A_{22}^2 D_{20}$ in $(a)'$ and no term $A_{22}^2 B_{11} D_{20}$ in $(b)'$, so that these will have to be replaced by others. Moreover since (17)–(18) is properly represented by $-A_{22}C_{31}$, the proper representatives of $(a)'$ and $(b)'$ are $-A_{22}B_{11}C_{31}$ and $-A_{22}B_{11}^2 C_{31}$.

The altered numerator will be

$$\begin{aligned} & 1 - A_{22}B_{11}C_{31} \\ & + A_{33}(1 - D_{20}) \\ & + B_{11}(1 - A_{22}B_{11}C_{31}) \\ & + \ldots\ldots\ldots\ldots \end{aligned}$$

in which the remaining terms of (I) have undergone no change.

Both numerator and denominator of the Real Generating Function, after it has thus been altered, are multiplied by $1 - B_{11}$; when the transformed numerator becomes

$$\text{(II)} \ldots \quad \begin{array}{lll} 1 & - B_{11}^3 - A_{22}B_{11}^2 & + A_{22}B_{11}^3 \\ + A_{33} & - A_{33}B_{11} - A_{33}D_{20} & + A_{33}B_{11}D_{20} \\ + B_{13} & - B_{11}B_{13} - A_{22}B_{13} & + A_{22}B_{11}B_{13} \\ + B_{23} & - B_{11}B_{23} - B_{22}D_{20} & + B_{11}B_{23}D_{20} \\ + B_{31} & - B_{11}^2B_{31} - A_{22}B_{31} & + A_{22}B_{11}^2B_{31} \\ + C_{11} & - B_{11}^2C_{11} - A_{22}C_{11} & + A_{22}B_{11}^2C_{11} \\ + C_{31} & - B_{11}C_{31} - A_{22}C_{31} & + A_{22}B_{11}^3C_{31} \\ + D_{40} & - B_{11}D_{40} - A_{22}D_{40} & + A_{22}B_{11}D_{40} \end{array}$$

The corresponding transformation of the Numerical Generating Function is obtained by simply multiplying both N and D by $1 - bcx$, when we have

$$\text{(III)} \quad N(1 - bcx) = \begin{array}{llll} 1 & - b^3c^3x^3 & - b^2c^4x^4 & + b^3c^5x^5 \\ + c^3x^3 & - bc^4x^4 & - b^3c^5x^3 & + b^4c^6x^4 \\ + bcx^3 & - b^2c^3x^4 & - bc^3x^5 & + b^2c^4x^6 \\ + bc^2x^2 & - b^3c^3x^3 & - b^4c^4x^2 & + b^5c^5x^3 \\ + bc^3x & - b^3c^5x^3 & - bc^5x^3 & + b^3c^7x^5 \\ + b^2cx & - b^4c^3x^3 & - b^2c^3x^3 & + b^4c^5x^5 \\ + b^2c^3x & - b^3c^4x^2 & - b^2c^5x^3 & + b^5c^8x^6 \\ + b^3c^4 & - b^4c^5x & - b^3c^6x^2 & + b^4c^7x^3 \end{array}$$

where the arrangement of the terms is the same as in (II).

The covariant B_{11} has now been transferred from the numerator into the denominator, and is henceforth to be reckoned among the denominator forms; while in consequence of this transference all the covariants, and all the syzygies, divisible by B_{11} have also disappeared from the numerator. Seven terms remain in the first column, or first positive block, of (II) or (III), all of them representing groundforms; the sixteen terms in the negative block all represent irreducible syzygies; and the existence of second syzygies is manifested by the appearance of a second positive block. There is reason to believe that one of the indicated second syzygies is compound, the remaining seven can be proved to be irreducible.

5. On account of the arbitrary selection of the negative terms in (I), previously mentioned, it is clear that, at any stage of the work, we may replace any combination of the 15 letters $A_{13} \dots D_{40}$ by a properly chosen combination in syzygy with it. Using this principle to obtain a further transformation of II, we have

B_{11}^3	in syzygy with		$A_{13}D_{20}$,	No. (23)
$A_{22}B_{11}^2$	"	"	$A_{13}C_{31}$,	No. (13)
$B_{11}^2B_{31}$	"	"	$A_{13}D_{40}$,	No. (27)
$B_{11}^2C_{11}$	"	"	$B_{13}D_{20}$,	No. (34)
$B_{11}B_{13}$	"	"	$A_{13}C_{11}$,	No. (8)
$B_{11}C_{31}$	"	"	$A_{22}D_{20}$,	No. (24)
$B_{11}D_{40}$	"	"	$B_{31}D_{20}$,	No. (37).

The indicated substitutions may be performed so as to obtain

(IV) . . .
$$\begin{aligned} &1 - A_{13}D_{20} - A_{22}D_{20} + A_{13}A_{22}D_{20} \\ &+ A_{33} - A_{33}B_{11} - A_{33}D_{20} + A_{33}B_{11}D_{20} \\ &+ B_{13} - B_{13}D_{20} - A_{22}B_{13} + A_{22}B_{13}D_{20} \\ &+ B_{22} - B_{11}B_{22} - B_{22}D_{20} + B_{11}B_{22}D_{20} \\ &+ B_{31} - B_{31}D_{20} - A_{22}B_{31} + A_{22}B_{31}D_{20} \\ &+ C_{11} - A_{13}C_{11} - A_{22}C_{11} + A_{13}A_{22}C_{11} \\ &+ C_{31} - A_{13}C_{31} - A_{22}C_{31} + A_{13}A_{22}C_{31}D_{20} \\ &+ D_{40} - A_{13}D_{40} - A_{22}D_{40} + A_{13}A_{22}D_{40} \end{aligned}$$

wherein some of the terms have been rearranged so that, as in (II), only denominator forms occur in the first row, and the last term of the seventh row is the last term of the first row multiplied by C_{31}. Each of the remaining six rows is divisible by a pair of denominator factors; *e. g.* the last is $D_{40}(1 - A_{13})(1 - A_{22})$.

This transformation proves that seven of the indicated second syzygies are irreducible, for (see Vol. VII, p. 337 of this Journal) any n^{th} syzygy that can be properly represented by a $(n + 1)$-ary combination of the groundforms is necessarily so. As an illustration, we have

$$
\begin{array}{ll}
A_{13}(11) & = A_{13}(B_{11}B_{22} - B_{02}B_{31} + 2A_{23}C_{11}) \\
-\ 2A_{23}(8) & -\ 2A_{23}(A_{13}C_{11} + B_{02}B_{22} + B_{11}B_{13}) \\
+\ B_{02}(3) & +\ B_{02}(A_{13}B_{31} - A_{33}B_{11} + 2A_{23}B_{22}) \\
-\ B_{11}(2) & -\ B_{11}(A_{13}B_{22} - A_{33}B_{02} - 2A_{23}B_{13}) = 0,
\end{array}
$$

which may be represented, with propriety, in several ways by a ternary combination. Thus the term $A_{23}B_{11}B_{13}$ represents it in (II), and the term $A_{13}A_{23}C_{11}$ in (IV); while (8) shows that $B_{11}B_{13}$ is in syzygy with $A_{13}C_{11}$ and forms the connecting link between the two modes of representation.

From (IV) we obtain immediately a set of names for the 44 irreducible syzygies, viz. these are the aggregate of the negative terms, and the binary combinations of the letters $A_{33}, B_{13}, \ldots D_{40}$ which form the first column. Names might of course be found with equal facility from (II), but these, though equally appropriate in other respects, would not consist exclusively of binary combinations.

In the following multiplication table each set of three numbers is the deg. deg. order of the syzygy whose name is the product of the letters standing one in the same row and the other in the same column.

	A_{33}	B_{13}	B_{22}	B_{31}	C_{11}	C_{31}	D_{40}	D_{20}
A_{33}	0.6.6							
B_{13}	1.4.6	2.2.6						
B_{22}	1.5.5	2.3.5	2.4.4					
B_{31}	1.6.4	2.4.4	2.5.3	2.6.2				
C_{11}	2.4.4	3.2.4	3.3.3	3.4.2	4.2.2			
C_{31}	2.6.4	3.4.4	3.5.3	3.6.2	4.4.2	4.6.2		
D_{40}	3.7.3	4.5.3	4.6.2	4.7.1	5.5.1	5.7.1	6.8.0	
A_{13}					2.2.4	2.4.4	3.5.3	3.3.3
A_{23}		1.3.5		1.5.3	2.3.3	2.5.3	3.6.2	3.4.2
B_{11}	1.4.4		2.3.3					
D_{20}	3.5.3	4.3.3	4.4.2	4.5.1				

The last four rows, in which the distribution seems somewhat irregular, correspond to the sixteen negative terms of (IV).

6. To obtain complete lists of the 2nd, 3rd, ... 9th syzygies, it will be necessary to multiply (IV) by each factor in succession of

$$(1-A_{33})(1-B_{13})(1-B_{22})(1-B_{31})(1-C_{11})(1-C_{31})(1-D_{40}),$$

and the form of the term $A_{13}A_{22}C_{31}D_{20}$ leads to the selection of $1-C_{31}$ for the first multiplier; for if this term represents the compound second syzygy $C_{31}(A_{13}A_{22}D_{20})$, it can be removed by this operation. To decide this question would involve some lengthy calculations, and, should the answer be in the affirmative, a precisely similar question would present itself at the next stage, and so on. In the absence of any information on this point we shall proceed on the assumption that $A_{13}A_{22}C_{31}D_{20}=C_{31}(A_{13}A_{22}D_{20})$, and similar assumptions will be made (one at each stage of the work) throughout. Or, what is the same, we shall assume that one of the terms of the last block, and one only, is removed at each multiplication. The result of the first multiplication is, however, given on both hypotheses.

Using, for the sake of brevity, P, Q, and R, to denote the groundforms, syzygies, and second syzygies represented in (IV), respectively; and writing

$$(1+P-Q+R)(1-C_{31})=1+P_1-Q_1+R_1-S_1,$$

we have in the two cases, viz.

$A_{13}A_{22}C_{31}D_{20}=C_{31}(A_{13}A_{22}D_{20})$	$A_{13}A_{22}C_{31}D_{20}$ irreducible
$P_1=P-C_{31}$	$P_1=P-C_{31}$
$Q_1=Q+C_{31}P$	$Q_1=Q+C_{31}P$
$R_1=(R-A_{13}A_{22}C_{31}D_{20})+C_{31}Q$	$R_1=R+C_{31}Q$
$S_1=(R-A_{13}A_{22}D_{20})\,C_{31}$	$S_1=C_{31}R.$

And in general, if after the n^{th} multiplication we have

$$1+P_n-Q_n+R_n-S_n+\dots$$

with $1-\mu$ for the $(n+1)^{\text{st}}$ multiplier, we shall find that

$$P_{n+1}=P_n-\mu$$
$$Q_{n+1}=Q_n+\mu P_n$$
$$R_{n+1}=R_n+\mu Q_n$$
$$\dots\dots\dots\dots$$

where, in virtue of our assumption, the last two of these will be reduced by a single term.

Writing $\mu=1$ in the above, P_n, Q_n, R_n, ... being treated as numbers, we get the number of terms in $1+P_n-Q_n+R_n-S_n+\dots$, viz. we have

$$\Delta P_n=-1$$
$$\Delta Q_n=P_n$$
$$\Delta R_n=Q_n$$
$$\dots\dots$$

the last two of these being reduced by unity.

The initial values obtained from (IV) by the consideration of the number of its terms are $P_0 = 7$, $Q_0 = 16$, $R_0 = 8$, $S_0 = 0$, $T_0 = 0$, etc.

Tabulating the values of P_n, Q_n, R_n, ... for different values of n, we find

n	0	1	2	3	4	5	6	7
P_n	7	6	5	4	3	2	1	0
Q_n	16	23	29	34	38	41	43	44
R_n	8	23	46	75	109	147	188	231
S_n	.	7	29	75	150	259	406	594
T_n	.	.	6	34	109	259	518	924
U_n	.	.	.	5	38	147	406	924
V_n	.	.	.	.	4	41	188	594
W_n	.	.	.	.	.	3	43	231
X_n	.	.	.	.	.	.	2	44
Y_n	.	.	.	.	.	.	.	1

and the number of terms in question is obtained by the addition of unity to the sum of the numbers in any column. The numbers in the last column show the total number of simple syzygies, 2nd syzygies, 3rd syzygies, etc., which are all of them irreducible (see *American Journal of Mathematics*, Vol. VII, p. 337, and apply the arguments used there to this case). Thus there are 44 simple syzygies, 231 second syzygies, and so on, provided only that the assumptions of the present article are legitimate; if they are not, P_n, Q_n, R_n, ... are still found by the same process, but the reduction of the last two numbers of a column by unity does not take place. It follows that if our assumptions are inadmissible the numbers in the last column must be increased.

7. Considering the successive multiplications at greater length, we have from (IV)

$$\begin{aligned} R = \; & A_{13}A_{22}D_{20} \\ & + A_{33}B_{11}D_{20} \\ & + A_{22}B_{13}D_{20} \\ & + B_{11}B_{22}D_{20} \\ & + A_{22}B_{31}D_{20} \\ & + A_{13}A_{22}C_{11} \\ & + A_{13}A_{22}C_{31}D_{20} \\ & + A_{13}A_{22}D_{40} \end{aligned} \qquad \text{whence } \begin{aligned} S_1 &= (R - A_{13}A_{22}D_{20})\,C_{31} \\ &= C_{31} \times \begin{bmatrix} A_{33}B_{11}D_{20} \\ + A_{22}B_{13}D_{20} \\ + B_{11}B_{22}D_{20} \\ + A_{22}B_{31}D_{20} \\ + A_{13}A_{22}C_{11} \\ + A_{13}A_{22}C_{31}D_{20} \\ + A_{13}A_{22}D_{40} \end{bmatrix} \end{aligned}$$

and six of the third syzygies represented by S_1 being certainly irreducible, the only question that arises is concerning the reducibility of $A_{13}A_{22}C_{31}^2D_{20}$.

Now observing that by No. (42) $C_{31}D_{20}$ is in syzygy with $C_{11}D_{40}$, and making this substitution in $A_{13}A_{22}C_{31}^2D_{20}$ without altering any other of the terms of S_1, we have

$$S_1 = C_{31} \times \begin{bmatrix} A_{33}B_{11}D_{20} \\ + A_{22}B_{13}D_{20} \\ + B_{11}B_{22}D_{20} \\ + A_{22}B_{31}D_{20} \\ + A_{13}A_{22}C_{11} \\ + A_{13}A_{22}C_{11}D_{40} \\ + A_{13}A_{22}D_{40} \end{bmatrix} \qquad \begin{aligned} T_2 &= (S_1 - A_{13}A_{22}C_{11}C_{31})\,D_{40} \\ &= C_{31}D_{40} \times \begin{bmatrix} A_{33}B_{11}D_{20} \\ + A_{22}B_{13}D_{20} \\ + B_{11}B_{22}D_{20} \\ + A_{22}B_{31}D_{20} \\ + A_{13}A_{22}C_{11}D_{40} \\ + A_{13}A_{22}D_{40} \end{bmatrix} \end{aligned}$$

where we assume, as in the first multiplication, that $A_{13}A_{22}C_{11}C_{31}D_{40}$ is reducible and $= D_{40} \times A_{13}A_{22}C_{11}C_{31}$; and where the multiplier used is $1 - D_{40}$. Observing now that the last term but one of T_2 is divisible by the last, making the assumption as to the reducibility of the corresponding 4th syzygy as before, and using the multiplier $1 - C_{11}$ we obtain

$$\begin{aligned} U_3 &= (T_2 - A_{13}A_{22}C_{31}D_{40}^2)\,C_{11} \\ &= C_{11}C_{31}D_{40}\,(A_{33}B_{11}D_{20} + A_{22}B_{13}D_{20} + B_{11}B_{22}D_{20} + A_{22}B_{31}D_{20} + A_{13}A_{22}C_{11}D_{40}) \end{aligned}$$

and, in the last term in brackets, writing $C_{31}D_{20}$ for $C_{11}D_{40}$, which is merely the reversal of the previous substitution,

$$U_3 = C_{11}C_{31}D_{20}D_{40}\,(A_{33}B_{11} + A_{22}B_{13} + B_{11}B_{22} + A_{22}B_{31} + A_{13}A_{22}C_{31}).$$

Here again, observing that $A_{13}C_{31}^2$ is in syzygy with $B_{13}B_{31}$, by No. (15), and making the corresponding substitution in the last term of U_3, we may write

$$U_3 = C_{11}C_{31}D_{20}D_{40}\,(A_{33}B_{11} + A_{22}B_{13} + B_{11}B_{22} + A_{22}B_{31} + A_{22}B_{13}B_{31})$$

where, after multiplying as before by $1 - B_{31}$ we obtain

$$V_4 = B_{31}C_{11}C_{31}D_{20}D_{40}\,(A_{33}B_{11} + B_{11}B_{22} + A_{22}B_{31} + A_{22}B_{13}B_{31})$$

and then using $1 - B_{13}$ for the next multiplier,

$$W_5 = B_{13}B_{31}C_{11}C_{31}D_{20}D_{40}\,(A_{33}B_{11} + B_{11}B_{22} + A_{22}B_{13}B_{31}).$$

The last term in brackets may be written $A_{22}B_{22}^2$, since No. (14) shows that $B_{13}B_{31}$ is in syzygy with B_{22}^2; and $A_{22}B_{22}^2$ itself may be further transformed into $A_{33}B_{11}B_{22}$, for $A_{22}B_{22}$ is in syzygy with $A_{33}B_{11}$ No. (3). Hence we may write

$$W_5 = B_{11}B_{13}B_{31}C_{11}C_{31}D_{20}D_{40}\,(A_{33} + B_{22} + A_{33}B_{22}).$$

Finally multiplying by $1 - B_{22}$ and $1 - A_{33}$ we obtain successively

$$X_6 = B_{11}B_{13}B_{22}B_{31}C_{11}C_{31}D_{20}D_{40}\,(B_{22} + A_{33}B_{22})$$

and

$$Y_7 = A_{33}^2B_{11}B_{13}B_{22}^2B_{31}C_{11}C_{31}D_{20}D_{40}.$$

In the course of the work the following reductions have been assumed.

			Deg. Deg. Order.
That of the 2nd syzygy		$A_{13}A_{22}C_{31}D_{20}$	5.8.6
3rd	"	$A_{13}A_{22}C_{11}C_{31}D_{40}$	7.11.7
4th	"	$A_{13}A_{22}C_{11}C_{31}D_{40}^2$	10.15.7
5th	"	$A_{22}B_{13}B_{31}C_{11}C_{31}D_{20}D_{40}$	12.16.8
6th	"	$A_{22}B_{13}B_{31}^2C_{11}C_{31}D_{20}D_{40}$	13.19.9
7th	"	$A_{33}B_{11}B_{22}B_{13}B_{31}C_{11}C_{31}D_{20}D_{40}$	14.20.12
8th	"	$A_{33}B_{11}B_{13}B_{22}^2B_{31}C_{11}C_{31}D_{20}D_{40}$	15.22.14

In every other case each n^{th} syzygy is represented by a $(n+1)$-ary combination of the groundforms, and is therefore irreducible. The case of Y_7 is only an apparent exception to this rule, for (Art. 2, Nos. 13 to 16) $A_{33}C_{11}$ is in syzygy with $A_{13}C_{31}$; so that we may write

$$Y_7 = A_{13}A_{33}B_{11}B_{13}B_{22}^2B_{31}C_{31}^2D_{20}D_{40}.$$

And since $B_{22}B_{31}$ is in syzygy with $A_{22}C_{31}$ (Art. 2, Nos. 17, 18),

$$Y_7 = A_{13}A_{22}A_{33}B_{11}B_{13}B_{22}C_{31}^3D_{20}D_{40}.$$

Lastly, since $A_{13}A_{22}B_{11}$ is in syzygy with $A_{33}B_{13}$ (Art. 2, No. 4),

$$Y_7 = A_{33}^2B_{13}^2B_{22}C_{31}^3D_{20}D_{40},$$

which is the product of 10 letters only, and may properly be taken as a name for the unique 9th syzygy of deg. deg. order 15.25.17.

Can a similar transformation be effected in the case of any one of the 7 assumed compounds? Certainly not for the first of them, for there is no ternary product of the groundforms of deg. deg. order 5.8.6. It follows then that if the 2nd syzygy $A_{13}A_{22}C_{31}D_{20}$ is in fact irreducible, and this would make all the rest so, the proof must proceed in some other way than the transformation of its name into a ternary product.

8. Proceeding now to the formation of a catalogue, *i. e.* a list with names, of the second syzygies, we have directly from (IV),

$$\begin{array}{lll}
P = A_{33} & Q = A_{13}D_{20} + A_{22}D_{20} & R - A_{13}A_{22}C_{31}D_{20} = A_{13}A_{22}D_{20} \\
\quad + B_{13} & \quad + A_{33}B_{11} + A_{33}D_{20} & \quad + A_{33}B_{11}D_{20} \\
\quad + B_{22} & \quad + B_{13}D_{20} + A_{22}B_{13} & \quad + A_{22}B_{13}D_{20} \\
\quad + B_{31} & \quad + B_{11}B_{22} + B_{22}D_{20} & \quad + B_{11}B_{22}D_{20} \\
\quad + C_{11} & \quad + B_{31}D_{20} + A_{22}B_{31} & \quad + A_{22}B_{31}D_{20} \\
\quad + C_{31} & \quad + A_{13}C_{11} + A_{22}C_{11} & \quad + A_{13}A_{22}C_{11} \\
\quad + D_{40} & \quad + A_{13}C_{31} + A_{22}C_{31} & \quad + A_{13}A_{22}D_{40} \\
 & \quad + A_{13}D_{40} + A_{22}D_{40} &
\end{array}$$

where the last column forms a portion of the catalogue.

The remaining portion is, with the notation of the preceding article, by means of the equations

$$\begin{aligned} R_1 &= (R - A_{13}A_{22}C_{31}D_{20}) + C_{31}Q \\ R_2 &= R_1 + D_{40}Q_1 \\ R_3 &= R_2 + C_{11}Q_2 \\ R_4 &= R_3 + B_{31}Q_3 \\ R_5 &= R_4 + B_{13}Q_4 \\ R_6 &= R_5 + B_{22}Q_5 \\ R_7 &= R_6 + A_{33}Q_6 \end{aligned}$$

found to be the additional part of R_7

$$= C_{31}Q + D_{40}Q_1 + C_{11}Q_2 + B_{31}Q_3 + B_{13}Q_4 + B_{22}Q_5 + A_{33}Q_6.$$

In which, since

$$\begin{aligned} Q_1 &= Q + C_{31}P \\ Q_2 &= Q_1 + D_{40}P_1 \\ Q_3 &= Q_2 + C_{11}P_2 \\ Q_4 &= Q_3 + B_{31}P_3 \\ Q_5 &= Q_4 + B_{13}P_4 \\ Q_6 &= Q_5 + B_{22}P_5 \end{aligned}$$

we may substitute these values, beginning with the last, and find for the remaining portion of the catalogue the expression,

$$\begin{aligned} &Q(C_{31} + D_{40} + C_{11} + B_{31} + B_{13} + B_{22} + A_{33}) + (D_{40} + C_{11} + B_{31} + B_{13} + B_{22} + A_{33})C_{31}P \\ &+ (C_{11} + B_{31} + B_{13} + B_{22} + A_{33})D_{40}P_1 + (B_{31} + B_{13} + B_{22} + A_{33})C_{11}P_2 \\ &+ (B_{13} + B_{22} + A_{33})B_{31}P_3 + (B_{22} + A_{33})B_{13}P_4 + A_{33}B_{22}P_5. \end{aligned}$$

In this we substitute for P and Q their given values, writing at the same time,

$$\begin{aligned} P_1 &= P - C_{31} = A_{33} + B_{13} + B_{22} + B_{31} + C_{11} + D_{40} \\ P_2 &= P_1 - D_{40} = A_{33} + B_{13} + B_{22} + B_{31} + C_{11} \\ P_3 &= P_2 - C_{11} = A_{33} + B_{13} + B_{22} + B_{31} \\ P_4 &= P_3 - B_{31} = A_{33} + B_{13} + B_{22} \\ P_5 &= P_4 - B_{13} = A_{33} + B_{22} \end{aligned}$$

then multiply out, add in the given value of $R - A_{13}A_{22}C_{31}D_{20}$, and arrange the terms in any convenient manner. In the arrangement adopted, terms, *i. e.* names of second syzygies, of the same deg. deg. order are written beneath it, and their frequency below them: thus

$$\begin{gathered} 1.6.8 \\ A_{22}A_{33}B_{13} \\ (1) \end{gathered}$$

is understood to mean that there is one second syzygy, of deg. deg. order 1.6.8, whose name is $A_{22}A_{33}B_{13}$.

9. Catalogue of the Second Syzygies:

1.6.8	1.7.7	1.7.9	1.8.6	1.8.8	1.9.7
$A_{22}A_{33}B_{13}$	$A_{33}^2B_{11}$	$A_{33}^2B_{13}$	$A_{22}A_{33}B_{31}$	$A_{33}^2B_{22}$	$A_{33}^2B_{31}$
(1)	(1)	(1)	(1)	(1)	(1)

2.4.6	2.4.8	2.5.7	2.5.9	2.6.6	2.6.8
$A_{13}A_{22}C_{11}$	$A_{22}B_{13}^2$	$A_{22}B_{13}B_{22}$	$A_{33}B_{13}^2$	$A_{22}B_{13}B_{31}$	$A_{33}B_{13}B_{22}$
.	.	$A_{33}B_{11}B_{13}$	.	$A_{22}A_{33}C_{11}$	$B_{13}B_{22}A_{33}$
.	.	$A_{13}A_{33}C_{11}$	.	$A_{22}B_{31}B_{13}$	.
.	.	.	.	$A_{33}B_{11}B_{22}$	.
.	.	.	.	$B_{11}B_{22}A_{33}$	.
(1)	(1)	(3)	(1)	(5)	(2)

2.7.5	2.7.7	2.8.4	2.8.6	2.9.5	2.9.7
$A_{22}B_{22}B_{31}$	$A_{13}A_{33}C_{31}$	$A_{22}B_{31}^2$	$A_{22}A_{33}C_{31}$	$A_{33}B_{31}^2$	$A_{33}^2C_{31}$
$A_{33}B_{11}B_{31}$	$A_{33}^2C_{11}$	.	$A_{33}B_{22}B_{31}$	.	.
.	$A_{33}B_{13}B_{31}$	.	$A_{33}B_{31}B_{22}$	.	.
.	$A_{33}B_{31}B_{13}$	.	.	.	.
.	$A_{33}B_{22}^2$	.	.	.	.
(2)	(5)	(1)	(3)	(1)	(1)

3.3.7	3.4.6	3.4.8	3.5.5	3.5.7	3.6.4
$A_{13}B_{13}C_{11}$	$A_{13}B_{22}C_{11}$	$B_{22}B_{13}^2$	$A_{13}A_{22}D_{20}$	$A_{13}B_{13}C_{31}$	$A_{22}B_{31}C_{11}$
.	$A_{22}B_{13}C_{11}$	.	$A_{22}B_{22}C_{11}$	$A_{33}B_{13}C_{11}$	$A_{22}C_{11}B_{31}$
.	$A_{22}C_{11}B_{13}$	.	$A_{33}B_{11}C_{11}$	$A_{33}C_{11}B_{13}$	$B_{11}B_{22}B_{31}$
.	$B_{11}B_{13}B_{22}$	.	$A_{13}B_{31}C_{11}$	$B_{13}B_{22}^2$	.
.	.	.	$B_{11}B_{22}^2$	$B_{13}^2B_{31}$	.
(1)	(4)	(1)	(5)	(5)	(3)

3.6.6	3.7.5	3.7.7	3.8.4	3.8.6	3.9.5	3.10.6
$A_{13}A_{33}D_{20}$	$A_{13}A_{22}D_{40}$	$A_{33}B_{13}C_{31}$	$A_{22}B_{31}C_{31}$	$A_{13}A_{33}D_{40}$	$A_{22}A_{33}D_{40}$	$A_{33}^2D_{40}$
$A_{13}B_{22}C_{31}$	$A_{22}A_{33}D_{20}$	$A_{33}C_{31}B_{13}$	$A_{22}C_{31}B_{31}$	$A_{33}^2D_{20}$	$A_{33}B_{31}C_{31}$	.
$A_{22}B_{13}C_{31}$	$A_{13}B_{31}C_{31}$	.	$B_{22}B_{31}^2$	$A_{33}B_{22}C_{31}$	$A_{33}C_{31}B_{31}$	.
$A_{22}C_{31}B_{13}$	$A_{22}B_{22}C_{31}$	.	.	$A_{33}C_{31}B_{22}$	.	.
$A_{33}B_{22}C_{11}$	$A_{33}B_{11}C_{31}$	.	.	.	.	.
$A_{33}C_{11}B_{22}$	$A_{33}B_{31}C_{11}$	.	.	.	.	.
$B_{13}B_{22}B_{31}$	$A_{33}C_{11}B_{31}$	.	.	.	.	.
$B_{13}B_{31}B_{22}$	$B_{13}B_{31}^2$	.	.	.	.	.
.	$B_{22}^2B_{31}$	.	.	.	.	.
(8)	(9)	(2)	(3)	(4)	(3)	(1)

4.3.5	4.3.7	4.4.4	4.4.6	4.5.5	4.5.7
$A_{13}C_{11}^{2}$	$B_{13}^{2}C_{11}$	$A_{22}C_{11}^{2}$	$A_{13}B_{13}D_{20}$	$A_{13}B_{22}D_{20}$	$B_{13}^{2}C_{31}$
.	.	$B_{11}B_{22}C_{11}$	$B_{13}B_{22}C_{11}$	$A_{13}C_{11}C_{31}$	.
.	.	.	$B_{13}C_{11}B_{22}$	$A_{13}C_{31}C_{11}$	.
.	.	.	.	$A_{22}B_{13}D_{20}$	.
.	.	.	.	$A_{22}D_{20}B_{13}$	.
.	.	.	.	$A_{33}C_{11}^{2}$	.
.	.	.	.	$B_{13}B_{31}C_{11}$	.
.	.	.	.	$B_{13}C_{11}B_{31}$	.
.	.	.	.	$B_{22}^{2}C_{11}$	.
(1)	(1)	(2)	(3)	(9)	(1)

4.6.4	4.6.6	4.7.3	4.7.5	4.8.4	4.8.6
$A_{13}B_{31}D_{20}$	$A_{13}B_{13}D_{40}$	$A_{22}B_{31}D_{20}$	$A_{13}B_{22}D_{40}$	$A_{13}B_{31}D_{40}$	$A_{33}B_{13}D_{40}$
$A_{33}B_{11}D_{20}$	$A_{33}B_{13}D_{20}$	$A_{22}D_{20}B_{31}$	$A_{13}C_{31}^{2}$	$A_{22}B_{22}D_{40}$	$A_{33}D_{40}B_{13}$
$A_{22}B_{22}D_{20}$	$A_{33}D_{20}B_{13}$	$B_{31}^{2}C_{11}$	$A_{22}B_{13}D_{40}$	$A_{22}C_{31}^{2}$	.
$A_{22}C_{11}C_{31}$	$B_{13}B_{22}C_{31}$	.	$A_{22}D_{40}B_{13}$	$A_{33}B_{11}D_{40}$	.
$A_{22}C_{31}C_{11}$	$B_{13}C_{31}B_{22}$	.	$A_{33}B_{22}D_{20}$	$A_{33}B_{31}D_{20}$	.
$B_{11}B_{22}C_{31}$	.	.	$A_{33}D_{20}B_{22}$	$A_{33}D_{20}B_{31}$	.
$B_{22}B_{31}C_{11}$	.	.	$A_{33}C_{11}C_{31}$	$B_{22}B_{31}C_{31}$	.
$B_{22}C_{11}B_{31}$	.	.	$A_{33}C_{31}C_{11}$	$B_{22}C_{31}B_{31}$	.
.	.	.	$B_{13}B_{31}C_{31}$	.	.
.	.	.	$B_{13}C_{31}B_{31}$	.	.
.	.	.	$B_{22}^{2}C_{31}$	.	.
(8)	(5)	(3)	(11)	(8)	(2)

4.9.3	4.9.5	4.10.4
$A_{22}B_{31}D_{40}$	$A_{33}C_{31}^{2}$	$A_{33}B_{31}D_{40}$
$A_{22}D_{40}B_{31}$	$A_{33}B_{22}D_{40}$	$A_{33}D_{40}B_{31}$
$B_{31}^{2}C_{31}$	$A_{33}D_{4}\,B_{22}$	.
(3)	(3)	(2)

5.3.5	5.4.4	5.4.6	5.5.3	5.5.5	5.6.4	5.6.6
$B_{13}C_{13}^{1}$	$A_{13}C_{11}D_{20}$	$B_{13}^{2}D_{20}$	$A_{22}C_{11}D_{20}$	$B_{13}B_{22}D_{20}$	$A_{13}C_{11}D_{40}$	$B_{13}^{2}D_{40}$
.	$B_{22}C_{11}^{2}$	.	$B_{11}B_{22}D_{20}$	$B_{13}D_{20}B_{22}$	$A_{13}C_{31}D_{20}$	.
.	.	.	$B_{31}C_{11}^{2}$	$B_{13}C_{11}C_{31}$	$A_{13}D_{40}C_{11}$	.
.	.	.	.	$B_{13}C_{31}C_{11}$	$A_{33}C_{11}D_{20}$	.
.	.	.	.	.	$B_{13}B_{31}D_{20}$	.
.	.	.	.	.	$B_{13}D_{20}B_{31}$	.
.	.	.	.	.	$B_{22}^{2}D_{20}$	.
.	.	.	.	.	$B_{22}C_{11}C_{31}$	.
.	.	.	.	.	$B_{22}C_{31}C_{11}$	.
(1)	(2)	(1)	(3)	(4)	(9)	(1)

5.7.3	5.7.5	5.8.2	5.8.4	5.9.3	5.10.2	5.10.4
$A_{22}C_{11}D_{40}$	$B_{13}B_{22}D_{40}$	$B_{31}^2D_{20}$	$A_{13}C_{31}D_{40}$	$A_{22}C_{31}D_{40}$	$B_{31}^2D_{40}$	$A_{33}C_{31}D_{40}$
$A_{22}D_{40}C_{11}$	$B_{13}D_{40}B_{22}$	.	$A_{13}D_{40}C_{31}$	$A_{22}D_{40}C_{31}$	.	$A_{33}D_{40}C_{31}$
$A_{22}C_{31}D_{20}$	$B_{13}C_{31}^2$	.	$A_{33}C_{11}D_{40}$	$B_{22}B_{31}D_{40}$	.	.
$B_{11}B_{22}D_{40}$	.	.	$A_{33}D_{40}C_{11}$	$B_{22}D_{40}B_{31}$	.	.
$B_{22}B_{31}D_{20}$	.	.	$A_{33}C_{31}D_{20}$	$B_{31}C_{31}^2$	.	.
$B_{22}D_{20}B_{31}$	.	.	$B_{13}B_{31}D_{40}$	.	.	.
$B_{31}C_{11}C_{31}$	.	.	$B_{13}D_{40}B_{31}$	.	.	.
$B_{31}C_{11}C_{31}$	.	.	$B_{22}^2D_{40}$	.	.	.
.	.	.	$B_{22}C_{31}^2$	.	.	.
(8)	(3)	(1)	(9)	(5)	(1)	(2)

6.4.4	6.5.3	6.6.2	6.6.4	6.7.3	6.8.2
$B_{13}C_{11}D_{20}$	$B_{22}C_{11}D_{20}$	$B_{31}C_{11}D_{20}$	$B_{13}C_{31}D_{20}$	$A_{13}D_{20}D_{40}$	$A_{22}D_{20}D_{40}$
.	$C_{11}^2C_{31}$	.	$B_{13}C_{11}D_{40}$	$B_{22}C_{31}D_{20}$	$B_{31}C_{31}D_{20}$
.	.	.	$B_{13}D_{40}C_{11}$	$B_{22}C_{11}D_{40}$	$B_{31}C_{11}D_{40}$
.	.	.	.	$B_{22}D_{40}C_{11}$	$B_{31}D_{40}C_{11}$
.	.	.	.	$C_{11}C_{31}^2$	.
(1)	(2)	(1)	(3)	(5)	(4)

6.8.4	6.9.3	6.10.2	6.11.3
$B_{13}C_{31}D_{40}$	$A_{13}D_{40}^2$	$A_{22}D_{40}^2$	$A_{33}D_{40}^2$
$B_{13}D_{40}C_{31}$	$A_{33}D_{20}D_{40}$	$B_{31}C_{31}D_{40}$	.
.	$B_{22}C_{31}D_{40}$	$B_{31}D_{40}C_{31}$	.
.	$B_{22}D_{40}C_{31}$	.	.
(2)	(4)	(3)	(1)

7.6.2	7.7.3	7.8.2	7.9.1	7.9.3	7.10.2	7.11.1
$C_{11}^2D_{40}$	$B_{13}D_{20}D_{40}$	$B_{22}D_{20}D_{40}$	$B_{31}D_{20}D_{40}$	$B_{13}D_{40}^2$	$B_{22}D_{40}^2$	$B_{31}D_{40}^2$
.	.	$C_{11}C_{31}D_{40}$	.	.	$C_{31}^2D_{40}$	.
.	.	$C_{11}D_{40}C_{31}$	.	.	.	.
(1)	(1)	(3)	(1)	(1)	(2)	(1)

8.9.1	8.11.1
$C_{11}D_{40}^2$	$C_{31}D_{40}^2$
(1)	(1)

10. It would be tedious to give an enumeration of the irreducible 3rd, 4th... syzygies, especially so since no new principle would be illustrated thereby. Besides, it would be very desirable to previously ascertain whether there is an irreducible second syzygy of deg. deg. order 5.8.6 or not. As this would also require a lengthy investigation, I close the present paper with a few remarks in conclusion.

First. If it should eventually be proved that there is an irreducible second syzygy of deg. deg. order 5.8.6, the catalogue would be the same as before, but the name $A_{13}A_{23}C_{31}D_{30}$ would have to be added to the 231 names contained in it.

Secondly. An inspection of the deg. deg. orders will show that there is no case of the coexistence of an irreducible simple syzygy with an irreducible second syzygy of the same deg. deg. order; but it would be unsafe to assume that this is so for every quantic or system of quantics. The case of the Quintic itself shows that this is not so; for the Quintic has both a simple syzygy and a second syzygy of deg. order 8.16.

Thirdly. If, as I have always found to be the case, simple syzygies may be represented by binary combinations and second syzygies by ternary combinations of the groundforms; there is an advantage in working with the Real Generating Function instead of the Numerical. For two terms, one of which is a binary and the other a ternary combination of the groundforms cannot cancel each other; but if, as in the case when the Numerical Generating Function is used, both are represented by the same symbol, they may so cancel each other. This advantage is of course limited to those cases in which irreducible first and second, or n^{th} and $(n+1)^{\text{st}}$ syzygies coexist; but the Real Generating Function gives names as well as the frequency of syzygies, whereas the Numerical Generating Function gives the frequency only, and this constitutes a special advantage in the use of the former over that of the latter.

Finally, the name of a second syzygy being PQR, where each letter denotes one of the groundforms, there cannot be more than two second syzygies of this name when P, Q, and R are three distinct letters, there cannot be more than one second syzygy of this name when P, Q, R are only two distinct letters (*i. e.* there cannot be more than one whose name is PQ^2), and there can be no second syzygy whose name contains only one distinct letter (P^3). This law is revealed by an inspection of the names in the catalogue, its significance is best seen by the following example.

Name $A_{33}^2 B_{11}$, deg. deg. order 1.7.7

$A_{33}(A_{13}B_{31} - A_{33}B_{11} + 2A_{22}B_{22})$ outside factor A_{33}, inside factor $A_{33}B_{11}$

$- B_{11}(A_{13}^2 A_{40} - 4A_{22}^3 - A_{33}^2)$ " " B_{11} " " A_{33}^2

$+ 2A_{22}(A_{13}A_{40}B_{02} + 2A_{13}A_{22}B_{20} - 2A_{22}^2 B_{11} - A_{33}B_{22})$

$+ A_{13}(A_{13}A_{40}B_{11} - 2A_{22}A_{40}B_{02} - 4A_{22}^2 B_{20} - A_{33}B_{31}) = 0.$

Observe that it is necessary to the existence of a second syzygy that its name should be divisible in two ways, into an outside and an inside factor, the outside factor being the name of a covariant and the inside factor the name of a first syzygy: for the name P^2Q, the only two possible ways are P outside, PQ inside, and Q outside, P^2 inside. Thus there will be one second syzygy, and only one, whose name is P^2Q, if both P^2 and PQ are the names of syzygies. But if there is no syzygy named P^2, or no syzygy named PQ, there will be no second syzygy named P^2Q. Again, P^3 can only be divided in one way, viz. P outside and P^2 inside: hence, there is no second syzygy named P^3.

For the second syzygy named PQR, we have the three divisions (1) P outside, QR inside; (2) Q outside, PR inside; (3) R outside, PQ inside. Suppose now that there exist three syzygies named QR, PR, and PQ. These may be written in the form $QR = \ldots$, $PR = \ldots$, and $PQ = \ldots$, in which the name of any one of them does not appear in the other two; and then we should have three second syzygies, viz.

$$P(QR \ldots) - Q(PR \ldots) + \ldots = 0$$
$$Q(PR \ldots) - R(PQ \ldots) + \ldots = 0$$
$$R(PQ \ldots) - P(QR \ldots) + \ldots = 0$$

where the terms written down may be called the distinctive portions of the second syzygies. But the distinctive portion of the third of these is the sum of the distinctive portions of the other two; proving that there are not more than two second syzygies named PQR. If PQ should not be the name of a syzygy the last two "second syzygies" would be non-existent, and there would remain only the first. The example given above illustrates this case also if we alter the name to $A_{13}A_{33}B_{31}$ (as we may if we choose). For there is no syzygy that can be named $A_{13}A_{33}$.

BUCKHURST HILL, ESSEX, Oct. 8, 1885.

On the Singularities of Curves of Double Curvature.

By Henry B. Fine.

The elements of a curve of double curvature may be either the points of which it is the carrier or the lines or planes by which it is enveloped. We shall find it convenient to call any curve C the *point-curve*, *line-curve*, or *plane-curve* C, according as it is regarded under the first, second, or third of these aspects.

Each of these curves—if analytical—can be represented in the neighborhood of any element by an expansion in positive integral powers of a parameter with Grassmann units for coefficients (§ 1); these units being points in the series which represents the point-curve, in part lines, in part linear complexes in the series for the line-curve, and planes in the plane-curve series.

A curve being thus represented, every projective singularity possible to a single curve element finds expression in linear relations among the coefficients of the opening terms in its expansion.

Three classes of such relations among the coefficients of the point-curve or plane-curve expansions are supposable, and these I make the definitions of a 1st, 2d and 3d class point singularity, and of a 1st, 2d and 3d class plane singularity (§§ 2, 3), viz: I define a point as singular in the 1st class to the κ_1 order when all the coefficients of its expansion through that of the κ_1 power of the parameter are congruent; as singular in the 2d class to the κ_2 order when the coefficients, through that of the $\kappa_1 + \kappa_2 + 1$ power of the parameter are linearly derivable from the leading coefficient and that of the $\kappa_1 + 1$ power; as singular in the 3d class to the κ_3 order when the coefficients, through that of the $\kappa_1 + \kappa_2 + \kappa_3 + 2$ power of the parameter, are linearly derivable from the coefficients of the 0, $\kappa_1 + 1$, and $\kappa_1 + \kappa_2 + 2$ powers; and have reciprocal definitions for a 1st, 2d and 3d class plane singularity with order indices μ_1, μ_2, μ_3.

The indices κ_1, κ_2, κ_3 play precisely the same rôle in the theory of higher singularities of curves of double curvature as do the "cuspidal" and "inflexional" indices (Cayley) in the corresponding theory for curves in a plane.*

Thus, for a pair of associated point and plane elements of any curve C, κ_1 is equal to μ_3, κ_2 to μ_2 and κ_3 to μ_1, (§ 4).

Of a set of associated point, line and plane elements of the curve—P, l and ε—κ_1 connects itself most closely with P, κ_2 with l and κ_3 with ε; these numbers being indeed the degrees to which P, l and ε respectively are stationary (§§ 2, 3, 6).

At P, a point tracing out the point-curve maintains the direction of its motion in l if κ_1 be even, but reverses it if κ_1 be odd; and the generating line and plane are controlled in like manner by κ_2 and κ_3. The eight combinations of odd and even values of the three numbers κ_1, κ_2, κ_3 give rise to the eight characteristic shapes—first noticed by von Standt†—possible to a curve of double curvature at any element (§§ 5, 6).

Finally κ_1, κ_2, κ_3 are respectively the number of simple stationary points, lines and planes to which, in its effect on the order, class, rank and genus of C, the singularity is equivalent‡ (§ 6).

As there are six linearly independent line elements in space, the number of singularity-defining relations supposable among the coefficients of the line-curve series is five. It is found (§ 6) that these are all controlled by the indices κ_1, κ_2, κ_3 when κ_1 and κ_3 are unequal, the first four of them when $\kappa_1 = \kappa_3$. An *independent* line singularity is possible, therefore, when $\kappa_1 = \kappa_3$ and only then. I call it the line class singularity and assign it the order index λ. This singularity is, geometrically speaking, that more consecutive elements of a line-curve, or developable, belong to the same linear complex than are forced into it by κ_1, κ_2, κ_3.

* Plücker was the first to define the higher singularities of algebraic plane curves by indices. He used two, an order and a class index. Prof. Cayley's cuspidal and inflexional indices are Plücker's order and class indices diminished each by unity. They give the number of simple cusps and inflexions which are to be regarded as entering into the composition of the higher singularity. Vid. Plücker's Theorie der algebraischen Curven, p. 205, and Cayley in Quarterly Journal, Vol. 7; also Smith in Proceedings of the London Math. Society, Vol. 6.

† Geom. d. Lage. Rückkehrelemente.

‡ There enter also into the composition of higher singularities in general, double points, double planes, etc., due to the mutual intersections of the projections of the several "partial branches" which unite in the singular element (as do the leaves of a Riemann surface at a spiral point). These we leave out of sight; the series with which we operate represent but single partial branches, and regarded as an element of such a branch, the singularity is completely defined by the indices κ_1, κ_2, κ_3, λ.

The highest projective singularity possible to an element of C is defined by its indices $\varkappa_1\,,\ \varkappa\ ,\ \varkappa_3\,,\ \lambda$.

In § 7 I have expressed the conditions of the singularity $\varkappa_1\,,\ \varkappa_2\,,\ \varkappa_3$ in terms of the coefficients in the expansions $\phi_i(t)\,,\ \psi_i(t)$ by which the point- or plane-curve is defined; and in § 8 the same conditions in terms of position relations among corresponding elements of the curve $\phi_i(t)$ (or $\psi_i(t)$) and its "differential curves" $\phi_i'(t)\,,\ \phi_i''(t)$, etc., ... ($\psi_i'(t)\,,\ \psi_i''(t)$, etc., ...). The use of the differential curves for this purpose was suggested to me by my friend Dr. Study of Leipsic, to whom I am indebted for other valuable suggestions, in particular for calling my attention to the singularity of six consecutive tangents lying in the same linear complex—the simplest case of the line-class singularity.

As Grassmann is not very generally read, I have presupposed an acquaintance with but the fundamental ideas of his algebra.* For his terms Punkt, Linientheil, Liniensumme, Ebenentheil, I have used simply point, line, linear complex, plane; for, though somewhat inexact, this terminology is simple and not liable to cause confusion.

The entire investigation admits of immediate extension to curves in space of n dimensions. It is only necessary to enlarge sufficiently the order of our region (Stufe). There are n classes of point and of "plane" singularities in a space of n dimensions; the one class for a space of one dimension being the Riemann spiral point (Verzweigungspunkt).

§ 1.

The equations

$$x_1 : x_2 : x_3 : x_4 = \phi_1(t) : \phi_2(t) : \phi_3(t) : \phi_4(t)\,,$$

where x_i are homogeneous point coordinates and ϕ_i analytical functions of the form

$$a_{i0} + a_{i1}(t - t_0) + a_{i2}(t - t_0)^2 + a_{i3}(t - t_0)^3 + \text{etc.} \dots,$$

which converge within a circle of the complex plane of t whose centre is t_0, define within this circle an analytical point-curve of double curvature. It is assumed that, generally speaking, to each point (x) but one value of t belongs. Of special interest among the analytical curves are the algebraic. These are defined in the first instance by equations of the form

$$x_1 : x_2 : x_3 : x_4 = \phi_1(\xi\,,\ \eta) : \phi_2(\xi\,,\ \eta) : \phi_3(\xi\,,\ \eta) : \phi_4(\xi\,,\ \eta)\,,$$

the ϕ_i being integral rational functions of ξ and η, and ξ and η being connected

* For a brief exposition of these, see an article by Prof. Hyde in Vol. VI of this Journal. Or see Grassmann: Ausdehnungslehre of 1844; Ausdehnungslehre of 1862, Part 1, Chapter 5.

by an algebraic relation $f(\xi, \eta) = 0$. But this algebraic relation enables us by a simple substitution to reduce the equations given to the general form $\rho x_i = \phi_i(t)$. For at a multiple point of the μ^{th} order on the curve $f(\xi, \eta) = 0$—we suppose its coordinates to be ξ_0, η_0—each of the μ values which $\eta - \eta_0$ can take may be developed in positive integral powers of $(\xi - \xi_0)^{\frac{1}{\mu}}$ if not in such powers of a lower root of $\xi - \xi_0$; so that it is only necessary to substitute $t - t_0$ for this radical to accomplish the desired reduction.*

If E_1, E_2, E_3, E_4 be Grassmann points, each of the weight 1, and situate one at each of the angular points of the tetraedron $x_1 = 0$, $x_2 = 0$, $x_3 = 0$, $x_4 = 0$, the point (x) is $P = x_1E_1 + x_2E_2 + x_3E_3 + x_4E_4$. For any point in space can be derived linearly from E_1, E_2, E_3, E_4, so that $P = \alpha_1E_1 + \alpha_2E_2 + \alpha_3E_3 + \alpha_4E_4$. Taking the progressive product of both sides of this equation into $E_2E_3E_4$, $PE_2E_3E_4 = \alpha_1 E_1E_2E_3E_4$. Each of the products $PE_2E_3E_4$ and $E_1E_2E_3E_4$ is scalar, and their ratio is that of the tetraedra $PE_2E_3E_4$ and $E_1E_2E_3E_4$; α_1, therefore, is in constant ratio to the perpendicular distance of P from the tetraedron face $E_2E_3E_4$ and is consequently equal to x_1. For the same reason $\alpha_2 = x_2$, $\alpha_3 = x_3$, $\alpha_4 = x_4$.

It follows, therefore, that within the circle of convergence common to the four functions $\phi_i(t)$,

$$P = \phi_1(t)E_1 + \phi_2(t)E_2 + \phi_3(t)E_3 + \phi_4(t)E_4$$

is the equation in Grassmann units of the point-curve

$$x_1 : x_2 : x_3 : x_4 = \phi_1(t) : \phi_2(t) : \phi_3(t) : \phi_4(t).$$

Developing $\phi_1(t)$, $\phi_2(t)$, $\phi_3(t)$, $\phi_4(t)$,

$$\begin{aligned} P_t = {} & a_{10}E_1 + a_{20}E_2 + a_{30}E_3 + a_{40}E_4 \\ & + (a_{11}E_1 + a_{21}E_2 + a_{31}E_3 + a_{41}E_4)(t - t_0) \\ & + \quad \cdot \quad \cdot \quad \cdot \quad \cdot \quad \cdot \quad \cdot \quad \cdot \quad \cdot \quad \cdot \quad \cdot \\ & \quad \cdot \quad \cdot \quad \cdot \quad \cdot \quad \cdot \quad \cdot \quad \cdot \quad \cdot \quad \cdot \quad \cdot \\ & + (a_{1n}E_1 + a_{2n}E_2 + a_{3n}E_3 + a_{4n}E_4)(t - t_0)^n \\ & + \text{etc.} \end{aligned}$$

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* If we suppose that neither of the lines ξ_0, η_0 touches any branch of the curve in the point ξ_0, η_0, these μ developments arrange themselves in groups of the type

$$\eta - \eta_0 = B_0(\xi - \xi_0) + B_1\omega^{\beta_1}(\xi - \xi_0)^{\frac{\beta_1}{\Delta}} + B_2\omega^{\beta_2}(\xi - \xi_0)^{\frac{\beta_2}{\Delta}} + \text{etc.} \ldots \quad \text{(a)}$$

where ω is any root of $\omega^{\Delta} = 1$ and the equations differ from one another only in containing different values of ω. $\Sigma\Delta = \mu$, the summation referring to the different groups. Each of the groups (a) defines a branch of the curve with the singularity indices $\Delta - 1$, $\beta_1 - \Delta - 1$. The branch is "linear" if $\Delta = 1$; "superlinear" if $\Delta > 1$; the terminology is Prof. Cayley's. To each branch (a) corresponds a branch of $\phi_i(t)$ of whose singularity indices it will generally be true that the first is $\Delta - 1$ and the second and third depend upon but Δ, β_1, β_2. See in this connection Prof. H. J. S. Smith's memoir on the "Higher Singularities of Plane Curves," Proceedings of London Math. Society, Vol. 6.

Or, if $a_{1j}E_1 + a_{2j}E_2 + a_{3j}E_3 + a_{4j}E_4 = A_j$,

$$P_t = A_0 + A_1(t - t_0) + A_2(t - t_0)^2 + \ldots + A_n(t - t_0)^n + \text{etc.} \ldots \qquad (1)$$

Reciprocally, if u_1, u_2, u_3, u_4 be the coordinates of a plane-element, the equations

$$u_1 : u_2 : u_3 : u_4 = \psi_1(t) : \psi_2(t) : \psi_3(t) : \psi_4(t)$$

yield the Grassmann equation of the plane-curve $\psi_i(t)$. It is

$$\varepsilon_t = \alpha_0 + \alpha_1(t - t_0) + \alpha_2(t - t_0)^2 + \ldots + \alpha_n(t - t_0)^n + \text{etc.} \ldots \qquad (2)$$

The coefficients α are pieces of fixed planes.

§ 2.

By Taylor's theorem, within the circle of convergence about t_0,

$$P_{t_1 + \delta} = P_{t_1} + P'_{t_1}\delta + P''_{t_1}\frac{\delta^2}{2!} + \ldots + P^n_{t_1}\frac{\delta^n}{n!} + \text{etc.}^*$$

I define the point P_{t_1} as *singular in the first class to the* κ_1 *order* when P_{t_1} is congruent with each of the differential points P'_{t_1}, P''_{t_1}, ... $P^{\kappa_1}_{t_1}$, but not with the point $P^{\kappa_1+1}_{t_1}$; or when $[P_{t_1}] \equiv P'_{t_1}, P''_{t_1}, \ldots P^{\kappa_1}_{t_1}, \not\equiv P^{\kappa_1+1}_{t_1}$.

I define a point P_{t_1} whose first class singularity is κ_1, as *singular in the second class to the* κ_2 *order*, when all the differential points $P^{\kappa_1+2}_{t_1}$, $P^{\kappa_1+3}_{t_1}$, ... $P^{\kappa_1+\kappa_2+1}_{t_1}$ lie in the same straight line with P_{t_1} and $P^{\kappa_1+1}_{t_1}$, but the point $P^{\kappa_1+\kappa_2+2}_{t_1}$ lies off this line; or when

$$[P_{t_1}, P^{\kappa_1+1}_{t_1}] \equiv P^{\kappa_1+2}_{t_1}, P^{\kappa_1+3}_{t_1}, \ldots P^{\kappa_1+\kappa_2+1}_{t_1}, \not\equiv P^{\kappa_1+\kappa_2+2}_{t_1}.$$

A point P_{t_1} whose first and second class singularities are of the orders κ_1 and κ_2 respectively, I define as *singular in the third class to the* κ_3 *order*, when all the differential points $P^{\kappa_1+\kappa_2+3}_{t_1}$, $P^{\kappa_1+\kappa_2+4}_{t_1}$, ... $P^{\kappa_1+\kappa_2+\kappa_3+2}_{t_1}$ lie in the same plane with P_{t_1}, $P^{\kappa_1+1}_{t_1}$, $P^{\kappa_1+\kappa_2+2}_{t_1}$, but the point $P^{\kappa_1+\kappa_2+\kappa_3+3}_{t_1}$ lies without this plane; or when $[P_{t_1}, P^{\kappa_1+1}_{t_1}, P^{\kappa_1+\kappa_2+2}_{t_1}] \equiv P^{\kappa_1+\kappa_2+3}_{t_1}, P^{\kappa_1+\kappa_2+4}_{t_1}, \ldots P^{\kappa_1+\kappa_2+\kappa_3+2}_{t_1} \not\equiv P^{\kappa_1+\kappa_2+\kappa_3+3}_{t_1}$.

As all points of space are linearly derivable from any four among which no linear relation exists, with the three classes just enumerated we exhaust the singularity-defining relations of the points P_{t_1}, P'_{t_1}, P''_{t_1}, etc.

1. *The First Class Singularity.* The infinitesmal change of position of a curve point P_t corresponding to an infinitesmal change δ in the value of t is of the same order as δ if κ_1, the index of the first class singularity of P_t, is zero; but if κ_1 have any other value than zero, this infinitesmal change of position of P_t is of a higher order than δ, being indeed of the $\kappa_1 + 1$ order when δ is of the first order. It is supposed of course that t is made to vary continuously.

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* Here, as in what follows, I indicate the order of a derived function by a simple exponent.

For the progressive product, $P_t P_{t+\delta}$ measures the order of the distance of the points P_t, $P_{t+\delta}$ apart; and $P_t P_{t+\delta} = P_t P_t^{\varkappa_1+1} \cdot \frac{\delta^{\varkappa_1+1}}{(\varkappa_1+1)!} +$ etc., since $P_t^{\varkappa_1+1}$ is the first differential point which is not congruent with P_t. Hence, *a point which is singular in the first class is stationary, and the index $\varkappa_1$ measures to what degree.*

2. *The Second Class Singularity.* Let P_{t_2}, P_{t_3} be any two points on the curve P_t, and let them be made, independently of each other, to approach P_{t_1}; the line $P_{t_2}P_{t_3}$, in its limiting position, is, by the ordinary definition, the *tangent* to the curve at the point P_{t_1}.

If $t_2 = t_1 + \delta.\rho_2$, $t_3 = t_1 + \delta.\rho_3$; by Taylor's theorem

$$P_{t_2} = P_{t_1} + P'_{t_1}\delta.\rho_2 + P''_{t_1}\frac{(\delta.\rho_2)^2}{2!} + \ldots P_{t_1}^{\varkappa_1+1}\frac{(\delta.\rho_2)^{\varkappa_1+1}}{(\varkappa_1+1)!} + \text{etc.} \ldots$$

$$P_{t_3} = P_{t_1} + P'_{t_1}\delta.\rho_3 + P''_{t_1}\frac{(\delta.\rho_3)^2}{2!} + \ldots P_{t_1}^{\varkappa_1+1}\frac{(\delta.\rho_3)^{\varkappa_1+1}}{(\varkappa_1+1)!} + \text{etc.} \ldots$$

Therefore,

$$P_{t_2}P_{t_3} = \begin{vmatrix} 1 & \frac{\rho_2^{\varkappa_1+1}}{(\varkappa_1+1)!} \\ 1 & \frac{\rho_3^{\varkappa_1+1}}{(\varkappa_1+1)!} \end{vmatrix} \delta^{\varkappa_1+1}.\, P_{t_1}P_{t_1}^{\varkappa_1+1} + \ldots,$$

so that the tangent at P_{t_1}, or $\lim\limits_{\delta \doteq 0}\left[\frac{P_{t_2}P_{t_3}}{\delta^{\varkappa_1+1}}\right]$, is congruent with $P_{t_1}P_{t_1}^{\varkappa_1+1}$.

The tangent to a curve at any point P is the line determined by P and the first of the differential points P', P'', ... which is not congruent with P; it is the line $PP^{\varkappa_1+1}$, if $\varkappa_1$ be the index of the first class singularity of the point.

Now the product of the point $P_{t+\delta}$ by $P_t P_t^{\varkappa_1+1}$ measures the order of its distance from the tangent; for the product $P_t P_t^{\varkappa_1+1} P_{t+\delta}$ is of the same order as the triangle $P_t P_t^{\varkappa_1+1} P_{t+\delta}$, and this of the same order as its altitude, the distance of $P_{t+\delta}$ from $P_t P_t^{\varkappa_1+1}$. But $P_t P_t^{\varkappa_1+1} P_{t+\delta} = P_t P_t^{\varkappa_1+1} P_t^{\varkappa_1+\varkappa_2+2} \frac{\delta^{\varkappa_1+\varkappa_2+2}}{(\varkappa_1+\varkappa_2+2)!} +$ etc., where $P_t^{\varkappa_1+\varkappa_2+2}$ is the first differential point which lies off the line $P_t P_t^{\varkappa_1+1}$; so that $P_{t+\delta}$ is distant from the tangent at P_t by an infinitesmal of the $\varkappa_1 + \varkappa_2 + 2$ order.

At a point of the first and second class singularity $\varkappa_1$, $\varkappa_2$, a curve has with its tangent a contact of the $\varkappa_1 + \varkappa_2 + 1$ order; in addition to the $\varkappa_1$ consecutive points which coalesce with P there lie still other $\varkappa_2 + 1$ consecutive points on the tangent through P.

3. *The Third Class Singularity.* If P_{t_2}, P_{t_3}, P_{t_4} be three curve points and these points be made independently of each other to approach the point P_t, the plane $P_{t_2}P_{t_3}P_{t_4}$ in its limiting position is the *osculating plane* to the curve at P_t.

By a simple extension of the proof in the last paragraph $\underset{\delta \doteq 0}{\text{limit}} \left[\frac{P_{t_2}P_{t_3}P_{t_4}}{\delta^{2\varkappa_1+\varkappa_2+3}}\right]$ is congruent with $P_t P_t^{\varkappa_1+1} P_t^{\varkappa_1+\varkappa_2+2}$; or

The osculating plane to a curve at any point P is that determined by P, $P^{\varkappa_1+1}$, the first differential point not congruent with P, and $P^{\varkappa_1+\varkappa_2+2}$, the first differential point off the line $PP^{\varkappa_1+1}$.

The progressive product $P_t P_t^{\varkappa_1+1} P_t^{\varkappa_1+\varkappa_2+2} P_{t+\delta}$ measures the order of the distance of $P_{t+\delta}$ from the osculating plane at P_t; but $P_t P_t^{\varkappa_1+1} P_t^{\varkappa_1+\varkappa_2+2} P_{t+\delta}$ $= P_t P_t^{\varkappa_1+1} P_t^{\varkappa_1+\varkappa_2+2} P_t^{\varkappa_1+\varkappa_2+\varkappa_3+3} \dfrac{\delta^{\varkappa_1+\varkappa_2+\varkappa_3+3}}{(\varkappa_1+\varkappa_2+\varkappa_3+3)\,!} + \text{etc.}$, since $P^{\varkappa_1+\varkappa_2+\varkappa_3+3}$ is the first differential point not linearly derivable from P, $P^{\varkappa_1+1}$, $P^{\varkappa_1+\varkappa_2+2}$. Hence $P_{t+\delta}$ is distant from the osculating plane at P_t by an infinitesmal of the $\varkappa_1+\varkappa_2+\varkappa_3+3$ order.

A third class singularity in P of the order $\varkappa_3$ adds $\varkappa_3$ to the order of contact of the curve with the osculating plane at P; the order of contact at a point whose three singularity indices are $\varkappa_1$, $\varkappa_2$, $\varkappa_3$, being $\varkappa_1+\varkappa_2+\varkappa_3+1$. The number of consecutive points in the osculating plane is $\varkappa_1+\varkappa_2+\varkappa_3+3$.

§ 3.

By Taylor's theorem,

$$\varepsilon_{t+\delta} = \varepsilon_t + \varepsilon_t'\delta + \varepsilon_t''\frac{\delta^2}{2!} + \ldots \varepsilon_t^n \frac{\delta^n}{n!} + \text{etc.}$$

The plane element ε_t I define as *singular in the first class of plane singularity, and to the μ_1 order,* when $[\varepsilon_t] \equiv \varepsilon_t', \varepsilon_t'', \ldots \varepsilon_t^{\mu_1} \not\equiv \varepsilon_t^{\mu_1+1}$.

The plane ε_t, whose first class singularity is of the order μ_1, I define as *singular in the second class to the μ_2 order* when

$$[\varepsilon_t,\ \varepsilon_t^{\mu_1+1}] \equiv \varepsilon_t^{\mu_1+2},\ \varepsilon_t^{\mu_1+3}, \ldots \varepsilon_t^{\mu_1+\mu_2+1}, \not\equiv \varepsilon_t^{\mu_1+\mu_2+2}.$$

The planes $\varepsilon_t^{\mu_1+2}, \ldots \varepsilon_t^{\mu_1+\mu_2+1}$ pass through the intersection of ε_t and $\varepsilon_t^{\mu_1+1}$, but $\varepsilon_t^{\mu_1+\mu_2+2}$ does not pass through this line.

The plane ε_t, of which the first and second class singularities are of the orders μ_1 and μ_2 respectively, I define as *singular in the third class to the order μ_3*, when $[\varepsilon_t,\ \varepsilon_t^{\mu_1+1},\ \varepsilon_t^{\mu_1+\mu_2+2}] \equiv \varepsilon_t^{\mu_1+\mu_2+3},\ \varepsilon_t^{\mu_1+\mu_2+4}, \ldots \varepsilon_t^{\mu_1+\mu_2+\mu_3+2}, \not\equiv \varepsilon_t^{\mu_1+\mu_2+\mu_3+3}$.

All the differential planes preceding $\varepsilon_t^{\mu_1+\mu_2+\mu_3+3}$ pass through the point in which ε_t, $\varepsilon_t^{\mu_1+1}$, $\varepsilon_t^{\mu_1+\mu_2+2}$ intersect, but $\varepsilon_t^{\mu_1+\mu_2+\mu_3+3}$ does not pass through this point.

With these three classes the singularity-defining relations among the coefficients of $\varepsilon_{t+\delta}$ are exhausted; for the space of planes is a region of but the fourth order, so that any plane in space is linearly derivable from any four between which no numerical relation exists.

1. *The First Class Singularity. A plane which is singular in the first class is stationary, and the index μ_1 measures to what degree.*

For the order of the (regressive) product of two planes of finite weight is the same as the order of the angle which they make with each other. If ABC be a triangle of one of the planes, its area the weight of the plane, and ABD a triangle of the other plane, its area the weight of its plane, the product of the planes is $[ABC.ABD] = [ABCD]\,AB$; $[ABCD]$ is scalar, being three times the volume of the tetraedron $ABCD$, and its order is that of the angle between ABC and ABD. But the order of the product $\varepsilon_t\varepsilon_{t+\delta}$, or of $\varepsilon_t\varepsilon_t^{\mu_1+1}\dfrac{\delta^{\mu_1+1}}{(\mu_1+1)!} +$ etc., is $\mu_1 + 1$.

2. *The Second Class Singularity.* The tangent line to the curve ε_t in the element ε_{t_1} is the line of intersection of any two elements ε_{t_2} and ε_{t_3} in what is its limiting position when ε_{t_2} and ε_{t_3} are made, independently of each other, to approach ε_{t_1}. By developing ε_{t_2} and ε_{t_3} and taking their regressive product, it is easily proved that the tangent line to a plane-curve in any plane ε is the line determined by ε and the first of the differential planes ε', ε'', . . . which is not congruent with ε: it is the line $\varepsilon\varepsilon^{\mu_1+1}$ if μ_1 be the index of the first class singularity of ε.

At a plane of the first and second class singularity μ_1, μ_2, a plane-curve has with its tangent a "contact" of the $\mu_1 + \mu_2 + 1$ order: in addition to the μ_1 consecutive planes which coincide with ε, there pass still other $\mu_2 + 1$ through the tangent in ε.

For the order of the angle which $\varepsilon_{t+\delta}$ makes with $\varepsilon\varepsilon^{\mu_1+1}$ is the order of the product $\varepsilon_{t+\delta}\varepsilon\varepsilon^{\mu_1+1}$ or $\mu_1 + \mu_2 + 2$.

3. *The Third Class Singularity.* The point of osculation of an element ε of a plane-curve being defined as the limiting position of the point of intersection of three planes which are made independently of one another to approach ε, it can be proved that it is the point determined by ε, ε^{μ_1+1}, $\varepsilon^{\mu_1+\mu_2+2}$, μ_1 and μ_2 being the first and second class singularity indices of ε.

A third class singularity in ε *of the order* μ_3 *adds* μ_3 *to the order of "contact" of the curve with the point of osculation of* ε; *the order of contact in a plane whose three singularity indices are* μ_1, μ_2, μ_3, *being* $\mu_1 + \mu_2 + \mu_3 + 1$. *The number of consecutive planes through the point of osculation is* $\mu_1 + \mu_2 + \mu_3 + 3$.

For the order of the distance of $\varepsilon_{t+\delta}$ from $\varepsilon\varepsilon^{\mu_1+1}\varepsilon^{\mu_1+\mu_2+2}$ is $\mu_1 + \mu_2 + \mu_3 + 3$.

§ 4.

The curve point P and its osculating plane ε are corresponding elements of a point-curve and a plane-curve. We are to prove that if

$\varkappa_1, \varkappa_2, \varkappa_3$ *be the singularity indices of* P,
$\varkappa_3, \varkappa_2, \varkappa_1$ *are the singularity indices of* ε.

And reciprocally, if

μ_1, μ_2, μ_3 *be the singularity indices of* ε_1,
μ_3, μ_2, μ_1 *are the singularity indices of* P_1.

By hypothesis and the theorem on pages 161 and 162,

$$\varepsilon = PP^{\kappa_1+1}P^{\kappa_1+\kappa_2+2}.$$

It must be proved that

1. $[\varepsilon] \equiv \varepsilon', \varepsilon'', \ldots \varepsilon^{\kappa_3}, \not\equiv \varepsilon^{\kappa_3+1}$.
2. $[\varepsilon, \varepsilon^{\kappa_3+1}] \equiv \varepsilon^{\kappa_3+2}, \ldots \varepsilon^{\kappa_3+\kappa_2+1} \not\equiv \varepsilon^{\kappa_3+\kappa_2+2}$.
3. $[\varepsilon, \varepsilon^{\kappa_3+1}, \varepsilon^{\kappa_3+\kappa_2+2}] \equiv \varepsilon^{\kappa_3+\kappa_2+3}, \ldots \varepsilon^{\kappa_3+\kappa_2+\kappa_1+2} \not\equiv \varepsilon^{\kappa_3+\kappa_2+\kappa_1+3}$.

1. In the series of planes into which $\varepsilon^j = [PP^{\kappa_1+1}P^{\kappa_1+\kappa_2+2}]^j$ can be developed the highest index which any point factor can have is $\varkappa_1 + \varkappa_2 + 2 + j$. So long therefore as $j <, = \varkappa_3$, ε^j will be congruent with ε. But if $j = \varkappa_3 + 1$ the term

$$PP^{\kappa_1+1}P^{\kappa_1+\kappa_2+\kappa_3+3}$$

enters into the expansion of ε^j, and this cannot be congruent with ε since $P^{\kappa_1+\kappa_2+\kappa_3+3}$ lies without ε by hypothesis.

2. $$\varepsilon^{\kappa_3+1+j} = [PP^{\kappa_1+1}P^{\kappa_1+\kappa_2+2}]^{\kappa_3+1+j}$$

which equals, numerical coefficients being omitted,

$$[PP^{\kappa_1+1}]^{\kappa_3+1+j}P^{\kappa_1+\kappa_2+2} + \ldots [PP^{\kappa_1+1}]^{\kappa_3+1+j-\kappa_3}P^{\kappa_1+\kappa_2+\kappa_3+2} \qquad (a)$$
$$+ [PP^{\kappa_1+1}]^{j}P^{\kappa_1+\kappa_2+\kappa_3+3} + \ldots [PP^{\kappa_1+1}]P^{\kappa_1+\kappa_2+\kappa_3+3+j}. \qquad (b)$$

Now for every value of j, from 1 through $\varkappa_2$, all the products (a) are of points whose indices are $<, = \varkappa_1 + \varkappa_2 + \varkappa_3 + 2$; so that for every j which is $<, = \varkappa_2$, the planes (a) are congruent with ε.

And again, under the same hypothesis with reference to j, every factor $[PP^{x_1+1}]^l$ occurring in the terms of (b) develops into a sum of lines which are congruent with PP^{x_1+1}; so that (b) itself is a sum of planes all of which pass through PP^{x_1+1}, the intersection of ε with ε^{x_3+1}.

If, on the other hand, $j = x_2 + 1$, there occurs in (a), in addition to terms which are congruent with ε, the term $PP^{x_1+x_2+x_3+3}P^{x_1+x_2+2}$ with the numerical coefficient 1; and in (b), in addition to planes through the intersection of ε and ε^{x_3+1}, the term

$$\frac{(x_3 + x_2 + 1)!}{(x_2 + 1)!\ (x_3 + 1)!}\, PP^{x_1+x_2+2}P^{x_1+x_2+x_3+3}.$$

Now, since $P^{x_1+x_2+x_3+3}$ is without ε and $P^{x_1+x_2+2}$ off PP^{x_1+1}, $PP^{x_1+x_2+x_3+3}P^{x_1+x_2+2}$ cannot pass through $\varepsilon\varepsilon^{x_3+1}$; nor can the term in any case disappear, since no integral positive values of x_1, x_2, x_3 can be found capable of solving the equation $\dfrac{(x_3 + x_2 + 1)!}{(x_2 + 1)!\ (x_3 + 1)!} = 1$.

$$
\begin{aligned}
3.\quad \varepsilon^{x_3+x_2+2+j} &= [PP^{x_1+1}P^{x_1+x_2+2}]^{x_3+x_2+2+j} \\
&= P^{x_3+x_2+2+j}[P^{x_1+1}P^{x_1+x_2+2} + \dots \\
&\quad + P^{x_3+x_2+2+j-x_3}[P^{x_1+1}P^{x_1+x_2+2}]^{x_3} \qquad \text{(a)} \\
&\quad + P^{x_2+1+j}[P^{x_1+1}P^{x_1+x_2+2}]^{x_3+1} + \dots \\
&\quad + P^{x_2+1+j-x_2}[P^{x_1+1}P^{x_1+x_2+2}]^{x_3+1+x_2} \qquad \text{(b)} \\
&\quad + P^{j}[P^{x_1+1}P^{x_1+x_2+2}]^{x_3+x_2+2} + \dots \\
&\quad + P\,[P^{x_1+1}P^{x_1+x_2+2}]^{x_3+x_2+2+j}. \qquad \text{(c)}
\end{aligned}
$$

For every value of j, from 0 through x_1—

1.) The sum of the terms (a) is a plane which is congruent with ε, for the highest index which any point factor can bear is $x_3 + x_2 + x_1 + 2$.

2.) The sum of the terms (b) is a plane which passes through the intersection of ε and ε^{x_3+1}. For each of these terms is the product of a point of the line PP^{x_1+1} into a line $[P^{x_1+1}P^{x_1+x_2+2}]^{x_3+1+l}$, $l = 0, 1, 2, \dots x_2$. But

$$
\begin{aligned}
[P^{x_1+1}P^{x_1+x_2+2}]^{x_3+1+l} &= P^{x_1+1}P^{x_1+x_2+x_3+3+l} + \dots + P^{x_1+x_2+1}P^{x_1+x_3+3+l} \qquad (a')_l \\
&\quad + P^{x_1+x_2+2}P^{x_1+x_3+2+l} + \dots + P^{x_1+x_3+2+l}P^{x_1+x_2+2}, \qquad (b')_l
\end{aligned}
$$

where all the lines $(a')_l$ meet PP^{x_1+1} and all the lines $(b')_l$ lie in the plane ε. It follows, therefore that (b) which is

$$= \sum_{l=0}^{l=x_2} P^{x_2+1+j-l}[P^{x_1+1}P^{x_1+x_2+2}]^{x_3+1+l}$$

is a plane through PP^{x_1+1}, *i. e.* through $\varepsilon\varepsilon^{x_3+1}$.

3.) Each of the planes in the sum (c) passes through P, since

$$P \equiv P', P'', \ldots P^{\varkappa_1}$$

by hypothesis. Each of the planes $\varepsilon^{\varkappa_3+\varkappa_2+2+j}$, therefore, where j takes in succession the values $0, 1, \ldots \varkappa_1$ passes through the point P.

On the other hand, the plane $\varepsilon^{\varkappa_3+\varkappa_2+\varkappa_1+3}$ cannot pass through P. For in its development occurs, in addition to planes through P, the term

$$P^{\varkappa_3+\varkappa_2+\varkappa_1+3} P^{\varkappa_1+1} P^{\varkappa_1+\varkappa_2+2}$$

with the numerical coefficient

$$1 - \frac{(\varkappa_3+\varkappa_2+\varkappa_1+3)!}{(\varkappa_2+\varkappa_1+2)!\,(\varkappa_3+1)!} - \frac{(\varkappa_3+\varkappa_2+\varkappa_1+3)!}{(\varkappa_1+1)!\,(\varkappa_3+\varkappa_2+2)!} + \frac{\varkappa_3+\varkappa_2+\varkappa_1+3}{(\varkappa_1+1)!\,\varkappa_2+1)!\,(\varkappa_3+1)!}$$

and only this term, and this term can in no case disappear.

§ 5.*

At a curve point P the generating point continues the direction of its motion along the tangent line at P, if $\varkappa_1$, the index of the first class singularity of P be even, but reverses the direction of this motion if $\varkappa_1$ be odd.

At the element ε of a plane curve the enveloping element maintains the sense of its turning about the tangent line in ε, if μ_1, the index of the first class singularity of ε be even, but reverses the sense of this turning if μ_1 be odd.

The generating point takes in succession the positions $P_{t-\delta}$, P_t, $P_{t+\delta}$. We wish to determine whether the projections of the points $P_{t-\delta}$, $P_{t+\delta}$ on the tangent line lie to the same side of P_t or to opposite sides of it. All depends on whether the two products $P_t P_{t+\delta}$, $P_t P_{t-\delta}$ have the same or opposite signs; but, on the hypothesis that the index of the first class singularity of P_t is $\varkappa_1$, these products are $P_t P_t^{\varkappa_1+1} \frac{\delta^{\varkappa_1+1}}{(\varkappa_1+1)!}$ and $P_t P_t^{\varkappa_1+1} \frac{(-\delta)^{\varkappa_1+1}}{(\varkappa_1+1)!}$, respectively, and so have the same sign when $\varkappa_1$ is odd, opposite signs when $\varkappa_1$ is even.

At the point P the projection of the generating point on the osculating plane at P crosses the tangent if $\varkappa_1+\varkappa_2$, the sum of the indices of the first and second class singularity of P, be an odd number, but remains to the same side of this line if $\varkappa_1+\varkappa_2$ be an even number.

At the element ε of a plane-curve the line in which the enveloping element cuts ε, maintains the sense of its turning in ε if $\mu_1+\mu_2$, the sum of the indices of the first and second class singularity of ε be an odd number, but reverses the sense of this turning if $\mu_1+\mu_2$ be an even number.

---

* Here and wherever else the *form* of the curve is discussed, it is supposed of course that the coefficients of the series as well as t itself are real.

The proof is in principle the same as that given for the preceding theorem. We take the product of the tangent $PP^{\varkappa_1+1}$ into the point $P_{t+\delta}$ and into the point $P_{t-\delta}$; on the hypothesis that P_t has the singularity indices $\varkappa_1$, $\varkappa_2$, these products are

$$PP^{\varkappa_1+1}P^{\varkappa_1+\varkappa_2+2}\,\frac{\delta^{\varkappa_1+\varkappa_2+2}}{(\varkappa_1+\varkappa_2+2)!}\quad\text{and}\quad PP^{\varkappa_1+1}P^{\varkappa_1+\varkappa_2+2}\,\frac{(-\delta)^{\varkappa_1+\varkappa_2+2}}{(\varkappa_1+\varkappa_2+2)!},$$

and have the same or opposite signs according as $\varkappa_1+\varkappa_2$ is even or odd.

At the point P the generating point crosses the osculating plane if $\varkappa_1+\varkappa_2+\varkappa_3$, the sum of the three singularity indices of P, be an even number, but remains to the same side of it if $\varkappa_1+\varkappa_2+\varkappa_3$ be an odd number.

At the element ε of a plane-curve the enveloping element crosses the point of osculation if $\mu_1+\mu_2+\mu_3$, the sum of the three singularity indices of ε be an even number, but does not cross it if $\mu_1+\mu_2+\mu_3$ be an odd number.

The product of $P_tP_t^{\varkappa_1+1}P_t^{\varkappa_1+\varkappa_2+2}$ into $P_{t+\delta}$ is

$$P_tP_t^{\varkappa_1+1}P_t^{\varkappa_1+\varkappa_2+2}P_t^{\varkappa_1+\varkappa_2+\varkappa_3+3}\,\frac{\delta^{\varkappa_1+\varkappa_2+\varkappa_3+3}}{(\varkappa_1+\varkappa_2+\varkappa_3+3)!},$$

and its product into $P_{t-\delta}$ is $P_tP_t^{\varkappa_1+1}P_t^{\varkappa_1+\varkappa_2+2}P_t^{\varkappa_1+\varkappa_2+\varkappa_3+3}\,\dfrac{(-\delta)^{\varkappa_1+\varkappa_2+\varkappa_3+3}}{(\varkappa_1+\varkappa_2+\varkappa_3+3)!}$; these products have opposite signs if $\varkappa_1+\varkappa_2+\varkappa_3$ be even, the same signs if $\varkappa_1+\varkappa_2+\varkappa_3$ be odd.

If a point-curve be projected into an axial pencil or a plane-curve into a range of points, to the singularities of the originals will correspond singularities in the projections, the singularities of the projections, however, depending also on the position of the line which is the base of the pencil or range. With reference to the classes of singularity possible to the two systems into which we project—the same considerations which prove three sorts of singularity to be possible to an element of a point- or plane-curve of double curvature prove that but one sort is possible to the element of an axial pencil or a range of points; in these systems there are but *two* linearly independent elements and the one source of singularity is that elements should coincide, which, in the general case, are distinct.

Let P be an element of a point-curve of double curvature with the singularity indices $\varkappa_1$, $\varkappa_2$, $\varkappa_3$, let l be the tangent line and ε the osculating plane to the curve at P, and a the base of the axial pencil into which the curve is projected; also, let ε_1 be an element of a plane-curve, μ_1, μ_2, μ_3 its singularity indices, l_1 its tangent line, P_1 its point of osculation, and a_1 the base of the range into which it is projected. Then

The singularity index of that element of the axial pencil a which is the projection of P, is $\varkappa_1$, provided the point $a\varepsilon$ does not lie in l, (1); $\varkappa_1 + \varkappa_2 + 1$ when $a\varepsilon$ is in l but is not P, (2); $\varkappa_2$ when $a\varepsilon$ is congruent with P, (3); $\varkappa_1 + \varkappa_2 + \varkappa_3 + 2$ when a is in ε but does not pass through P, (4); $\varkappa_3 + \varkappa_{2+1}$ when a is in ε and through P, (5); and $\varkappa_3$ when a is itself l, (6).

The singularity index of that point of the range a_1 which is the projection of ε_1 is μ_1, provided the plane a_1P_1 does not pass through l_1 (1); is $\mu_1 + \mu_2 + 1$ when a_1P_1 passes through l_1 but does not coincide with ε_1, (2); μ_2 when a_1P_1 is congruent with ε_1, (3); $\mu_1 + \mu_2 + \mu_3 + 2$ when a_1 passes through P_1 but does not lie in ε_1, (4); $\mu_3 + \mu_2 + 1$ when a_1 is in ε_1 and contains P_1, (5); and μ_3 when a_1 is itself l_1, (6).

For if $P_{t+\delta}$ be a curve point consecutive to P_t, the projections of P_t and $P_{t+\delta}$, *i. e.* aP_t and $aP_{t+\delta}$, are consecutive elements of the axial pencil a. In case (1), *i. e.* when a does not meet l, $aP_{t+\delta} = aP + aP'.\delta +$ etc.; here $aP \equiv aP'$, $aP'', \ldots aP^{\varkappa_1}, \not\equiv aP^{\varkappa_1+1}$, and the singularity index of aP is $\varkappa_1$. In case (2) $a = P_0(\alpha P_t + \beta P_t^{\varkappa_1+1})$, P_0 being a point without ε; hence $aP_{t+\delta} = P_0(\alpha P_t + \beta P_t^{\varkappa_1+1})(P_t + P'\delta +$ etc.) and all the planes of this expansion through

$$P_0(\alpha P_t + \beta P_t^{\varkappa_1+1}) \frac{P^{\varkappa_1+\varkappa_2+1}\delta^{\varkappa_1+\varkappa_2+1}}{(\varkappa_1+\varkappa_2+1)!}$$

are congruent with $P_0(\alpha P_t + \beta P_t^{\varkappa_1+1})P_t$, but not the plane next following; so that the singularity index of aP is $\varkappa_1 + \varkappa_2 + 1$. When a passes through P, (3), it is equal to P_0P_t, P_0 having the same meaning as before. And

$$aP_{t+\delta} = P_0P_t(P_t + P_t'\delta + \text{etc.}) = P_0P_tP_t^{\varkappa_1+1}\frac{\delta^{\varkappa_1+1}}{(\varkappa_1+1)!} + \text{etc., or } \frac{aP_{t+\delta}}{\delta^{\varkappa_1+1}} = \frac{P_0P_tP_t^{\varkappa_1+1}}{(\varkappa_1+1)!}$$

+ etc. Of the terms of this expansion the first $(\varkappa_1 + \varkappa_2 + 1) - (\varkappa_1 + 1)$ are congruent, so that the singularity index of aP is $\varkappa_2$. The other cases admit of similar treatment. It seems necessary to remark only that in (4) $a = \alpha P_tP_t^{\varkappa_1+1} + \beta P_t^{\varkappa_1+1}P_t^{\varkappa_1+\varkappa_2+2} + \gamma PP^{\varkappa_1+\varkappa_2+2}$, in (5) $a = \alpha P_tP_t^{\varkappa_1+1} + \beta P_tP_t^{\varkappa_1+\varkappa_2+2}$, and in (6) $a \equiv PP^{\varkappa_1+1}$.

The demonstrations of the theorems in the right-hand column are simply the reciprocals of those just given. We have but to substitute the regressive products of planes for the progressive products of points.

Evidently the singularity of P is determined when the singularities of its projections in any three of the axial pencils (1), (2), ... (6) are known; and the like of ε_1.

The indices $\varkappa_1$, $\varkappa_2$, $\varkappa_3$ determine whether the projections of P in the several pencils a are "Rückkehrelemente"* or not; for the generating element of any of these systems maintains or reverses the sense of its turning about a as it reaches any fixed element, according as the singularity index of that element is even or odd; which in the case before us depends solely on the values of $\varkappa_1$, $\varkappa_2$, $\varkappa_3$. A similar relation holds between the numbers μ_1, μ_2, μ_3 and the motion of the generating element of each of the several ranges into which we project ε_1.

Besides projecting into one dimensional system, we may project both of our curves into two dimensional systems also; the point-curve into a pencil of rays, the plane-curve into the system of lines in a plane. The picture of the point-curve is then a cone, and the picture of the plane-curve a line-curve. Here, again, to singularities in the originals correspond singularities in the projections—the exact character of the correspondence depending on the position of the base of the pencil, or of the field of lines. The classes of singularity possible to an element of a cone or of a line-curve in a plane are two, the number of linearly independent elements in each system being but three. A ray l is singular in the first class to the order λ_1, and in the second class to the order λ_2, when

$$l \equiv l', l'', \ldots l^{\lambda_1} \mathrel{\equiv\!\!|\!\!\equiv} l^{\lambda_1+1} \text{ and } [l, l^{\lambda_1+1}] \equiv l^{\lambda_1+2}, \ldots l^{\lambda_1+\lambda_2+1}, \mathrel{\equiv\!\!|\!\!\equiv} l^{\lambda_1+\lambda_2+2}.$$

Let the symbols P, l, ε, P_1, l_1, ε_1 have the same meaning as in the preceding theorem; and let the point-curve P_t be projected into the pencil of rays whose centre is A and the plane-curve ε_t into the field α. Then

The two singularity indices of the ray which is the projection of P are $\varkappa_1$, $\varkappa_2$, when A does not lie in ε; $\varkappa_1$, $\varkappa_2 + \varkappa_3 + 1$, when A lies in ε but not in l; $\varkappa_1 + \varkappa_2 + 1$, $\varkappa_3$ when A lies in l but is not P; $\varkappa_2$, $\varkappa_3$ when A is P.

The two singularity indices† of the line in which ε_1 intersects α are μ_1, μ_2 when α does not pass through P_1; μ_1, $\mu_2 + \mu_3 + 1$ when α passes through P_1 but not through l_1; $\mu_1 + \mu_2 + 1$, μ_3 when α passes through l_1 but is not ε_1; μ_2, μ_3 when α is ε_1.

For when A does not lie in ε, in the product $AP_{t+\delta}$, or $AP + AP'\delta +$ etc., $AP \equiv AP', \ldots AP^{\varkappa_1}, \mathrel{\equiv\!\!|\!\!\equiv} AP^{\varkappa_1+1}$ and $[AP, AP^{\varkappa_1+1}] \equiv AP^{\varkappa_1+2}, \ldots AP^{\varkappa_1+\varkappa_2+1}, \mathrel{\equiv\!\!|\!\!\equiv} AP^{\varkappa_1+\varkappa_2+2}$. If A lie in ε but not in l, $AP \equiv AP', \ldots AP^{\varkappa_1}, \mathrel{\equiv\!\!|\!\!\equiv} AP^{\varkappa_1+1}$ and $[AP, AP^{\varkappa_1+1}] \equiv AP^{\varkappa_1+2}, \ldots AP^{\varkappa_1+\varkappa_2+\varkappa_3+2}, \mathrel{\equiv\!\!|\!\!\equiv} AP^{\varkappa_1+\varkappa_2+\varkappa_3+3}$; and the indices are $\varkappa_1$ and $\varkappa_1 + \underline{\varkappa_2 + \varkappa_3 + 2} - \underline{(\varkappa_1 + 1)} = \varkappa_2 + \varkappa_3 + 1$. When A lies in l and is not P, $AP = AP', \ldots AP^{\varkappa_1+\varkappa_2+1}, \mathrel{\equiv\!\!|\!\!\equiv} AP^{\varkappa_1+\varkappa_2+2}$ and $[AP, AP^{\varkappa_1+\varkappa_2+2}] \equiv AP^{\varkappa_1+\varkappa_2+3}, \ldots$

*Conf. von Staudt, Geom. d. Lage, p. 113.

† The two singularity indices of the element of a line-curve in a plane are its "inflexional" and "cuspidal" indices respectively.

$AP^{\kappa_1+\kappa_2+\kappa_3+2}$, $\equiv AP^{\kappa_1+\kappa_2+\kappa_3+3}$; and the indices are $\kappa_1+\kappa_2+1$ and $(\kappa_1+\kappa_2+\kappa_3+2) - (\kappa_1+\kappa_2+2) = \kappa_3$. Finally, when A is P all the terms $AP, \dots AP^{\kappa_1}$ vanish, so that the value of $AP_{t+\delta}$ expresses itself in a series of which the first term is $AP^{\kappa_1+1}\dfrac{\delta^{\kappa_1+1}}{(\kappa_1+1)!}$; with this term all those immediately following are congruent, while from it and $AP^{\kappa_1+\kappa_2+2}$ all the terms through $AP^{\kappa_1+\kappa_2+\kappa_3+2}$ are linearly derivable; in this case, therefore, the singularity indices are κ_2 and κ_3.

The theorems in the right-hand column are the reciprocals of those just demonstrated.

§6.

By a "line-curve," I mean a developable, *i. e.* not any continuous ∞^1 aggregate of the general line elements (Liniensummen) of space—such as the point-curve is of points and the plane-curve of planes—but only aggregates whose elements are the tangents to a point- or plane-curve. The general element of a line-curve δ then is $[PP^{\kappa_1+1}]_t$ or $[\epsilon\epsilon^{\kappa_3+1}]_t$. Its singularity finds expression in linear relations among the coefficients in the expansion of $[PP^{\kappa_1+1}]_{t+\delta}$. As there are in space but six linearly independent line elements, the possible number of these singularity-defining relations among the coefficients is five. We are to investigate to what extent they are consequent on the singularity of P_t and to what extent they are independent.

The general term of the expansion of $[PP^{\kappa_1+1}]_{t+\delta}$ is $\dfrac{\delta^j}{j!}[PP^{\kappa_1+1}]^j$. Omitting numerical coefficients

$$[PP^{\kappa_1+1}]^j = PP^{\kappa_1+1+j} + P'P^{\kappa_1+1+j-1} + \dots P^{\kappa_1}P^{\kappa_1+1+j-\kappa_1} \qquad \text{(a)}$$

$$+ P^{\kappa_1+1}P^j + P^{\kappa_1+2}P^{j-1} + \dots P^{\kappa_1+\kappa_2+1}P^{j-\kappa_2} \qquad \text{(b)}$$

$$\left.\begin{aligned} &+ P^{\kappa_1+\kappa_2+2}P^{j-(\kappa_2+1)} + P^{\kappa_1+\kappa_2+3}P^{j-(\kappa_2+2)} \\ &\qquad + \dots P^{\kappa_1+\kappa_2+\kappa_3+2}P^{j-(\kappa_2+\kappa_3+1)} \end{aligned}\right\} \qquad \text{(c)}$$

$$\left.\begin{aligned} &+ P^{\kappa_1+\kappa_2+\kappa_3+3}P^{j-(\kappa_2+\kappa_3+2)} + P^{\kappa_1+\kappa_2+\kappa_3+4}P^{j-(\kappa_2+\kappa_3+3)} \\ &\qquad + \dots P^{\kappa_1+2\kappa_2+\kappa_3+3}P^{j-(2\kappa_2+\kappa_3+2)} \end{aligned}\right\} \qquad \text{(d)}$$

$$+ P^{\kappa_1+2\kappa_2+\kappa_3+4}P^{j-(2\kappa_2+\kappa_3+3)} + \dots P^jP^{\kappa_1+1}. \qquad \text{(e)}$$

1. It will be noticed that the coefficients $[PP^{\kappa_1+1}]'$, $[PP^{\kappa_1+1}]'' \dots [PP^{\kappa_1+1}]^{\kappa_2}$ are congruent with PP^{κ_1+1}; but not so the next following coefficient. For so long as j does not exceed κ_2, $[PP^{\kappa_1+1}]^j$ is a sum of products of points which all lie on the line PP^{κ_1+1}; $[PP^{\kappa_1+1}]^{\kappa_2+1}$, on the other hand, contains in its expansion $PP^{\kappa_1+\kappa_2+2}$.

2. The coefficients to $[PP^{\kappa_1+1}]^{\kappa_2+\kappa_1+1}$ inclusive belong to the pencil of rays through P; but not so the coefficient next following. For so long as j does not

exceed $\varkappa_2 + \varkappa_1 + 1$, the sum into which $[PP^{\varkappa_1+1}]^j$ expands terminates in the row (b); the lines (a) all pass through P and the lines (b) coincide with $PP^{\varkappa_1+1}$: $[PP^{\varkappa_1+1}]^{\varkappa_2+\varkappa_1+2}$, on the other hand, contains in its expansion $P^{\varkappa_1+1}P^{\varkappa_1+\varkappa_2+2}$, which cannot pass through P.

3. The coefficients to $[PP^{\varkappa_1+1}]^{\varkappa_2+\varkappa_3+1}$ inclusive belong to the field of lines $PP^{\varkappa_1+1}P^{\varkappa_1+\varkappa_2+2}$, *i. e.* ε; but the next coefficient lies without this field. For all the points $P, \ldots P^{\varkappa_1+\varkappa_2+\varkappa_3+2}$ lie in ε, but $P^{\varkappa_1+\varkappa_2+\varkappa_3+3}$ without ε.

From 2 and 3 it follows that the coefficients to $[PP^{\varkappa_1+1}]^{\varkappa_2+\left\{\varkappa_1 \atop \varkappa_3\right\}+1}$ inclusive belong to the flat pencil $PP^{\varkappa_1+1}$, $PP^{\varkappa_1+\varkappa_2+2}$; it being understood that that one of the indices $\varkappa_1$, $\varkappa_3$ is to be read which is not the greater.

4. Every coefficient $[PP^{\varkappa_1+1}]^{\varkappa_1+\varkappa_2+\varkappa_3+2}$ belongs to one of the two systems, the pencil P or the field ε, or to both of them; but the coefficient $[PP^{\varkappa_1+1}]^{\varkappa_1+\varkappa_2+\varkappa_3+3}$ belongs to neither of them. For so long as j does not exceed $\varkappa_1 + \varkappa_2 + \varkappa_3 + 2$, all the terms (a) in the expansion of $[PP^{\varkappa_1+1}]^j$ pass through P, and all the terms from $P^{\varkappa_1+1}P^j$ on lie in ε. $[PP^{\varkappa_1+1}]^{\varkappa_1+\varkappa_2+\varkappa_3+3}$, on the other hand, contains $P^{\varkappa_1+1}P^{\varkappa_1+\varkappa_2+\varkappa_3+3}$, which does neither the one nor the other.

5. The coefficients to $[PP^{\varkappa_1+1}]^{\varkappa_1+2\varkappa_2+\varkappa_3+3}$ inclusive belong to the linear complex of which the axis is $PP^{\varkappa_1+1}$; but not the coefficient $[PP^{\varkappa_1+1}]^{\varkappa_1+2\varkappa_2+\varkappa_3+4}$. For so long as j does not exceed $\varkappa_1 + 2\varkappa_2 + \varkappa_3 + 3$, the expansion of $[PP^{\varkappa_1+1}]^j$ consists only of terms out of (a), (b), (c), (d); and of these the terms (a), (b), (d) contain a point of $PP^{\varkappa_1+1}$ as one of their factors and the terms (c) are the products of points which all lie in the plane ε. On the other hand, the expansion of $[PP^{\varkappa_1+1}]^{\varkappa_1+2\varkappa_2+\varkappa_3+4}$ contains the term $P^{\varkappa_1+\varkappa_2+2}P^{\varkappa_1+\varkappa_2+\varkappa_3+3}$, a line joining a point not on $PP^{\varkappa_1+1}$ with a point not in ε.

Therefore, the defining characteristics (defining because sufficient as well as necessary) of l, the element of the developable l_t corresponding to the point $P(\varkappa_1, \varkappa_2, \varkappa_3)$, are:

I. (1) *l is a line through P.*

(2) *$l^{\varkappa_2+1}$ is also a line through P, but not coincident with l.*

(3) *$l^{\varkappa_2+\varkappa_1+2}$ (if $\varkappa_1 < \varkappa_3$) is a line in the plane of l and $l^{\varkappa_2+1}$, but not through P.*

(4) *$l^{\varkappa_2+\varkappa_3+2}$ is the sum of a line through P and not in the plane of l and $l^{\varkappa_2+1}$, and a line in the plane of l and $l^{\varkappa_2+1}$ and not through P.*

(5) *$l^{\varkappa_1+\varkappa_2+\varkappa_3+3}$ is an element of the linear complex l which is not resolvable into a line through P and a line in the plane of l and $l^{\varkappa_2+1}$.*

(6) *$l^{\varkappa_1+2\varkappa_2+\varkappa_3+4}$ is not an element of the linear complex l.*

II. (1) $l \equiv l', l'', \ldots l^{\kappa_2}$.

(2) $[l, l^{\kappa_2+1}] \equiv l^{\kappa_2+2}, l^{\kappa_2+3}, \ldots l^{\kappa_2+\kappa_1+1}$.

(3) $[l, l^{\kappa_2+1}, l^{\kappa_2+\kappa_1+2}] \equiv l^{\kappa_2+\kappa_1+3}, l^{\kappa_2+\kappa_1+4}, \ldots l^{\kappa_2+\kappa_3+1}$.

(4) $[l, l^{\kappa_2+1}, l^{\kappa_2+\kappa_1+2}, l^{\kappa_2+\kappa_3+2}] \equiv l^{\kappa_2+\kappa_3+3}, \ldots l^{\kappa_1+\kappa_2+\kappa_3+2}$.

(5) $[l, l^{\kappa_2+1}, l^{\kappa_2+\kappa_1+2}, l^{\kappa_2+\kappa_3+2}, l^{\kappa_1+\kappa_2+\kappa_3+3}] \equiv l^{\kappa_1+\kappa_2+\kappa_3+4}, \ldots l^{\kappa_1+2\kappa_2+\kappa_3+3}$.

N. B. If $\kappa_1 > \kappa_3$, the indices κ_1 and κ_3 are to change places in both I and II. $l^{\kappa_2+\kappa_3+2}$ is then a line through P, but not in the plane of l and l^{κ_2+1}, and $l^{\kappa_2+\kappa_1+2}$ the sum of two lines, the one through P, the other in the plane of l and l^{κ_2+1}. If $\kappa_1 = \kappa_3$, both I (3) and II (3) disappear. The only one of the l^j's which *must* be the general line element, complex, is (4).

We may regard II (1), (2), (3), (4), (5) as the definitions of line singularities of four or five classes with order indices κ_2, κ_1, etc. They are conditioned solely by the singularity of P.

The possibility of an independent line singularity depends entirely on whether κ_1 and κ_3 be equal or unequal. For when $\kappa_1 \neq \kappa_3$, there are among the coefficients to $l^{\kappa_1+2\kappa_2+\kappa_3+4}$ inclusive six linearly independent lines, and from six linearly independent lines all the lines and complexes of space can be derived; in this case no singularity beyond II (5) is possible. But if $\kappa_1 = \kappa_3$, (3) falls away, and there are but five linearly independent lines among the coefficients to $l^{\kappa_1+2\kappa_2+\kappa_3+4}$ inclusive. These determine a linear complex to which a number of the coefficients $l^{\kappa_1+2\kappa_2+\kappa_3+5}$, $l^{\kappa_1+2\kappa_2+\kappa_3+6}$, etc., may belong.

I define an element of a developable as *singular in the line-class to the order* λ *when* $\kappa_1 = \kappa_3$, *and*

$$[l, l^{\kappa_2+1}, l^{\kappa_2+\kappa_1+2}, l^{\kappa_2+2\kappa_1+3}, l^{2\kappa_2+2\kappa_1+4}] \equiv l^{2\kappa_1+2\kappa_2+5}, \ldots l^{2\kappa_1+2\kappa_2+\lambda+4},$$
$$\neq l^{2\kappa_1+2\kappa_2+\lambda+5}.$$

The geometrical character of the singularities of the developable can be most easily studied by aid of its projection in a point-space of five dimensions. Let $l_1, l_2, \ldots l_6$ be any six linearly independent lines or linear complexes of an S_3, and let the six linearly independent points $P_1, P_2, \ldots P_6$ be their projections in S_5; every line and linear complex of S_3 is then of the form

$$\alpha_1 l_1 + \alpha_2 l_2 + \alpha_3 l_3 + \alpha_4 l_4 + \alpha_5 l_5 + \alpha_6 l_6,$$

and its projection in S_5 is $\alpha_1 P_1 + \alpha_2 P_2 + \ldots \alpha_6 P_6$. To linear relations among elements of the one space correspond linear relations among their projections in the other. Thus a ray of the flat pencil $\alpha_1 l_1 + \alpha_2 l_2$ (l_1 and l_2 being intersecting lines) projects into a point of the range $\alpha_1 P_1 + \alpha_2 P_2$, the rays of a pencil or the lines of a plane in S_3 project into the points of a plane in S_5, the elements of a linear complex in S_3, into the points of an S_4 in S_5. In particular, a developable in S_3 projects into a point-curve in S_5, its singular elements into singular elements of the point-curve with the same singularity indices. Applying now to this point-curve the methods developed in §2 and interpreting the results for the developable, we infer the following theorems: P, ε, l are supposed to be corresponding elements of the point-, plane- and line-curve C, with singularity indices $\varkappa_1$, $\varkappa_2$, $\varkappa_3$, λ.

1. *l is stationary to the degree $\varkappa_2$.*

2. *The developable has in l a contact of the $\varkappa_2 + \varkappa_1 + 1$ order with P; $\varkappa_2 + \varkappa_1 + 2$ consecutive elements of the developable belong to the pencil P.*

3. *The developable has in l a contact of the $\varkappa_2 + \varkappa_3 + 1$ order with ε; $\varkappa_2 + \varkappa_3 + 2$ consecutive elements of the developable belong to the field ε.*

4. *The developable has in l a contact of the $\varkappa_1 + 2\varkappa_2 + \varkappa_3 + 1$ order with the linear complex of which l is the axis; $\varkappa_1 + 2\varkappa_2 + \varkappa_3 + 4$ consecutive elements of the developable belong to this complex.*

5. *In particular, when $\varkappa_1 = \varkappa_3$, the developable has in l a contact of the order $2\varkappa_1 + 2\varkappa_2 + \lambda + 1$ with the linear complex determined by l, $l^{\varkappa_2+1}$, $l^{\varkappa_2+\varkappa_1+2}$, $l^{2\varkappa_1+\varkappa_2+3}$, $l^{2\varkappa_1+2\varkappa_2+4}$; $2\varkappa_1 + 2\varkappa_2 + \lambda + 5$ consecutive elements of the developable belong to this complex.*

In case $\varkappa_1 = \varkappa_3$, as the generating element of the developable moves away from l it first turns about P in ε, then splits into two lines, one through P and not in ε, the other in ε and not through P, becomes in succession a general element of the complex l and of the complex $(l, \ldots l^{2\varkappa_1+2\varkappa_2+4})$, and finally a general element of line space. In the other cases, before splitting it becomes a line in ε not through P, or a line through P not in ε; and passes directly from the complex l into unrestricted line space.

Again, at l the generating element

1. *maintains the sense of its turning in ε about P if $\varkappa_2$ be even, reverses it if $\varkappa_2$ be odd.*
2. *in turning about P, crosses ε or not according as $\varkappa_2 + \varkappa_3$ is odd or even.**
3. *in its motion in ε crosses P or not according as $\varkappa_2 + \varkappa_1$ is odd or even.*
4. *moves across l or not according as $\varkappa_1 + \varkappa_3$ is odd or even.*

The intersection of the developable with any plane α is a point-curve in α; and its projection in the pencil of planes through any point A is a cone of this pencil.

* *I. e.* every point on it crosses ε or not, as the case may be.

The two singularity indices of the point in α which is the trace of l are $\varkappa_2, \varkappa_3$ when α does not pass through P; $\varkappa_2 + \varkappa_1 + 1, \varkappa_3$ when α passes through P but not through l; $\varkappa_1, \varkappa_2 + \varkappa_3 + 1$ when α passes through l but is not ε; $\varkappa_1, \varkappa_2$ when α is ε.

The two singularity indices of the plane through A which is the projection of l are $\varkappa_2, \varkappa_1$ when A does not lie in ε; $\varkappa_2 + \varkappa_3 + 1, \varkappa_1$ when A lies in ε but not in l; $\varkappa_3, \varkappa_2 + \varkappa_1 + 1$ when A lies in l but is not P; $\varkappa_3, \varkappa_2$ when A is P.

These theorems have been already demonstrated (p. 169). They may also be demonstrated independently by aid of the development $l_{t+\delta} = l + l'\delta +$ etc.

From the Plücker equations connecting the ordinary singularities of the projections of the developable in α and A, Prof. Cayley has deduced equations connecting the ordinary singularities of the developable itself.* In the Plücker equations for the trace of the developable in α (α not passing through P) the singularity $\varkappa_1, \varkappa_2, \varkappa_3$ counts for $\varkappa_2$ simple cusps and $\varkappa_3$ simple inflexions; in the Plücker equations for the cone of planes which is the projection of the developable in A (A not in ε), it counts for $\varkappa_2$ simple inflexions and $\varkappa_1$ simple cusps. Therefore—

In respect to its influence on the order, class, rank and genus of its curve, the singularity $\varkappa_1, \varkappa_2, \varkappa_3$ is equivalent to $\varkappa_1$ points, $\varkappa_2$ lines, $\varkappa_3$ planes, all stationary to the first degree.

N. B. No account is here taken of the double points, double planes, etc., which enter also into the composition of higher singularities—due to the mutual intersections of the projections of the several partial branches which unite in the singular element. The series with which we operate and which define our singularities represent but single partial branches.†

§7.

An analytical point-curve of double curvature being defined in the neighborhood of the point $(x)_0$ by the equations

$$x_1 : x_2 : x_3 : x_4 = \phi_1(t) : \phi_2(t) : \phi_3(t) : \phi_4(t),$$

where $$\phi_i(t) = a_{i0} + a_{i1}(t - t_0) + \ldots a_{in}(t - t_0)^n + \text{etc.}$$

* *Vid.* Salmon's Geometry of Three Dimensions, p. 291.

† Conf. note at the bottom of page 157; also Cayley in Quarterly Journal, Vol. 7, and Smith in Proceedings London Math. Society, Vol. 6.

1. *The sufficient and necessary condition that the point $(x)_0$ be stationary to the $\varkappa_1$ degree is the vanishing of all the determinants of the second order in the matrix:*

$$\left\| \begin{array}{cccc} a_{10} & a_{20} & a_{30} & a_{40} \\ a_{11} & a_{21} & a_{31} & a_{41} \\ \cdot & \cdot & \cdot & \cdot \\ \cdot & \cdot & \cdot & \cdot \\ a_{1\varkappa_1} & a_{2\varkappa_1} & a_{3\varkappa_1} & a_{4\varkappa_1} \end{array} \right\|$$

For, by hypothesis, (x_0) is singular in the 1st class to the $\varkappa_1$ order; that is, $A_0 \equiv A_1, A_2, \ldots A_{\varkappa_1}$.* Therefore each of the products $A_0A_1, \ldots A_0A_{\varkappa_1}$, etc., must vanish.

But $A_0A_1 = [a_{10}E_1 + a_{20}E_2 + a_{30}E_3 + a_{40}E_4]\ [a_{11}E_1 + a_{21}E_2 + a_{31}E_3 + a_{41}E_4]$.

$$= \begin{vmatrix} a_{10} & a_{20} \\ a_{11} & a_{21} \end{vmatrix} E_1E_2 + \begin{vmatrix} a_{10} & a_{30} \\ a_{11} & a_{31} \end{vmatrix} E_1E_3 + \begin{vmatrix} a_{10} & a_{40} \\ a_{11} & a_{41} \end{vmatrix} E_1E_4$$

$$+ \begin{vmatrix} a_{30} & a_{40} \\ a_{31} & a_{41} \end{vmatrix} E_3E_4 + \begin{vmatrix} a_{40} & a_{20} \\ a_{41} & a_{21} \end{vmatrix} E_4E_2 + \begin{vmatrix} a_{20} & a_{30} \\ a_{21} & a_{31} \end{vmatrix} E_2E_3.$$

The points $E_1E_2E_3E_4$ are linearly independent, so that A_0A_1 can only be 0 when all the determinants $\begin{vmatrix} a_{10} & a_{20} \\ a_{11} & a_{21} \end{vmatrix}$, etc., vanish, and the same reasoning applies to all the remaining products $A_0A_{\varkappa_1}$, etc.

2. *That, in addition to being itself stationary to the $\varkappa_1$ degree, its tangent line be stationary to the $\varkappa_2$ degree, the necessary and sufficient conditions are* 1, *and the vanishing of all the determinants of the third order in the matrix:*

$$\left\| \begin{array}{cccc} a_{10} & a_{20} & a_{30} & a_{40} \\ a_{1,\varkappa_1+1} & a_{2,\varkappa_1+1} & a_{3,\varkappa_1+1} & a_{4,\varkappa_1+1} \\ a_{1,\varkappa_1+2} & a_{2,\varkappa_1+2} & a_{3,\varkappa_1+2} & a_{4,\varkappa_1+2} \\ \cdot & \cdot & \cdot & \cdot \\ \cdot & \cdot & \cdot & \cdot \\ a_{1,\varkappa_1+\varkappa_2+1} & a_{2,\varkappa_1+\varkappa_2+1} & a_{3,\varkappa_1+\varkappa_2+1} & a_{4,\varkappa_1+\varkappa_2+1} \end{array} \right\|$$

For the hypothesis requires that all the products

$$A_0A_{\varkappa_1+1}A_{\varkappa_1+2}, \ldots A_0A_{\varkappa_1+1}A_{\varkappa_1+\varkappa_2+1}$$

vanish, and that can only be when the determinants of the third order in the matrix 2 all vanish.

3. *That, in addition to the singularities $\varkappa_1$, $\varkappa_2$, it may have the further singularity that its osculating plane be stationary to the $\varkappa_3$ degree, the sufficient and necessary*

* *Vid.* §1.

conditions are 1, 2, *and the vanishing of all the determinants of the fourth order in the matrix:*

$$\left\| \begin{matrix} a_{10} & a_{20} & a_{30} & a_{40} \\ a_{1,\kappa_1+1} & a_{2,\kappa_1+1} & a_{3,\kappa_1+1} & a_{4,\kappa_1+1} \\ a_{1,\kappa_1+\kappa_2+2} & a_{2,\kappa_1+\kappa_2+2} & a_{3,\kappa_1+\kappa_2+2} & a_{4,\kappa_1+\kappa_2+2} \\ a_{1,\kappa_1+\kappa_2+3} & a_{2,\kappa_1+\kappa_2+3} & a_{3,\kappa_1+\kappa_2+3} & a_{4,\kappa_1+\kappa_2+3} \\ \cdots & \cdots & \cdots & \cdots \\ \cdots & \cdots & \cdots & \cdots \\ a_{1,\kappa_1+\kappa_2+\kappa_3+2} & a_{2,\kappa_1+\kappa_2+\kappa_3+2} & a_{3,\kappa_1+\kappa_2+\kappa_3+2} & a_{4,\kappa_1+\kappa_2+\kappa_3+2} \end{matrix} \right\|$$

1, 2, 3, express equally well the conditions of plane singularity: it is only necessary to suppose the given equations to be

$$u_1 : u_2 : u_3 : u_4 = \psi_1(t) : \psi_2(t) : \psi_3(t) : \psi_4(t).$$

The line singularities admit also of expressions of this sort, but for the general case they are very cumbrous and it does not seem worth while to give them.

§ 8.

If P_t be the Grassmann equation of the point-curve of double curvature

$$x_1 : x_2 : x_3 : x_4 = \phi_1(t) : \phi_2(t) : \phi_3(t) : \phi_4(t),$$

P_t', P_t'', etc., are the equations in the same analysis of the "differential curves"*

$$x_1 : x_2 : x_3 : x_4 = \phi_1'(t) : \phi_2'(t) : \phi_3'(t) : \phi_4'(t)\text{; etc.}$$

So also are ε_t', ε_t'', etc., the equations of the first, second, etc., differential (plane-) curves of ε_t.

No addition is made to our knowledge of singularities by regarding P_t', P_t'', etc., in this light, but it leads to a graphical representation of the conditions of singularity which is worth noticing.

Call the curve of which P_t^j is the point equation ${}_P\_C^j$, the curve of which ε_t^j ($\varepsilon_t = P_t P_t^{\kappa_1+1} P_t^{\kappa_1+\kappa_2+2}$) is the plane equation ${}_\varepsilon\_C^j$, and understand by "corresponding" elements of the curves such as are determined by the same value of t. Then

*A familiar example of such a differential curve is Hamilton's hodograph. If in the expansion $P_t = A_0 + A_1(t - t_0) + A_2(t - t_0)^2 +$ etc., only A_0 is a point and the rest of the coefficients, A_1, etc., are vectors (Strecken), the vector P_t' represents, both in direction and magnitude, the velocity of P_t; and if one extremity of it be fixed, the other will trace the hodograph of P_t. Or the curve which P_t' traces in the plane at infinity is the hodograph, the position of the points of this curve representing the direction, their weight the magnitude, of the velocity, of P_t. Conf. Grassmann, Ausdehnungslehre (1884), p. 146.

In a point of which the singularity indices are κ_1, κ_2, κ_3, a curve C is met by ${}_{p-}C'$, ${}_{p-}C''$, ... ${}_{p-}C^{\kappa_1}$ in the corresponding points of these curves; its tangent line passes through the corresponding points of ${}_{p-}C^{\kappa_1+1}$, ... ${}_{p-}C^{\kappa_1+\kappa_2+1}$; and its osculating plane through the corresponding points of ${}_{p-}C^{\kappa_1+\kappa_2+2}$, ... ${}_{p-}C^{\kappa_1+\kappa_2+\kappa_3+2}$: and conversely.

With a plane element of C of which the singularity indices are κ_3, κ_2, κ_1, the corresponding planes of ${}_{e-}C'$, ... ${}_{e-}C^{\kappa_3}$ coincide; through its tangent line pass the corresponding planes of ${}_{e-}C^{\kappa_3+1}$, ... ${}_{e-}C^{\kappa_3+\kappa_2+1}$; and through its point of osculation the corresponding planes of ${}_{e-}C^{\kappa_3+\kappa_2+2}$, ... ${}_{e-}C^{\kappa_3+\kappa_2+\kappa_1+2}$: and conversely.

These corresponding points of ${}_{p-}C'$, ${}_{p-}C''$, etc., have singularities of which the indices are determined within certain limits by κ_1, κ_2, κ_3. For the first κ_1 of them the indices are $\kappa_1 - 1$, κ_2, κ_3; $\kappa_1 - 2$, κ_2, κ_3; ...; 0, κ_2, κ_3 respectively. The rest are not determined absolutely; *e. g.* of the point on ${}_{p-}C^{\kappa_1+1}$ it is only required that the sum of its first and second-class indices be $\kappa_2 - 1$.

The curves ${}_{p-}C'$, ... ${}_{p-}C^{\kappa_1}$ have contacts of the orders $\kappa_1 + \kappa_2$, ... $\kappa_2 + 1$ with the tangent to C in the point $P(\kappa_1, \kappa_2, \kappa_3)$, and contacts of the orders $\kappa_1 + \kappa_2 + \kappa_3$, ..., $\kappa_2 + \kappa_3 + 1$ with its osculating plane. The curves ${}_{p-}C^{\kappa_1+1}$, ... ${}_{p-}C^{\kappa_1+\kappa_2}$ also are generally touched by this tangent and the curves ${}_{p-}C^{\kappa_1+1}$ ... ${}_{p-}C^{\kappa_1+\kappa_2+\kappa_3}$ by this osculating plane. If, however, the point corresponding to P on one of them, ${}_{p-}C^{j}$, have the first-class singularity index $\kappa_2 + \kappa_1 + 1 - j$, it and the curves ${}_{p-}C^{j+1}$, ... ${}_{p-}C^{\kappa_1+\kappa_2+1}$ are touched by the line joining those points of ${}_{p-}C^{j}$ and ${}_{p-}C^{\kappa_1+\kappa_2+2}$ which correspond to P. Even in this case the curves ${}_{p-}C'$, ... ${}_{p-}C^{\kappa_1+\kappa_2}$ are touched by the osculating plane of C, unless the corresponding point on ${}_{p-}C^{\kappa_1+\kappa_2+2}$ have the first-class singularity index κ_3. It is evident that the curves ${}_{p-}C^{\kappa_1+\kappa_2+j}$ ... ${}_{p-}C^{\kappa_1+\kappa_2+\kappa_3}$ also are not touched by the osculating plane of C if the first-class singularity index of the point on ${}_{p-}C^{\kappa_1+\kappa_2+j}$ be $\kappa_3 - j + 1$, or the sum of its first and second-class singularity indices $\kappa_3 - j + 2$.

The projections of ${}_{p-}C'$ ... ${}_{p-}C^{\kappa_1}$ upon the osculating plane of C at P touch the projection of C upon the same plane; the first, third, fifth, etc., of them crossing the projection of C, the second, fourth, sixth, etc., not crossing it.

There are reciprocal theorems for the plane-curves ${}_{e-}C'$, ${}_{e-}C''$, etc.

${}_{p-}C'$, ${}_{p-}C''$, etc., and ${}_{e-}C'$, ${}_{e-}C''$, etc., are two distinct sets of curves. By constructing the point equations of ${}_{e-}C'$, ${}_{e-}C''$, etc., it is found that all of these curves to ${}_{e-}C^{\kappa_1+\kappa_2+\kappa_3}$ inclusive meet C in $P(\kappa_1, \kappa_2, \kappa_3)$; but that the corresponding point of the envelope of ${}_{e-}C^{\kappa_1+\kappa_2+\kappa_3+1}$ may lie anywhere in space.

LEIPSIC, May 27, 1885.

A Proof of the Theorem

THE EQUATION $f(z) = 0$ HAS A ROOT WHERE $f(z)$ IS ANY HOLOMORPHIC FUNCTION OF z.

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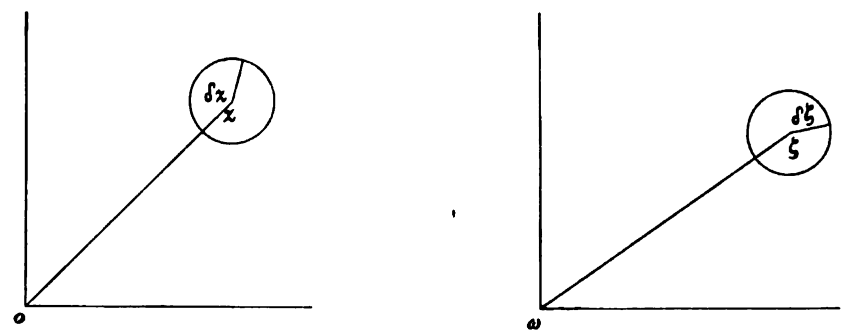
Represent z by a point in one plane and $f(z) = \zeta$ by a point in another plane.

If $f(z) = \zeta$ cannot become equal to zero for any value of z, there must be some minimum distance from the origin ω within which ζ cannot fall. Suppose ζ at such minimum distance from ω.

We can suppose this, for, $f(z)$ becoming infinite along with z, this minimum must be for a finite value of z, and can therefore be reached.



To z give an increment δz; the corresponding increment of ζ is $\delta\zeta = \frac{f^r(z)}{\underline{|r}}(\delta z)^r$ where $f^r(z)$ is the first of the successive derivatives of $f(z)$ which does not vanish for this value of z. Now varying δz, make it describe a closed curve round z; $\delta\zeta$ will at the same time describe a closed curve r times over round the point ζ, and will therefore come between the point ζ and origin ω. The point ζ has then no such minimum distance from the origin ω as was supposed; the function $f(z) = \zeta$ is therefore capable of becoming $= 0$, and the equation $f(z) = 0$ has therefore a root.

The above statement might be slightly varied, thus: to $f(z)$ we can give an increment in any direction we may choose; for δz, $\delta\zeta$, being any two corresponding increments of z, ζ, respectively; if the required increment is to be in a direction inclined to $\delta\zeta$ at an angle α, give to δz the rotation $\frac{\alpha}{r}$ and $\delta\zeta$ takes the required direction.

If mod (δz) remain constant while rotating round z, $\delta\zeta$ at the same time describes a circle round ζ, and we give ζ as much of an increment in one direction as in another.

We can always for a polynomial take $r = 1$, for if there be no root of $f(z) = 0$, ζ will have a minimum along each of radiating lines drawn through origin ω (for $f(z)$ can easily be shown to have some value on each of these lines), and as $f'(z)$ cannot vanish for more than $n - 1$ values of z, we can always choose our minimum along a line on which $f'(z)$ cannot vanish.

[Just as this note was about to go to press, I discovered that practically the same proof as above had been given by Hoüel in his Cours de Calcul Infinitésimal.—J. C. F.]

On a Linear Differential Equation of the Second Order.

By Thomas Craig.

I.

The differential equation to be studied is

$$\frac{d^2y}{dx^2}+\frac{A_{11}x^2+A_{12}x+A_{13}}{x(1-x^2)}\frac{dy}{dx}+\frac{A_{21}x^4+A_{22}x^3+A_{23}x^2+A_{24}x+A_{25}}{x^2(1-x^2)^2}y=0,$$

or for brevity

$$\frac{d^2y}{dx^2}=P_1\frac{dy}{dx}+P_2y.$$

The coefficients P_1 and P_2 have for points of (polar) discontinuity

$$x=0,\ x=1,\ x=-1.$$

The domain of the point $x=0$ is a circle of radius unity having the origin as center, that of the point $x=1$ is a circle of radius unity having the point $x=1$ as center, and similarly for the point $x=-1$. The domain of the point $x=\infty$ is the entire infinite plane lying outside the circle having the origin as center and radius unity. These several domains will be denoted by C_0, C_1 C_{-1}, C_∞, and the portion of the plane common to two domains C_i and C_j will be denoted by C_{ij}. The fundamental integrals of the equation in the domains C_i and C_j must coincide in the common domain C_{ij}.

In the domain of the point $x=0$ write the differential equation in the form

$$\frac{d^2y}{dx^2}=\frac{Q_1}{x}\frac{dy}{dx}+\frac{Q_2}{x^2}y.$$

The fundamental determinant equation for the point $x=0$ is

$$r(r-1)-rQ_1(0)-Q_2(0)=0,$$

or, on substituting for $Q_1(0)$ and $Q_2(0)$ their values

$$r(r-1)+rA_{13}+A_{25}$$

the roots of which are

$$r_1=\tfrac{1}{2}\{(1-A_{13})+\sqrt{(1-A_{13})^2-4A_{25}}\}$$
$$r_2=\tfrac{1}{2}\{(1-A_{13})-\sqrt{(1-A_{13})^2-4A_{25}}\}.$$

Assume as a special case $A_{13} = 1$, $A_{25} = 0$, then the roots are each equal to zero, and consequently the fundamental integrals of the equation in the domain of $x = 0$ are

$$y_1 = \phi_{11}(x),$$
$$y_2 = \phi_{21}(x) + \phi_{22}(x) \log x.$$

Where ϕ_{11}, ϕ_{21}, ϕ_{22} are, in the domain C_0, uniform and continuous functions of x and do not vanish for $x = 0$; and where ϕ_{22} differs only by a constant factor from ϕ_{11}.

Take the first integral and write for brevity $\phi_{11} = u$. Being a uniform and continuous function of x in the region of the point $x = 0$ and not vanishing for $x = 0$, we have for u the form

$$u = \sum_{a=0}^{a=\infty} C_a x^a,$$

a convergent series in which C_0 is not equal to zero. Substitute u in the differential equation and develop $(1 - x^2)^{-1}$ and $(1 - x^2)^{-2}$ in series going according to ascending powers of x. The product of $\frac{du}{dx}$ and the development of $(1 - x^2)^{-1}$ is

$$\sum_{n=0}^{n=\infty} A_n x^n,$$

where

$$A_n = \sum_{k=1}^{k=n+1} \frac{1 + e^{(n+k+1)i\pi}}{2} k C_k,$$

and multiplying this by the factor

$$-(A_{11}x^2 + A_{12}x + 1),$$

we have for the first term in the right-hand side of the differential equation

$$x \frac{d^2u}{dx^2} = -\frac{(A_{11}x^2 + A_{12}x + 1)}{1 - x^2} \frac{du}{dx} - \frac{(A_{21}x^3 + A_{22}x^2 + A_{23}x + A_{24})}{(1 - x^2)^2} u$$

the value

$$-\sum_{n=0}^{n=\infty} [A_{11}A_{n-2} + A_{12}A_{n-1} + A_n] x^n.$$

Multiplying u by the development of $(1 - x^2)^{-2}$, we have

$$\sum_{n=0}^{n=\infty} x^n \sum_{k=0}^{k=n} \frac{1 + e^{(n+k)i\pi}}{2} \left(\frac{n-k}{2} + 1\right) C_k,$$

or, writing for brevity

$$\sum_{k=0}^{k=n} \frac{1 + e^{(n+k)i\pi}}{2} \left(\frac{n-k}{2} + 1\right) C_k = B_n,$$

simply
$$\sum_{n=0}^{n=\infty} B_n x^n.$$
On multiplying by the remaining factor
$$-(A_{21}x^3 + A_{22}x^2 + A_{23}x + A_{24}),$$
the second term on the right-hand side of the differential equation is
$$-\sum_{n=0}^{n=\infty}\left[A_{21}B_{n-3} + A_{22}B_{n-2} + A_{23}B_{n-1} + A_{24}B_n\right]x^n.$$
The whole equation is now
$$\sum_{n=0}^{n=\infty} n(n+1)C_{n+1}x^n = -\sum_{n=0}^{n=\infty}\left[A_{11}A_{n-2} + A_{12}A_{n-1} + A_n\right]x^n$$
$$-\sum_{n=0}^{n=\infty}\left[A_{21}B_{n-3} + A_{22}B_{n-2} + A_{23}B_{n-1} + A_{24}B_n\right]x^n.$$

Equating the coefficients of x^n in this and we obtain a series of equations for the determination of C_0, C_1, C_2 .... It is clear that ultimately every C will be merely C_0 multiplied by a determinate constant. We have for C_{n+1} after transposing one term to the left-hand side of the equation
$$(n+1)^2 C_{n+1} = -A_{21}\sum_{k=0}^{k=n-3}\frac{1+e^{(n-3+k)i\pi}}{2}\left(\frac{n-3+k}{2}+1\right)C_k$$
$$-A_{22}\sum_{0}^{n-2}\frac{1+e^{(n-2+k)i\pi}}{2}\left(\frac{n-2+k}{2}+1\right)C_k$$
$$-A_{23}\sum_{0}^{n-1}\frac{1+e^{(n-1+k)i\pi}}{2}\left(\frac{n-1+k}{2}+1\right)C_k$$
$$-A_{24}\sum_{0}^{n}\frac{1+e^{(n+k)i\pi}}{2}\left(\frac{n+k}{2}+1\right)C_k$$
$$-A_{11}\sum_{1}^{n-1}\frac{1+e^{(n-1+k)i\pi}}{2}kC_k$$
$$-A_{12}\sum_{1}^{n}\frac{1+e^{(n+k)i\pi}}{2}kC_k$$
$$-\sum_{1}^{n}\frac{1+e^{(n+k+1)i\pi}}{2}kC_k.$$

In particular,

$$C_1 = -A_{24}C_0$$
$$C_2 = \tfrac{1}{4}\left[-A_{23} + A_{24}^2 + A_{13}A_{24}\right]C_0$$
$$C_3 = \tfrac{1}{36}\left[2A_{13}A_{23} - 3A_{13}A_{24}^2 - 2A_{13}^2A_{24} + 5A_{23}A_{24} - A_{24}^3 - 4A_{11}A_{24} - 4A_{24} - 4A_{22}\right],$$

etc.

If we assume

$$A_{24} = 0,\ A_{23} = -1,\ A_{22} = 0,\ A_{21} = 1,\ A_{13} = 1,\ A_{12} = 0,\ A_{11} = -3,$$

the differential equation becomes

$$\frac{d^2y}{dx^2} + \frac{1-3x^2}{x(1-x^2)}\frac{dy}{dx} - \frac{1}{1-x^2}y = 0,$$

which has for one integral the elliptic integral, K, of the first kind with modulus x. The values of the constants (computed from the general formula) are readily found to be

$$C_0,\ C_1 = 0,\ C_2 = C_0\tfrac{1}{4},\ C_3 = 0,\ C_4 = C_0\frac{1^2.3^2}{2^2.4^2},\ C_5 = 0,\ C_6 = C_0\frac{1^2.3^2.5^2}{2^2.4^2.6^2},\ \text{etc.}$$

Assuming then $C_0 = \frac{\pi}{2}$ we have for y_1 the value

$$y_1 = \frac{\pi}{2}\left[1 + \frac{1^2}{2^2}.x^2 + \frac{1^2.3^2}{2^2.4^2}x^4 + \frac{1^2.3^2.5^2}{2^2.4^2.6^2}x^6 + \ldots\right]$$

which is the known development of K in terms of the modulus.

I have similarly verified the general formula in the case of the differential equation, giving the elliptic integral of the second kind.

Another relation between five consecutive coefficients is at once obtained if we multiply the given equation through by $x(1-x^2)^2$ and then equate coefficients of x^n: this is

$$(n+1)^2C_{n+1} + (nA_{13} + A_{24})C_n + \left[-2(n-2)(n-1) + (n-1)(A_{11}-1) + A_{23}\right]C_{n-1}$$
$$+ \left[A_{22} - (n-2)A_{12}\right]C_{n-2} + \left[(n-3)(n-4) - (n-3)A_{11} + A_{21}\right]C_{n-3} = 0,$$

or say $\Phi(n, C) = 0$. Considered as a function of n we may write

$$\frac{d\Phi(n, C)}{dn} = 2(n+1)C_{n+1} + A_{13}C_n + \left[1 + A_{11} - 4(n-1)\right]C_{n-1}$$
$$- A_{12}C_{n-2} + \left[2(n-3) - (1 + A_{11})\right]C_{n-3}.$$

For the second integral of the equation we have

$$y_2 = \phi(x) + u\log x,$$

where $\phi(x)$ is a uniform continuous function of x in the domain C_0 and does not vanish for $x = 0$, we have then

$$\phi(x) = \sum_{n=0}^{n=\infty} c_n x^n.$$

Substituting in the differential equation and equating coefficients of x^n, we find as before the relation

$$2(n+1)C_{n+1}+A_{13}C_n+[1+A_{11}-4(n-1)]C_{n-1}-A_{12}C_{n-2}$$
$$+[2(n-3)-(1+A_{11})]C_{n-3}+(n+1)^2c_{n+1}+(nA_{13}+A_{24})c_n$$
$$+[-2(n-2)(n-1)+(n-1)(A_{11}-1)+A_{23}]c_{n-1}$$
$$+[A_{22}-(n-2)A_{12}]c_{n-2}+[(n-3)(n-4)-(n-3)A_{11}+A_{21}]c_{n-3}=0,$$

or, as this may obviously be written

$$\Phi(n,c)+\frac{d\Phi(n,C)}{dn}=0.$$

A simple application of the formulæ is to the differential equation

$$\frac{d^2y}{dx^2}+\frac{1-(\alpha+\beta+1)x}{x(1-x)}\frac{dy}{dx}-\frac{\alpha\beta}{x(1-x)}y=0,$$

of which a solution is the hypergeometric series $u=F(\alpha,\beta,1,x)$. Here we have

$$A_{11}=-(\alpha+\beta+1),\ A_{12}=-(\alpha+\beta),$$
$$A_{21}=\alpha\beta,\ A_{22}=\alpha\beta,\ A_{23}=-\alpha\beta,\ A_{24}=-\alpha\beta.$$

Applying the formula for $C_1, C_2, C_3\ldots$ and writing $C_0=1$, we find readily

$$C_1=\frac{\alpha.\beta}{1.1},\ C_2=\frac{\alpha(\alpha+1)\beta(\beta+1)}{1.2.1.2},$$
$$C_3=\frac{\alpha(\alpha+1)(\alpha+2)\beta(\beta+1)(\beta+2)}{1.2.3.1.2.3},\ \text{etc.}$$

Also for the coefficients $c_1, c_2\ldots$

$$c_1=\alpha\beta\left[c_0+\frac{1}{\alpha}+\frac{1}{\beta}-2(1)\right],$$
$$c_2=\frac{\alpha(\alpha+1)\beta(\beta+1)}{1.2.1.2}\left[c_0+\frac{1}{\alpha}+\frac{1}{\alpha+1}+\frac{1}{\beta}+\frac{1}{\beta+1}-2\left(1+\frac{1}{2}\right)\right],\ \text{etc.}$$

We have now found a system of fundamental integrals of the equation

$$\frac{d^2y}{dx^2}+\frac{A_{11}x^2+A_{12}x+A_{13}}{x(1-x^2)}\frac{dy}{dx}\frac{(A_{21}x^4+A_{22}x^3+A_{23}x^2+A_{24}x+A_{25})}{x^2(1-x^2)^2}y=0$$

in the domain of the point $x=0$, and for this case the roots of the fundamental determinant equation were taken to be equal and each equal to zero, *i. e.* we made $A_{13}=1$ and $A_{25}=0$. Removing this restriction now, we have for the roots of the fundamental determinant equation

$$r_1=\tfrac{1}{2}\{(1-A_{13})+\sqrt{(1-A_{13})^2-4A_{25}}\}$$
$$r_2=\tfrac{1}{2}\{(1-A_{13})-\sqrt{(1-A_{13})^2-4A_{25}}\}$$

or say

$$r_1=\alpha+\sqrt{\beta}$$
$$r_2=\alpha-\sqrt{\beta}.$$

We will assume first that β, *i. e.* $(1-A_{13})^2-4A_{25}$ is not equal to zero; now two cases arise, either r_1-r_2 is zero or an integer or the same difference is neither zero nor an integer. By hypothesis the difference cannot be zero, so we have only to consider the cases when $\sqrt{(1-A_{13})^2-4A_{25}}$ is and is not an integer.

Assume first that $\sqrt{(1-A_{13})^2-4A_{25}}$ is not an integer, then the fundamental integrals will be of the form

$$y_1=x^{r_1}\phi_1(x),$$
$$y_2=x^{r_2}\phi_2(x).$$

Where $\phi_1(x)$ and $\phi_2(x)$ are uniform and continuous functions of x in the domain C_0 and are not equal to zero for $x=0$. We have therefore

$$\phi_1(x),=u,=\sum_{n=0}^{n=\infty}\Delta_n x^n,\ \Delta_0 \text{ not } = \text{zero.}$$

The differential equation for u is

$$\frac{d^2u}{dx^2}=\frac{Q_1}{x}\frac{du}{dx}+\frac{Q_2}{x}u,$$

writing

$$Q_1'(x)=\frac{Q_1(x)-Q(0)}{x},$$

and substituting for Q_1 and Q_2 their values we have for the differential equation in u

$$x\frac{d^2u}{dx^2}+(A_{13}+2r)\frac{du}{dx}=-\left[\frac{x^2(A_{11}+A_{13}-2r)+A_{12}x+2r}{1-x^2}\right]\frac{du}{dx}$$
$$-\left[\begin{matrix}x^3\left[A_{21}-rA_{11}+r(r-1)\right]+x^2\left[A_{22}-rA_{12}\right]\\ +x\left[A_{23}-rA_{13}+rA_{11}-2r(r-1)\right]+\left[A_{24}+rA_{12}\right]\end{matrix}\right]\frac{u}{(1-x^2)^2}.$$

Substituting now for u its value, and equating the coefficients of x^n, we have, after some easy reductions,

$$\begin{aligned}(n+1)(n+A_{13})\Delta_{n+1}=&\left[-A_{21}+rA_{11}-r(r-1)\right]B_{n-3}+\left[rA_{12}-A_{22}\right]B_{n-2}\\&+\left[rA_{13}-rA_{11}-A_{23}+2r(r-1)\right]B_{n-1}-\left[A_{24}+rA_{12}\right]B_n\\&+\left[-A_{11}-A_{13}+2r\right]A_{n-2}-A_{12}A_{n-1}-2rA_n,\end{aligned}$$

where

$$B_n=\sum_{k=0}^{k=n}\frac{1+e^{(n+k)i\pi}}{2}\left(\frac{n-k}{2}+1\right)\Delta_k$$

$$A_n=\sum_{k=0}^{k=n}\frac{1+e^{(n+k)i\pi}}{2}(k+1)\Delta_{k+1}.$$

(It will be noticed that the coefficient A_{25} does not appear explicitly in the above formula.)

In particular make $r=0$, $A_{13}=1$; then for $n=0$ we have

$$\Delta_1 = -A_{24}\Delta_0,$$

for $n=1$ $$\Delta_2 = (-A_{23}+A_{24}^2+A_{13}A_{24})\,\Delta_0, \text{ etc.},$$

results agreeing with those obtained above.

If we compute the constants Δ from the formula, they will all be found to be of the form $$\Delta_n = \Gamma_n\Delta_0,$$

then giving to r the values r_1 and r_2 we find the two functions above designated as $\phi_1(x)$ and $\phi_2(x)$: the fundamental integrals of the differential equation in the domain C_0 are therefore

$$y_1 = x^{r_1}\phi_1(x),$$
$$y_2 = x^{r_2}\phi_2(x),$$

in the case where r_1 and r_2 are different and their difference is not an integer. A particular case of the above formula is when

$$A_{11} = -(\alpha+\beta+1),\quad A_{12}=\gamma-\alpha-\beta-1,\quad A_{13}=\gamma,$$
$$A_{21}=\alpha\beta,\quad A_{22}=\alpha\beta,\quad A_{23}=-\alpha\beta,\quad A_{24}=-\alpha\beta,\quad A_{25}=0.$$

The roots of the fundamental determinant equation are now

$$r_1 = 0,$$
$$r_2 = 1-\gamma,$$

and the differential equation is

$$\frac{d^2y}{dx^2}+\frac{\gamma-(\alpha+\beta+1)x}{x(1-x)}\frac{dy}{dx}+\frac{\alpha\beta}{x(1-x)}y=0,$$

which, when γ is not a negative integer, has for the integral corresponding to $r_1=0$ the hypergeometric series $F(\alpha, \beta, \gamma, x)$
and, corresponding to $r_2=1-\gamma$, when $2-\gamma$ is not a negative integer, the hypergeometric series

$$x^{1-\gamma}F(\alpha+1-\gamma,\ \beta+1-\gamma,\ 2-\gamma,\ x).$$

Substituting the above values of the A's in the formula we have first for $r_1=0$

$$\Delta_1 = \frac{\alpha\beta}{1.\gamma},$$
$$\Delta_2 = \frac{\alpha(\alpha+1)\beta(\beta+1)}{1.2.\gamma.\gamma+1}, \text{ etc.}$$

For the case of $r_2=1-\gamma$ the formula is similarly verified.

The relation between five consecutive coefficients is found by clearing the equation of fractions; doing this and substituting for n its value, we find

$$\{n(n+1)+(n+1)(A_{12}+2r)+[A_{25}+rA_{13}+r(r-1)]\}\Delta_{n+1}$$
$$+\{nA_{12}+A_{24}+rA_{12}\}\Delta_n$$
$$+\{-2(n-1)(n-2)+(n-1)(A_{11}-A_{13}-4rA_{13})$$
$$+[A_{23}+r(A_{11}-A_{13})-2r(r-1)]\Delta_{n-1}$$
$$+\{-(n-2)A_{12}+A_{22}-rA_{12}\}\Delta_{n-2}$$
$$+\{(n-3)(n-4)+[-A_{11}+2r](n-3)+[A_{21}-rA_{11}+r(r-1)]\}\Delta_{n-3}=0.$$

Take now the case where r_1-r_2 is an integer, *i. e.*

$$\sqrt{(1-A_{13})^2-4A_{25}}=\text{integer},$$

then of course $4A_{25}$ must be equal to the sum of two squares, one of which is an integer.

The two integrals now are of the form

$$y_1=x^{r_1}\phi_{11}$$
$$y_2=x^{r_2}[\phi_{21}+\phi_{22}\log x],$$

or say

$$y_1=x^{r_1}u$$
$$y_2=x^{r_2}[v+u\log x],$$

since ϕ_{22} only differs from ϕ_{11} by a constant factor, and since Cy_2 is an integral as well as y_2.

The first of these integrals is of course the same as the one obtained above. To find the second substitute in the differential equation: the coefficient of $\log x$ will be zero, and we have left

$$x\left\{\frac{d^2v}{dx^2}+\left[\frac{A_{11}x^2+A_{12}x+A_{13}}{x(1-x^2)}+\frac{2r}{x}\right]\frac{dv}{dx}\right.$$
$$\left.+\left[\frac{A_{21}x^4+A_{22}x^3+A_{23}x^2+A_{24}x+A_{25}}{x^2(1-x^2)^2}+\frac{r[A_{11}x^2+A_{12}x+A_{13}]}{x^2(1-x^2)}+\frac{r(r-1)}{x^2}\right]u\right\}$$
$$=-\left\{2\frac{du}{dx}+\left[\frac{2r-1}{x}+\frac{A_{11}x^2+A_{12}x+A_{13}}{x(1-x^2)}\right]u\right\}$$

or

$$x^2\frac{d^2v}{dx^2}+\frac{(A_{11}-2r)x^3+A_{12}x^2+(A_{13}+2r)x}{1-x^2}\frac{dv}{dx}$$
$$+\{x^4[A_{21}-rA_{11}+r(r-1)]+x^3[A_{22}-rA_{12}]+x^2[A_{23}+rA_{11}-rA_{13}-2r(r-1)]$$
$$+x[A_{24}+rA_{12}]\}\frac{1}{(1-x^2)^2}$$
$$=-2x\frac{du}{dx}+\frac{(A_{11}-2r+1)x^2+A_{12}x+A_{13}+2r-1}{1-x^2}u.$$

Multiplying out by $(1-x^2)^2$: writing

$$u=\sum_{n=0}^{n=\infty}\Delta_nx^n,\quad v=\sum_{n=0}^{n=\infty}\delta_nx^n,$$

and equating coefficients of x^{n+1}, we have for the determination of the coefficients in v, the relation

$$\begin{aligned}
&\{n(n+1)+(n+1)(A_{13}+2r)+[A_{25}+rA_{13}+r(r-1)]\}\,\delta_{n+1}\\
&+\{nA_{12}+A_{24}+rA_{12}\}\,\delta_n\\
&+\{-2(n-1)(n-2)+(n-1)(A_{11}-A_{13}-4rA_{13})\\
&\qquad\qquad+A_{23}+r(A_{11}-A_{13})-2r(r-1)\}\,\delta_{n-1}\\
&+\{-A_{12}(n-2)+A_{22}-rA_{12}\}\,\delta_{n-2}\\
&+\{(n-3)(n-4)+[-A_{11}+2r](n-3)+A_{21}-rA_{11}+r(r-1)\}\,\delta_{n-3}\\
&=[-2(n+1)-(A_{13}+2r-1)]\,\Delta_{n+1}-A_{12}\Delta_n\\
&+[4(n-1)-A_{11}+A_{13}+4r-2]\,\Delta_{n-1}+A_{12}\Delta_{n-2}-2(n-3)\,\Delta_{n-3}.
\end{aligned}$$

Here we must of course replace r by r_2, in order to find the values of δ.

The relation satisfied by the five consecutive coefficients Δ_{n+1}, Δ_n, Δ_{n-1}, Δ_{n-2}, Δ_{n-3} is what the above equation becomes when on the left-hand side δ is replaced by Δ, and the right-hand side is made equal to zero; using the same notation as that employed when $r_1=r_2=0$, this may be written

$$\Phi(n,\,r_1,\,\Delta)=0,$$

and consequently the relation giving the δ's may be written

$$\Phi(n,\,r_2,\,\delta)+\frac{d}{dn}\Phi(n,\,r_2,\,\Delta)=0.$$

(It will be observed that the coefficients of Δ_{n+1} and δ_{n+1} are integers, since $2r=1-A_{13}=$ integer.)

Consider now the domain C_1 of the point $x=1$. The fundamental determinant equation for this point is

$$r(r-1)+r\left[\frac{A_{11}+A_{12}+A_{13}}{2}\right]+\frac{A_{21}+A_{22}+A_{23}+A_{24}+A_{25}}{4}=0;$$

or writing

$$\frac{A_{11}+A_{12}+A_{13}}{2}=J_1,$$

$$\frac{A_{21}+A_{22}+A_{23}+A_{24}+A_{25}}{4}=J_2,$$

$$r^2-r(1-J_1)+J_2=0,$$

the roots of which are

$$r_1=\tfrac{1}{2}\{1-J_1+\sqrt{(1-J_1)^2-4J_2}\}$$
$$r_2=\tfrac{1}{2}\{1-J_1-\sqrt{(1-J_1)^2-4J_2}\}.$$

Suppose, first, $J_1=0$ and $J_2=0$, *i. e.*,

$$r_1=1,\ r_2=0.$$

The integrals of the differential equation are now of the form

$$y_1=(x-1)\,u$$
$$y_2=v+u\log(x-1),$$

where u and v are in the domain C_1 uniform and continuous functions of x, which for $x = 1$ do not vanish. We have then

$$u = \sum_{n=0}^{n=\infty} C_n (x-1)^n,$$

$$v = \sum_{n=0}^{n=\infty} c_n (x-1)^n.$$

The differential equation becomes now

$$\frac{d^2y}{dx^2} - \frac{A_{11}x + A_{11} + A_{12}}{x(1+x)} \frac{dy}{dx} - \Big[A_{21}x^3 + (A_{21} + A_{22})x^2 + (A_{21} + A_{22} + A_{23})x$$
$$+ \frac{(A_{21} + A_{22} + A_{23} + A_{24})}{x^2(x-1)(1+x)^2}\Big] y,$$

or $$\frac{d^2y}{dx^2} = \frac{A_{11}x - A_{13}}{x(x+1)} \frac{dy}{dx} + \frac{A_{21}x^3 + (A_{21} + A_{22})x^2 + (A_{21} + A_{22} + A_{23})x - A_{25}}{x^2(x-1)(x+1)^2} y.$$

Change the variable by the transformation

$$x = x' + 1.$$

Change also, for brevity, the notation for the coefficients, by writing

$$A_{11} = \alpha_1, \quad A_{11} - A_{13} = \delta_1,$$

$$A_{21} = \alpha_2,\ 4A_{21} + A_{22} = \beta_2,\ 6A_{21} + 3A_{22} + A_{23} = \gamma_2,\ 3A_{21} + 2A_{22} + A_{23} - A_{25} = \delta_2,$$

and the equation becomes

$$\frac{d^2y}{dx'^2} = \frac{\alpha_1 x' + \delta_1}{(x'+1)(x'+2)} \frac{dy}{dx'} + \frac{\alpha_2 x'^3 + \beta_2 x'^2 + \gamma_2 x' + \delta_2}{x'(x'+1)^2(x'+2)^2} y,$$

also

$$y_1 = x'u,$$

$$y_2 = v + u \log x',$$

$$u = \sum_{n=0}^{n=\infty} C_n x'^n,$$

$$v = \sum_{n=0}^{n=\infty} c_n x'^n.$$

For convenience, again write the equation in the form

$$\frac{d^2y}{dx^2} = \frac{P_1}{x'} \frac{dy}{dx'} + \frac{P_2}{x'^2} y,$$

where

$$P_1 = x' \frac{(\alpha_1 x' + \delta_1)}{(x'+1)(x'+2)},$$

$$P_2 = x' \frac{(\alpha_2 x'^3 + \beta_2 x'^2 + \gamma_2 x' + \delta_2)}{(x'+1)^2(x'+2)}.$$

It is at once seen that the roots of the fundamental determinant equation for the above differential equation are, in the domain of $x'=0$, $r_1=1$, $r_2=0$, as they should be. Write now $y=x'u$; substituting in

$$\frac{d^2y}{dx'^2}=\frac{\alpha_1x'+\delta_1}{(x'+1)(x'+2)}\frac{dy}{dx'}+\frac{\alpha_2x'^3+\beta_2x'^2+\gamma_2x'+\delta_2}{x'(x'+1)^2(x'+2)^2}y,$$

we have for u the equation

$$\frac{d^2u}{dx^2}=\left[\frac{(\alpha_1-2)x^2+(\delta_1-6)x-4}{x(x+1)(x+2)}\right]\frac{du}{dx}$$
$$+\left[\frac{(\alpha_1+\alpha_2)x^4+(\beta_1+3\alpha_1+\delta_1)x^3+(\gamma_1+2\alpha_1+\delta_1)x^2+(2\delta_1+\delta_2)x}{x^2(x+1)^2(x+2)^2}\right]u.$$

I write, for convenience (as no confusion is likely to arise), x instead of x'.

Clearing this of fractions, substituting for u its value, and equating the coefficients of x^{n+1}, we find, without much difficulty, the relation

$$\begin{aligned}&4(n+1)(n+2)C_{n+1}+[12n(n-1)-n(2\delta_1-24)-(2\delta_1+\delta_2)]C_n\\&+[13(n-1)(n-2)-(n-1)(2\alpha_1+3\delta_1-26)-(\gamma_2+2\alpha_1+3\delta_1)]C_{n-1}\\&+[6(n-2)(n-3)-(n-2)(\delta_1+3\alpha_1-12)-(\beta_2+3\alpha_1+\delta_1)]C_{n-2}\\&+[(n-3)(n-4)-(\alpha_1-2)(n-3)-(\alpha_1+\alpha_2)]C_{n-3}=0.\end{aligned}$$

As an example of this, take the case of

$$\alpha_1=-1,\quad \delta_1=-2,$$
$$\alpha_2=-1,\quad \beta_2=-4,\quad \gamma_2=-4,\quad \delta_2=0.$$

The differential equation in y now becomes

$$\frac{d^2y}{dx^2}+\frac{1}{x+1}\frac{dy}{dx}+\frac{1}{(x+1)^2}y=0,$$

the integrals of which are

$$y_1=\sin\log(x+1),$$
$$y_2=\cos\log(x+1).$$

The first of these is the one to be considered. Expanding $\sin\log(x+1)$ in a series going according to ascending powers of x, we have

$$\sin\log(x+1)=u_0+u_1x+u_2x^2+\dots$$

where $\qquad n(n+1)u_{n+1}+n(2n-1)u_n+(n^2-2n+2)u_{n-1}=0;$

further, since for $x=0$ we have $\sin\log(x+1)=0$, it follows that $u_0=0$.

The next few coefficients are u_1 (which is unity—but may obviously be replaced by an arbitrary constant in the expression for the integral), $u_2=-\frac{1}{2}u_1$, $u_3=\frac{1}{6}u_1$, $u_4=0$, $u_5=-\frac{1}{12}u_1$, $u_6=\frac{1}{8}u_1$, etc.

We ought now to have

$$C_0=u_1,\quad C_1=u_2,\quad C_2=u_3,\quad C_3=0,\quad C_4=u_5,\ \text{etc.},$$

and by substitution of the above values of $\alpha_1, \delta_1, \alpha_2, \ldots \delta_2$ in the formula, it is easily seen that these conditions are all satisfied. Another relation connecting the coefficients C in this particular case is found by expanding $\frac{1}{1+x}$ and $\frac{1}{(1+x)^2}$ in the differential equation

$$\frac{d^2(xu)}{dx^2} + \frac{1}{1+x}\frac{d(xu)}{dx} + \frac{xu}{(1+x)^2} = 0,$$

and equating to zero the coefficient of x^n: this is

$$(n+1)(n+2)C_{n+1} + \sum_{k=0}^{k=n-1}(-)^{n+k-1}[5(k+1)C_{k+1} + 2(n-k)C_k] + \sum_{k=0}^{k=n}(-)^{n+k}(n-k+1)C_k = 0,$$

or, replacing n by $n-1$ and C_n by u_{n+1}, we have for the relation connecting the coefficients $u_1, u_2, \ldots$

$$n(n+1)u_{n+1} + \sum_{k=0}^{k=n-2}(-)^{(n+k-2)}[5(k+1)u_{k+2} + 2(u-k+1)u_{k+1}] + \sum_{k=0}^{k=n-1}(-)^{n+k-1}(n-k)\,u_{k+1} = 0.$$

Having found the values of the coefficients C, and consequently the value of the function u, it is only necessary to replace x by $x-1$ in order to get the first integral

$$y_1 = (x-1)\,u$$

of the original differential equation. The second integral is in general of the form

$$y_2 = v + u\log(x-1),$$

or, changing the variable again, simply

$$y_2 = v + u\log x,$$

where u and v go according to ascending powers of x. In the simple case just noted, viz.

$$\frac{d^2y}{dx^2} + \frac{1}{(x+1)}\frac{dy}{dx} + \frac{1}{(x+1)^2}\,y = 0,$$

there is evidently no logarithm—the two integrals being in fact, as already mentioned,

$$y_1 = \sin\log(x+1),\ (= xu),$$
$$y_2 = \cos\log(x+1),$$

the coefficients in the developments being in each case connected by the relation

$$n(n+1)\,u_{n+1} + n(2n-1)\,u_n + (n^2-2n+2)\,u_{n-1} = 0,$$

with, in the case of y_1, $u_0 = 0$.

To determine whether or not logarithms exist in the case at present treated, *i. e.* when the roots of the fundamental determinant equation are $r_1 = 1$, $r_2 = 0$, and where consequently $r_1 - r_2 = 1$, we form the differential equation satisfied by the second derivatives of the integrals y_1 and y_2; according as there exist or do not exist negative roots of the fundamental determinant equation belonging to this new differential equation, there exist or do not exist logarithms in the original equation in y.

Resuming now the differential equation

$$\frac{d^2y}{dx^2} = \frac{\alpha_1 x + \delta_1}{(x+1)(x+2)}\frac{dy}{dx} + \frac{\alpha_2 x^3 + \beta_2 x^2 + \gamma_2 x + \delta_2}{x(x+1)^2(x+2)^2} y,$$

write

$$y = v + u \log x$$

(here, as before, x is written for x'). After some easy reductions, we find for v the equation

$$[x(x+1)(x+2)]^2 \frac{d^2v}{dx^2} - x^2(x+1)(x+2)(\alpha_1 x + \delta_1)\frac{dv}{dx} - x(\alpha_2 x^3 + \beta_2 x^2 + \gamma_2 x + \delta_2) v$$

$$= -2x(x+1)^2(x+2)^2\frac{du}{dx} + [(x+1)^2(x+2)^2 + x(x+1)(x+2)(\alpha_1 x + \delta_1)]\, u.$$

Substituting for u and v their values, viz.,

$$u = \sum_{n=0}^{n=\infty} C_n x^n, \quad v = \sum_{n=0}^{n=\infty} c_n x^n,$$

and equating coefficients of x^{n+1}, we have for the relation connecting the coefficients C and c

$$\begin{aligned} 4n(n+1)c_{n+1} &+ [12n(n-1) - 2\delta_1 n - \delta_2]\, c_n \\ &+ [13(n-1)(n-2) - (n-1)(2\alpha_1 + 3\delta_1) - \gamma_2]\, c_{n-1} \\ &+ [6(n-2)(n-3) - (n-2)(3\alpha_1 + \delta_1) - \beta_2]\, c_{n-2} \\ &+ [(n-3)(n-4) - \alpha_1(n-3) - \alpha_2]\, c_{n-3} \end{aligned}$$

$$= -4(2n+1)C_{n+1} + [-24n + 12 + 2\delta_1]\, C_n + [-26(n-1) + 13 + 2\alpha_1 + 3\delta_1]\, C_{n-1}$$
$$+ [-12(n-2) + 6 + 3\alpha_1 + \delta_1]\, C_{n-2} + [-2(n-3) + (1 + \alpha_1)]\, C_{n-3}.$$

In the particular case already referred to, viz.

$$\frac{d^2y}{dx} + \frac{1}{(x+1)}\frac{dy}{dx} + \frac{1}{(x+1)^2} y = 0$$

(*i. e.* $\alpha_1 = -1$, $\delta_1 = -2$, $\alpha_2 = -1$, $\beta_2 = -4$, $\gamma_2 = -4$, $\delta_2 = 0$), the integral which we are in search of is simply the function v, consequently the values of the constants c are found by writing the left-hand member of the above equation $= 0$; it is easy to see then that the coefficients c of the function v are the same as those in the development of $\cos \log (x + 1)$.

PHOTOGRAPH

OF THE

NORMAL SOLAR SPECTRUM

MADE BY

PROFESSOR H. A. ROWLAND.

AT THE

JOHNS HOPKINS UNIVERSITY, BALTIMORE.

This photographic map of the Solar Spectrum is now complete from wave length 3680. to 5790., and the portion above 3680. to the extremity of the ultra violet, wave length about 3100., is nearly ready. Negatives have also been prepared down to and including *B*, and it is possible they may be prepared for publication.

These photographs have been made with one of Professor Rowland's concave gratings of 21½ ft. radius of curvature and 6 in. diameter, mounted so as to preserve the focus constant and give a normal spectrum of the same scale for any given spectrum. A scale of wave lengths has been added, so that the whole makes a map of the spectrum, down to wave length 5790., more exact and giving greater detail than any other map now in existence. The error in the wave length at no part exceeds $\frac{1}{50000}$ of the whole, and is generally caused by a slight displacement of the scale, which is easily corrected. The wave lengths of more than 200 lines in the spectrum have been accurately determined to about $\frac{1}{500000}$ part, and these can serve as standards to correct any small error of the scale. It is to be noted that the photograph of the spectrum can have none of the local irregularities of wave length which occur in all engraved maps, and which amount to more than $\frac{1}{20000}$ part in all so far published. This often so distorts a group of lines as, in conjunction with the imperfect intensities, to render them almost unrecognizable.

The definition of the spectrum is more than equal in every part, down at least to wave length 5325., to any map so far published. The 1474 line is widely double, as also b_3 and b_4, while *E* is given so nearly double as to be recognized as such by all persons familiar with spectrum observation. Above the green the superiority increases very quickly, so that at *H* we have 120 lines between *H* and *K*, while the original negatives show 150 lines. The photographs show more at this point than the excellent map of Lockyer of this region. Above *H* to wave lengths 3200. the number of lines in excess of all published maps is so great as to make all comparison useless. However, above *H* the determination of the wave length is more uncertain than in the visible parts, and must remain so until a special investigation can be made.

The plates all contain two strips of the spectrum, except No. 2, which contains three. They are three feet long and one foot wide.

Plate No. 1.	Wave length	3710. to 4135.	Scale 4 times	Angström.
2.	" "	3100. to 3730.	" 3 "	"
3.	" "	3680. to 4135.	" 3 "	"
4.	" "	4075. to 4530.	" 3 "	"
5.	" "	4480. to 4935.	" 3 "	"
6.	" "	4875. to 5325.	" 3 "	"
7.	" "	5210. to 5790.	" 2 "	"

These can now all be furnished to order except No. 2, the negative of which is being made.

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Take up now the case where r_1 and r_2 differ from each other and the difference is not an integer. We may say for convenience that r_1 and r_2 are *like* when they differ by either zero or an integer, and *unlike* when the difference $r_1 - r_2$ is neither zero nor an integer. We have now r_1 and r_2 unlike, and

$$r_1 = \tfrac{1}{2}\{(1-J_1) + \sqrt{(1-J_1)^2 - 4J_2}\},$$
$$r_2 = \tfrac{1}{2}\{(1-J_1) - \sqrt{(1-J_1)^2 - 4J_2}\},$$
$$r_1 - r_2 = \sqrt{(1-J_1)^2 - 4J_2},$$
$$J_1 = \frac{A_{11} + A_{12} + A_{13}}{2},$$
$$J_2 = \frac{A_{21} + A_{22} + A_{23} + A_{24} + A_{25}}{4}.$$

The integrals of the differential equation are therefore of the form

$$y_1 = (x-1)^{r_1} u$$
$$y_2 = (x-1)^{r_2} v,$$

where

$$u = \sum_{n=0}^{n=\infty} \Delta_n (x-1)^n, \quad v = \sum_{n=0}^{n=\infty} \delta_n (x-1)^n.$$

The differential equation is

$$\frac{d^2y}{dx^2} + \frac{A_{11}x^2 + A_{12}x + A_{13}}{x(1-x^2)}\frac{dy}{dx} + \frac{A_{21}x^4 + A_{22}x^3 + A_{23}x^2 + A_{24}x + A_{25}}{x^2(1-x^2)^2}y = 0;$$

change the variable by the relation

$$x = x' + 1$$

and in the result write for convenience x instead of x': also write

$$A_{11} = -\alpha_1, \; 2A_{11} + A_{12} = -\beta_1, \; A_{11} + A_{12} + A_{13} = -\eta_1,$$
$$A_{21} = -\alpha_2, \; 4A_{21} + A_{22} = -\beta_2, \; 6A_{21} + 3A_{22} + A_{23} = -\gamma_2,$$
$$4A_{21} + 3A_{22} + 2A_{23} + A_{24} = -\zeta_2, \; A_{21} + A_{22} + A_{23} + A_{24} + A_{25} = -\eta_2,$$

and we have

$$\frac{d^2y}{dx^2} = \frac{\alpha_1 x^2 + \beta_1 x + \eta_1}{x(x+1)(x+2)}\frac{dy}{dx} + \frac{\alpha_2 x^4 + \beta_2 x^3 + \gamma_2 x^2 + \zeta_2 x + \eta_2}{x^2(x+1)^2(x+2)^2}y,$$

and also

$$y_1 = x^{r_1}u, \quad u = \sum_{n=0}^{n=\infty} \Delta_n x^n,$$

$$y_2 = x^{r_2}v, \quad v = \sum_{n=0}^{n=\infty} \delta_n x^n.$$

The form of the integral y_1 (*i. e.* the values of the coefficients Δ) having been obtained, it will be only necessary to change r_1 into r_2 in order to find the values of the coefficients δ—that is, in order to find the integral y_2. Writing now $y = x^r u$, we find for u the differential equation

$$\frac{d^2u}{dx^2} = \frac{(\alpha_1 - 2r)x^2 + (\beta_1 - 6r)x + (\eta_1 - 4r)}{x(x+1)(x+2)}\frac{du}{dx} + \frac{1}{x^2(x+1)^2(x+2)^2}\{[\alpha_2 + r\alpha_1 - r(r-1)]x^4$$
$$+ [\beta_2 + r(\beta_1 + 3\alpha_1) - 6r(r-1)]x^3 + [\gamma_2 + r(2\alpha_1 + 3\beta_1 + \eta_1) - 13r(r-1)]x^2$$
$$+ [\zeta_2 + r(2\beta_1 + 3\eta_1) - 12r(r-1)]x + [\eta_2 + 2r\eta_1 - 4r(r-1)]\}u.$$

Multiplying out by $[x(x+1)(x+2)]^2$ we have

$$\{x^6 + 6x^5 + 13x^4 + 12x^3 + 4x^2\}\frac{d^2u}{dx^2}$$
$$= \{(\alpha_1 - 2r)x^3 + (\beta_1 + 3\alpha_1 - 12r)x^4 + (\eta_1 + 3\beta_1 + 2\alpha_1 - 26r)x^3 + (3\eta_1 + 2\beta_1 - 24r)x^2$$
$$+ (2\eta_1 - 8r)\}\frac{du}{dx} + \{[\alpha_2 + r\alpha_1 - r(r-1)]x^4 + [\beta_2 + r(\beta_1 + 3\alpha_1) - 6r(r-1)]x^3$$
$$+ [\gamma_2 + r(2\alpha_1 + 3\beta_1 + \eta_1) - 13r(r-1)]x^2 + [\zeta_2 + r(2\beta_1 + 3\eta_1) - 12r(r-1)]x$$
$$+ [\eta_2 + 2r\eta_1 - 4r(r-1)]\}u.$$

Substituting for u its value, viz.

$$u = \sum_{n=0}^{n=\infty} \Delta_n x^n,$$

and equating the coefficients of x^{n+1}, we find for the relation connecting five consecutive coefficients Δ

$$\{4n(n+1) - (n+1)(2\eta_1 - 8r) - [\eta_2 + 2r\eta_1 - 4r(r-1)]\}\Delta_{n+1}$$
$$+ \{12n(n-1) - n(3\eta_1 + 2\beta_1 - 24r) - [\zeta_2 + r(2\beta_1 + 3\eta_1) - 12r(r-1)]\}\Delta_n$$
$$+ \{13(n-1)(n-2) - (n-1)(\eta_1 + 3\beta_1 + 2\alpha_1 - 26r)$$
$$- [\gamma_2 + r(2\alpha_1 + 3\beta_1 + \eta_1) - 13r(r-1)]\}\Delta_{n-1}$$
$$+ \{6(n-2)(n-3) - (n-2)(\beta_1 + 3\alpha_1 - 12r)$$
$$- [\beta_2 + r(\beta_1 + 3\alpha) - 6r(r-1)]\}\Delta_{n-2}$$
$$+ \{(n-3)(n-4) - (n-3)(\alpha_1 - 2r) - [\alpha_2 + r\alpha_1 - r(r-1)]\}\Delta_{n-3} = 0.$$

Computing the coefficients Δ by aid of this formula, replacing x by $x - 1$ and the coefficients α, β, etc., by their values in $A_{11}, A_{12} \ldots A_{25}$, we have the desired integral of the original differential equation in the region C_1.

Suppose, now, that the roots of the fundamental determinant equations differ by an integer, *i. e.* $r_1 - r_2 =$ (positive) integer. The integrals are of the form $Y_1 = x^{r_1}\log x$, $Y_2 = x^{r_2}[v + n\log x]$.

The integral J_1 has its coefficients determined by the last formula. Write

$$v = \sum_{n=0}^{n=\infty} \delta_n x^n$$

and substitute Y_2 in the differential equation; we find (writing r for r_2)

$$\{x^6 + 6x^5 + 13x^4 + 12x^3 + 4x^2\} \frac{d^2v}{\delta x^2},$$

$$\{(\alpha_1 - 2r)x^5 + (\beta_1 + 3\alpha_1 - 12r)x^4 + (\eta_1 + 3\beta_1 + 2\alpha_1 - 26r)x^3 + (3\eta_1 + 2\beta_1 - 24r)x^2$$
$$+ (2\eta_1 - 8r)x\} \frac{dv}{\delta x} - \{[\alpha_2 + r\alpha_1 - r(r-1)]x^4 + [\beta_2 + r(\beta_1 + 3\alpha_1) - 6r(r-1)]x^3$$
$$+ [\gamma_2 + r(2\alpha_1 + 3\beta_1 + \eta_1) - 13r(r-1)]x^2 + [\zeta_2 + r(2\beta_1 + 3\eta_1) - 12r(r-1)]x$$
$$+ [\eta_2 + 2r\eta_1 - 4r(r-1)]\}v = -[\{2x^5 + 12x^4 + 26x^3 + 24x^2 + 8x\} \frac{du}{\delta x}$$
$$+ \{(2r - 1 - \alpha_1)x^4 + [6(2r-1) - 3\alpha_1 - \beta_1]x^3 + [13(2r-1) - \eta_1 - 3\beta_1 - 2\alpha_1]x^2$$
$$+ [12(2r-1) - 3\eta_1 - 2\beta_1]x + [4(2r-1) - 2\eta_1]\}]\, u.$$

Equating coefficients of x^{n+1}, we have as the relation connecting the Δ's with the δ's,

$$\{4n(n+1) - (n+1)(2\eta_1 - 8r) - [\eta_2 + 2r\eta_1 - 4r(r-1)]\}\,\delta_{n+1}$$
$$+ \{12n(n-1) - n(3\eta_1 + 2\beta_1 - 24r) - [\zeta_2 + r(2\beta_1 + 3\eta_1) - 12r(r-1)]\}\delta_n$$
$$+ \{13(n-1)(n-2) - (n-1)(\eta_1 + 3\beta_1 + 2\alpha_1 - 26r) - [\gamma_2 + r(2\alpha_1 + 3\beta_1 + \eta_1)$$
$$- 13r(r-1)]\}\,\delta_{n-1}$$
$$+ \{6(n-2)(n-3) - (n-2)(\beta_1 + 3\alpha_1 - 12r) - [\beta_2 + r(\beta_1 + 3\alpha_1) - 6r(r-1)]\}\delta_{n-2}$$
$$+ \{(n-3)(n-4) - (n-3)(\alpha_1 - 2r) - [\alpha_2 + r\alpha_1 - r(r-1)]\}\delta_{n-3}$$
$$= -\{[8n + 4 + 8r - 2\eta_1]\,\Delta_{n+1} + [24n + 24r - 12 - 3\eta_1 - 2\beta_1]\,\Delta_n$$
$$+ [26n + 26r - 39 - \eta_1 - 3\beta_1 - 2\alpha_1]\,\Delta_{n-1}$$
$$+ [12n - 30 + 12r - 3\alpha_1 - \beta_1]\,\Delta_{n-2} + [2n - 7 + 2r - \alpha_1]\,\Delta_{n-3}\}.$$

It will be noticed that the coefficients of δ and Δ in this formula (as in all the preceding ones) are symmetrical in r and n; all the terms in the coefficient of δ_{n+1} which contain only n or r, or both, unite into the one term $4(n+r)(n+r+1)$; the similar terms in the coefficient of δ_n unite into $12(n+r)(n+r-1)$; those in the coefficient of δ_{n-1} into $13(n+r)(n+r-3)$, etc.

(*To be continued.*)

Lectures on the Theory of Reciprocants.

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[Reported by J. HAMMOND.]

The lectures here reproduced were delivered, or are still in the course of delivery, before a class of graduates and scholars in the University of Oxford during the present year. They are to be regarded as easy lessons in the new Theory of Reciprocants of which an outline will be found in Nature for January 7, which contains a report of a Public Lecture on the subject delivered before the University of Oxford in December of the preceding year.

They are designed as a practical introduction to an enlarged theory of Algebraical Forms, and as such are not constructed with the rigorous adhesion to logical order which might be properly expected in a systematic treatise. The object of the lecturer was to rouse an interest in the subject, and in pursuit of this end he has not hesitated to state many results, by way of anticipation, which might, with stricter regard to method, have followed at a later point in the course.

There will be found also occasional repetitions and intercalations of allied topics which are to be justified by the same plea, and also by the fact that the lectures were not composed in their entirety previous to delivery, but gradually evolved and written between one lecture and another in the way that seemed most likely to the lecturer to secure the attention of his auditors.

Since the delivery of his public lecture in December last, papers have been contributed on the subject to the Proceedings of the Mathematical Society of London by Messrs. Hammond, MacMahon, Elliott, Leudesdorf and Rogers, and one to the Comptes Rendus de l'Institut by M. George Perrin. It may therefore be inferred that the lectures have not altogether failed in attaining the desired end of drawing attention to a subject which, in the opinion of the lecturer, constitutes a very considerable extension of the previous limits of algebraical science.

LECTURE I.

A new world of Algebraical forms, susceptible of important geometrical applications, has recently come into existence, of which I gave a general account in a public lecture at the end of last term. I propose in the following brief course to go more fully into the subject and lay down the leading principles of the theory so far as they are at present known to me. The parallelism between the theory of what may be called pure reciprocants and that of invariants is so remarkable that it will be frequently expedient to pass from one theory to the other or to treat the two simultaneously. It may be as well therefore at once to give notice that the term invariant will hereafter be applied alike to invariants ordinarily so called and to those more general algebraical forms which have been termed sources of covariants, differentiants, seminvariants, or subinvariants. A form which is an invariant in the old sense will be termed, when necessary to specify it, a satisfied invariant, an expression which the chemicographical representation of invariants or covariants will serve to explain and justify.

In an elucidatory course of lectures such as the present, it will be advisable to follow a freer order of treatment than would be suitable to the presentation of it in a systematic memoir. My object is to make the theory known, to excite curiosity regarding it, and to invite co-operation in the task of its development.

By way of introduction to the subject, let us begin with an investigation of the properties of a differential expression involving only the first, second and third differential coefficients of either of two variables in respect to the other. For this purpose let us consider not what I have called the Schwarzian itself, which is an integral rational function of these three quantities, but the fractional expression

$$\frac{\dfrac{d^3y}{dx^3}}{\dfrac{dy}{dx}} - \frac{3}{2}\left(\frac{\dfrac{d^2y}{dx^2}}{\dfrac{dy}{dx}}\right)^2$$

which becomes the Schwarzian when cleared of fractions, and which after Cayley we may call the Schwarzian Derivative and denote by

$$(y, x);$$

(x, y) will then of course mean

$$\frac{\dfrac{d^3x}{dy^3}}{\dfrac{dx}{dy}} - \frac{3}{2}\left(\frac{\dfrac{d^2x}{dy^2}}{\dfrac{dx}{dy}}\right)^2.$$

It is easy to establish the identical equation

$$(y, x) = -\left(\frac{dy}{dx}\right)^2 (x, y). \tag{1}$$

Using for brevity y', y'', y''' to denote, as usual, $\dfrac{dy}{dx}$, $\dfrac{d^2y}{dx^2}$, $\dfrac{d^3y}{dx^3}$,

and $x_{\prime}$, $x_{\prime\prime}$, $x_{\prime\prime\prime}$ to denote $\dfrac{dx}{dy}$, $\dfrac{d^2x}{dy^2}$, $\dfrac{d^3x}{dy^3}$, respectively, the relation to be verified is

$$\frac{2y'y''' - 3y''^2}{y'^2} = -y'^2 \cdot \frac{2x_{\prime}x_{\prime\prime\prime} - 3x_{\prime\prime}^2}{x_{\prime}^2}.$$

Now,

$$x_{\prime} = \frac{1}{y'}$$

$$x_{\prime\prime} = \frac{d}{dy}(x_{\prime}) = \frac{1}{y'} \cdot \frac{d}{dx}\left(\frac{1}{y'}\right) = -\frac{y''}{y'^3}$$

and

$$x_{\prime\prime\prime} = \frac{d}{dy}(x_{\prime\prime}) = \frac{1}{y'} \cdot \frac{d}{dx}\left(-\frac{y''}{y'^3}\right) = -\frac{y'''}{y'^4} + \frac{3y''^2}{y'^5}.$$

Whence we obtain

$$2x_{\prime}x_{\prime\prime\prime} - 3x_{\prime\prime}^2 = \left(-\frac{2y'''}{y'^5} + \frac{6y''^2}{y'^6}\right) - \frac{3y''^2}{y'^6}$$

$$= -\frac{1}{y'^6}(2y'y''' - 3y''^2),$$

and the truth of (1) is manifest.

This may be put under the form

$$\frac{2y'y''' - 3y''^2}{y'^3} = -\frac{2x_{\prime}x_{\prime\prime\prime} - 3x_{\prime\prime}}{x_{\prime}^3},$$

showing that a certain function of the first, second and third derivatives of one variable in respect to another remains unaltered, save as to algebraical sign, when the variables are interchanged. An example of a similar kind with which we are all familiar is presented by the well-known function $\dfrac{d^2y}{dx^2} \div \left(\dfrac{dy}{dx}\right)^{\frac{3}{2}}$, which is equal to $-\dfrac{d^2x}{dy^2} \div \left(\dfrac{dx}{dy}\right)^{\frac{3}{2}}$.

We are thus led to inquire whether there may not be an infinite number of algebraical functions of differential derivatives which possess a similar property, and by prosecuting this inquiry to lay the foundations of the theory of Reciprocation or Reciprocants.

Having regard to the fact that the present theory originated in that of the Schwarzian Derivative, I shall proceed to demonstrate, although this is not strictly necessary for the theory of Reciprocants, the remarkable identity

$$(y,\, x) - (z,\, x) = \left(\frac{dz}{dx}\right)^2 . (y,\, z).$$

This identical relation is the fundamental property of Schwarzians, and from it every other proposition concerning their form is an immediate deduction.

In the following proof,* y and z are regarded as two given functions of any variable t, and x as a variable function of the same: so that y and z are functions of x for any given function that x is of t.

It will be seen that

$$((y,\, x) - (z,\, x))\left(\frac{dx}{dz}\right)^2$$

remains unaltered by any infinitesimal variation θ of x, *i. e.* when x becomes $x + \varepsilon\phi(x)$, ε being an infinitesimal constant and $\phi(x)$ an arbitrary finite function of x.

For brevity, let accents denote differential derivation in regard to x, and let any function of x enclosed in a square parenthesis signify the augmented value of that function when x becomes $x + \theta$. In calculating such augmented values, since we suppose that $\theta = \varepsilon\phi(x)$, it is clear that θ, θ', θ''. . . . are each of them infinitesimals of the first order, and consequently that all products, and all powers higher than the first of these quantities, may be neglected.

We have therefore

$$[y'] = \frac{dy}{dx + d\theta} = \frac{y'}{1 + \theta'} = y' - \theta' y'$$

$$[y''] = \frac{d[y']}{dx + d\theta} = \frac{\frac{d}{dx}(y' - \theta' y')}{1 + \theta'} = \frac{y''(1 - \theta') - \theta'' y'}{1 + \theta'}$$

$$= y'' - 2\theta' y'' - \theta'' y'$$

$$[y'''] = \frac{d[y'']}{dx + d\theta} = \frac{\frac{d}{dx}(y'' - 2\theta' y'') - \theta'' y')}{1 + \theta'} = \frac{y'''(1 - 2\theta') - 3\theta'' y'' - \theta''' y'}{1 + \theta'}$$

$$= y''' - 3\theta' y''' - 3\theta'' y'' - \theta''' y'.$$

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* As originally given in the *Messenger of Mathematics* (Vol. XV,   ), this was defaced by so many errata as to render expedient its reproduction in a corrected form.

Hence

$$[y'y'''] = y'y''' - 4\theta'y'y''' - 3\theta''y'y'' - \theta'''y'^2$$
$$\tfrac{3}{2}[y''^2] = \tfrac{3}{2}y''^2 - 6\theta'y''^2 - 3\theta''y'y''$$
$$[y'^2] = y'^2 - 2\theta'y'^2.$$

And since by definition

$$(y, x) = \frac{y'y''' - \frac{3}{2}y''^2}{y'^2},$$

we readily obtain

$$[(y, x)] = \frac{(y, x)}{1 - 2\theta'} - 4\theta'(y, x) - \theta''' = (y, x)(1 - 2\theta') - \theta'''.$$

So also

$$[(z, x)] = (z, x)(1 - 2\theta') - \theta'''.$$

Whence by subtraction

$$[(y, x) - (z, x)] = (1 - 2\theta')\{(y, x) - (z, x)\}.$$

Dividing the left-hand side of this by $[z'^2]$, and the right-hand side by $z'^2(1 - 2\theta')$ which is the equivalent of $[z'^2]$, our final result is

$$\left[\frac{(y, x) - (z, x)}{z'^2}\right] = \frac{(y, x) - (z, x)}{z'^2}.$$

Thus, then, we have seen that the expression

$$\frac{(y, x) - (z, x)}{\left(\frac{dz}{dx}\right)^2}$$

does not vary when x receives an infinitesimal variation $\epsilon\phi(x)$, from which it follows, by the general principle of successive continuous accumulation, that the same will be true when x undergoes any finite arbitrary variation, and consequently this expression has a value which is independent of the form of x regarded as a function of t; it will, of course, be remembered that y and z are supposed to be invariable functions of t. Let x become z, then (y, x) becomes (y, z), while at the same time (z, x) vanishes and $\frac{dz}{dx}$ becomes unity: so that we obtain

$$\frac{(y, x) - (z, x)}{\left(\frac{dz}{dx}\right)^2} = (y, z).$$

Hence, *whatever* function x may be of t,

$$(y, x) - (z, x) = \left(\frac{dz}{dx}\right)^2 . (y, z). \qquad (2)$$

To this fundamental proposition the equation marked (1), itself the important point in regard to the Theory of Reciprocants, is an immediate corollary. For if in (2) we interchange y and z, it becomes

$$(z,\ x)-(y,\ x)=\left(\frac{dy}{dx}\right)^2.(z,\ y);$$

and now, making $x=z$, we have

$$-(y,\ z)=\left(\frac{dy}{dz}\right)^2.(z,\ y),$$

which is the same as (1), except that z occupies the place of x.

But (1) may be obtained more immediately from (2) by substituting in it x for y and y for z, leaving x unaltered; when it becomes

$$-(y,\ x)=\left(\frac{dy}{dx}\right)^2.(x,\ y).$$

This is equivalent to saying that

$$2y'y'''-3y''^2=-y'^6(2x_{\prime}x_{\prime\prime\prime}-3x_{\prime\prime}^2),$$

a verification of which has been given already.

Observe that $\dfrac{y'y'''-\frac{3}{2}y''^2}{y'^2}$ or $(y,\ x)$ contains $\left(\dfrac{dy}{dx}\right)^2$ in its denominator and $(x,\ y)$ contains $\left(\dfrac{dx}{dy}\right)^2$ in its denominator, which is the same as $\left(\dfrac{dy}{dx}\right)^2$ in the numerator. Thus it is that the *square* of $\dfrac{dy}{dx}$ enters three times.

Let me insist for a moment on the import of the fact brought to light in the course of this investigation, that $\dfrac{(y,\ x)-(z,\ x)}{\left(\dfrac{dz}{dx}\right)^2}$ is invariable when x, y and z being regarded as functions of t, x alters its form, but y and z retain theirs. Of course we might write $\left(\dfrac{dy}{dx}\right)^2$ in the denominator instead of $\left(\dfrac{dz}{dx}\right)^2$, and then make the same affirmation as before; as will be evident if we only remember that by hypothesis y and z are both of them constant functions of t, and that therefore $\left(\dfrac{dz}{dy}\right)^2$ must also be so. This is tantamount to saying that when the same conditions are fulfilled $((y,\ x)-(z,\ x))(dx)^2$ is invariable, *i. e.* that when x becomes X in virtue of any substitution (including a homographic one) impressed upon it,

$$\{(y,\ x)-(z,\ x)\}(dx)^2=\{(y,\ X)-(z,\ X)\}(dX)^2,$$

and thus we see that when x becomes X,

$$(y,\ x)-(z,\ x)$$

remains unaltered except that it takes to itself the factor $\left(\frac{dX}{dx}\right)^2$ which depends solely on the particular substitution impressed on x.

If $$y = f(x),\ z = \phi(x),\ \text{and } X = \omega(x),$$
our formula becomes
$$\{(fx, x) - (\phi x, x)\}(dx)^2 = \{(f\omega^{-1}X, X) - (\phi\omega^{-1}X, X)\}(dX)^2,$$
so that, speaking of Quantics and Covariants with respect to a single variable x, $(fx, x) - (\phi x, x)$ is to all intents and purposes a Covariant to the simultaneous forms $f(x)$ and $\phi(x)$, in a sense comprehending but far transcending that in which the term is ordinarily employed; for it remains a persistent factor of its altered self when for x any arbitrary function of x is substituted, the new factor taken on depending wholly and solely on the particular substitution impressed upon x. In the ordinary theory of invariants, the substitution impressed is limited to be homographic; in this case it is absolutely general. We might, moreover, add as a corollary that if (y, x), (z, x), (u, x) .... are regarded as roots of any Binary Quantic, every invariant of that Binary Quantic is a covariant in the extended sense in which the word has just been used, in respect to the system of simultaneous forms $f(x)$, $\phi(x)$, $\psi(x)$ .... For every such invariant will be a function of $(y, x) - (z, x)$, $(y, x) - (u, x)$, $(z, x) - (u, x)$, ... and will therefore remain a persistent factor of its altered self, taking on a power of $\frac{dX}{dx}$ as its extraneous factor.

Calling (fx, x) the Schwarzian Derivative of $f(x)$, our theorem may be stated in general terms as follows:

All invariants of a Binary Quantic whose roots are the Schwarzian Derivatives of a given set of functions of the same variable are Covariants (in an extended sense) of that set of functions.

The theory of the Schwarzian derivative originates in that of the linear differential equation of the second order,
$$u'' + 2Pu' + Qu = 0,$$
which becomes, when we write $u = ve^{-\int P dx}$,
$$v'' + Iv = 0,$$
where $$I = Q - P^2 - P'.$$
Now, suppose that u_1 and u_2 are any two particular solutions of the first of these

equations, and let z denote their mutual ratio; so that, when v_1 and v_2 are the corresponding particular solutions of the second equation, we readily obtain

$$z = \frac{u_2}{u_1} = \frac{v_2}{v_1},$$

and therefore,

$$z' = \frac{v_1 v_2' - v_2 v_1'}{v_1^2}.$$

A second differentiation gives

$$z'' = \frac{v_1 v_2'' - v_2 v_1''}{v_1^2} - \frac{2v_1'(v_1 v_2' - v_2 v_1')}{v_1^3}.$$

But since

$$\frac{v_1''}{v_1} = \frac{v_2''}{v_1} = -I,$$

the first term of the expression just found vanishes identically, and we have

$$z'' = -\frac{2v_1' z'}{v_1},$$

or,

$$v_1' = -\frac{z'' v_1}{2z'}.$$

Differentiating this again, we find

$$-2v_1'' = \left(\frac{z'''}{z'} - \frac{z''^2}{z'^2}\right) v_1 + \frac{z''}{z'} v_1'$$

$$= \left(\frac{z'''}{z'} - \frac{3}{2}\frac{z''^2}{z'^2}\right) v_1.$$

Hence

$$\frac{z'''}{z'} - \frac{3}{2}\frac{z''^2}{z'^2} = 2I,$$

where the left-hand side of the equation is "the Schwarzian Derivative" with z written in the place of y.

LECTURE II.

The expression $2y'y''' - 3y''^2$, which we have called the Schwarzian, may be termed a reciprocant, meaning thereby that on interchanging y', y'', y''' with $x_{,}, x_{,,}, x_{,,,}$ its form remains unaltered, save as to the acquisition of what may be called an extraneous factor, which, in the case before us, is a power of y' (with a multiplier -1). Before we proceed to consider other examples of reciprocants it will be useful to give formulae by means of which the variables may be readily interchanged in any differential expression.

We shall write t for y' and τ for its reciprocal x, using the letters $a, b, c, \ldots$ to denote the second, third, fourth, etc., differential derivatives of y with respect to x, and $\alpha, \beta, \gamma, \ldots$ to denote those of x with respect to y. The advantage of this notation will be seen in the sequel.

The values of $\alpha, \beta, \gamma, \ldots$ in terms of $t, a, b, c, \ldots$ are given by the formulae

$$\alpha = -a \div t^3,$$
$$\beta = -bt + 3a^2 \div t^5,$$
$$\gamma = -ct^2 + 10abt - 15a^3 \div t^7,$$
$$\delta = -dt^3 + (15ac + 10b^2)t^2 - 105a^2bt + 105a^4 \div t^9,$$
$$\varepsilon = -et^4 + (21ad + 35bc)t^3 - (210a^2c + 280ab^2)t^2 + 1260a^3bt - 945a^5 \div t^{11},$$

. . . .

If, in these equations, we write $a = 1.2.a_0$, $b = 1.2.3.a_1$, $c = 1.2.3.4.a_2, \ldots$ and $\alpha = 1.2.\alpha_0$, $\beta = 1.2.3.\alpha_1$, $\gamma = 1.2.3.4.\alpha_2, \ldots$ they become

$$\alpha_0 = -a_0 \div t^3,$$
$$\alpha_1 = -a_1t + 2a_0^2 \div t^5,$$
$$\alpha_2 = -a_2t^2 + 5a_0a_1t - 5a_0^3 \div t^7,$$
$$\alpha_3 = -a_3t^3 + (6a_0a_2 + 3a_1^2)t^2 - 21a_0^2a_1t + 14a_0^4 \div t^9,$$
$$\alpha_4 = -a_4t^4 + (7a_0a_3 + 7a_1a_2)t^3 - (28a_0^2a_2 + 28a_1^2)t^2 + 84a_0^3a_1t - 42a_0^5 \div t^{11},$$

. . . .

Any one of the formulae in either set may be deduced from the formula immediately preceding it by a simple process of differentiation.

Thus, since $\quad \beta = \dfrac{-bt + 3a^2}{t^5}$ and $\dfrac{d}{dy} = \dfrac{1}{t} \cdot \dfrac{d}{dx}$,

we have $\quad \dfrac{d\beta}{dy} = \dfrac{1}{t} \cdot \dfrac{d}{dx}\left(\dfrac{-bt + 3a^2}{t^5}\right).$

But $\quad \dfrac{d\beta}{dy} = \gamma$ and $\dfrac{d}{dx} = a\partial_t + b\partial_a + c\partial_b + \ldots,$

so that
$$\gamma = \frac{1}{t}(a\partial_t + b\partial_a + c\partial_b + \ldots)\left(\frac{-bt + 3a^2}{t^5}\right)$$
$$= \frac{1}{t^7}(-ct^2 + 10abt - 15a^3).$$

By continually operating with $\dfrac{1}{t}(a\partial_t + b\partial_a + c\partial_b + \ldots)$ the table may be extended as far as we please, the expressions on the right-hand side being the successive values of

$$\left\{\frac{1}{t}(a\partial_t + b\partial_a + c\partial_b + \ldots)\right\}^n \left(-\frac{a}{t^3}\right)$$

found by giving to n the values $0, 1, 2, 3, \ldots$

Precisely similar reasoning shows that, when the modified letters $\alpha_0, \alpha_1, \alpha_2, \ldots .$ are used,

$$(n+2)\alpha_n = \frac{1}{t}\left(2a_0\partial_t + 3a_1\partial_{a_0} + 4a_2\partial_{a_1} + \ldots .\right)\alpha_{n-1},$$

and that

$$\alpha_n = \frac{\left\{\frac{1}{t}\left(2a_0\partial_t + 3a_1\partial_{a_0} + 4a_2\partial_{a_1} + \ldots .\right)\right\}^n\left(-\frac{a_0}{t^3}\right)}{3.4.5.\ldots .(n+2)}.$$

A proof of the formula

$$\alpha_n = -t^{-n-3}\left(e^{-\frac{V}{t}}\right)a_n,$$

obtained by Mr. Hammond, in which

$$V = 4\cdot\frac{a_0^2}{2}\partial_{a_1} + 5a_0a_1\partial_{a_2} + 6\left(a_0a_2 + \frac{a_1^2}{2}\right)\partial_{a_3} + 7\left(a_0a_3 + a_1a_2\right)\partial_{a_4} + \ldots .$$

will be given later on, when we treat of this operator, which, in the theory of Reciprocants, is the analogue of the operator $a\partial_b + 2b\partial_c + 3c\partial_d + , \ldots .$ with which we are familiarly acquainted in the theory of Invariants.

Consider the expression $\qquad ct - 5ab.$

If, in $\gamma\tau - 5\alpha\beta$, which may be called its transform, we write

$$\tau = \frac{1}{t}, \quad \alpha = -\frac{a}{t^3}, \quad \beta = \frac{-bt + 3a^2}{t^5}, \quad \gamma = \frac{-ct^2 + 10abt - 15a^3}{t^7},$$

this becomes a fraction whose denominator is t^8, while its numerator is

$$-ct^2 + 10abt - 15a^3 + 5a(-bt + 3a^2) = -ct^2 + 5abt.$$

Removing the common factor t from the numerator and denominator of this fraction, we have

$$\gamma\tau - 5\alpha\beta = -\frac{ct - 5ab}{t^7}.$$

Here, then, as in the case of the well-known monomial for which

$$a = -t^3\alpha,$$

and the Schwarzian for which

$$2bt - 3a^2 = -t^6(2\beta\tau - 3\alpha^2),$$

the expression $\qquad ct - 5ab = -t^7(\gamma\tau - 5\alpha\beta)$

changes its sign on reciprocation.

That reciprocation is not always accompanied with a change of sign will be clear if we consider the product of any pair of the three expressions given above. Or we may take, as an example of a reciprocant in which this change of sign does not occur, the form

$$3ac - 5b^2.$$

Here $\qquad 3\alpha\gamma - 5\beta^2 = \dfrac{3a(ct^2 - 10abt + 15a^3) - 5(bt - 3a^2)^2}{t^{10}}.$

In the fraction on the right-hand side the only surviving terms of the numerator are those containing the highest power of t, the rest destroying one another.

Thus $$3a\gamma - 5\beta^2 = \frac{1}{t^6}(3ac - 5b^2).$$

Reciprocants which change their sign when the variables x and y are interchanged, will be said to be of odd character; those, on the contrary, which keep their sign unchanged will be said to be of even character. The distinction is an important one, and will be observed in what follows.

Forms such as the one just considered, where t does not appear in the form itself, but only in the extraneous factor, will be called Pure Reciprocants, in order to distinguish them from those forms (of which the Schwarzian $2tb - 3a^2$ is an example) into which t enters, which will be called Mixed Reciprocants. It will be seen hereafter that Pure Reciprocants are the analogues of the invariants of Binary Quantics.

With modified letters (*i. e.* writing $a = 2a_0$, $b = 6a_1$, and $c = 24a_2$)

$$3ac - 5b^2 \text{ becomes } 144a_0a_2 - 180a_1^2 = 36(4a_0a_2 - 5a_1^2).$$

Operating on this with

$$V = 2a^2\partial_{a_1} + 5a_0a_1\partial_{a_2} + \ldots\ldots,$$

we have $$V(4a_0a_2 - 5a_1^2) = 0.$$

We shall prove subsequently that all Pure Reciprocants are, in like manner, subject to annihilation by the operator V.

Hitherto we have only considered homogeneous; let us now take as an example of a non-homogeneous reciprocant the expression

$$(1 + t^2)b - 3a^2t.$$

Here $$(1 + \tau^2)\beta - 3\alpha^2\tau = \left(1 + \frac{1}{t^2}\right)\left(\frac{-bt + 3a^2}{t^5}\right) - \frac{3a^2}{t^7}$$

$$= \frac{(1 + t^2)(-bt + 3a^2) - 3a^2}{t^7}.$$

In the numerator of this fraction the terms $+3a^2$ and $-3a^2$ cancel, a factor t divides out, and we have finally

$$(1 + \tau^2)\beta - 3\alpha^2\tau = -\frac{(1 + t^2)b - 3a^2t}{t^6}.$$

In general, a Reciprocant may be defined to be a function F of such a kind that $F(\tau, \alpha, \beta, \gamma, \ldots\ldots)$ contains $F(t, a, b, c, \ldots\ldots)$ as a factor. An important special case is that in which the other factor is merely numerical; the function F is then said to be an Absolute Reciprocant.

When we limit ourselves to the case where F is a rational integral function of the letters, it may be proved that

$$F(t, a, b, c, \ldots) = \pm t^\mu F(\tau, \alpha, \beta, \gamma, \ldots).$$

For, in the first place, since any one of the letters $\alpha, \beta, \gamma, \ldots$ is a rational function of $t, a, b, c, \ldots$ and integral with respect to all of them except t, containing only a power of this letter in the denominator, it is clear that any rational integral function of $\tau, \alpha, \beta, \gamma, \ldots$ such as $F(\tau, \alpha, \beta, \gamma, \ldots)$ is supposed to be, must be a rational integral function of $t, a, b, c, \ldots$ divided by some power of t. But since F is a reciprocant, $F(\tau, \alpha, \beta, \gamma, \ldots)$ must contain $F(t, a, b, c, \ldots)$ as a factor; and if we suppose the other factor to be $\dfrac{\phi(t, a, b, c, \ldots)}{t^\lambda}$ we must have

$$F(\tau, \alpha, \beta, \gamma, \ldots) = \frac{\phi(t, a, b, c, \ldots)}{t^\lambda} F(t, a, b, c, \ldots),$$

where ϕ is rational and integral with respect to all the letters.

Moreover,

$$F(t, a, b, c, \ldots) = \frac{\phi(\tau, \alpha, \beta, \gamma, \ldots)}{\tau^\lambda} F(\tau, \alpha, \beta, \gamma, \ldots).$$

Hence we must have identically

$$\phi(t, a, b, c, \ldots)\, \phi(\tau, \alpha, \beta, \gamma, \ldots) = 1,$$

where, on the supposition that the functions ϕ contain other letters besides t and τ, $\phi(t, a, b, c, \ldots)$ is, and $\phi(\tau, \alpha, \beta, \gamma, \ldots)$ can be expressed as, a rational function integral as regards the letters $a, b, c, \ldots$. But this supposition is manifestly inadmissible, for the product of two integral rational functions of $a, b, c, \ldots$ cannot be identically equal to unity. Hence t is the only letter that can appear in the extraneous factor and we may write

$$F(\tau, \alpha, \beta, \gamma, \ldots) = \frac{\psi(t)}{t^\lambda} F(t, a, b, c, \ldots)$$

where $\psi(t)$ is a rational integral function.

The same reasoning as before shows that we must have identically

$$\psi(t)\, \psi(\tau) = 1.$$

But this cannot be true if $\psi(t)$ has any root different from zero; for if we give t such a value as will make $\psi(t)$ vanish, this value must also make $\psi(\tau)$ infinite; and since

$$\begin{aligned}\psi(\tau) &= A + B\tau + C\tau^2 + \ldots + M\tau^m \\ &= A + \frac{B}{t} + \frac{C}{t^2} + \ldots + \frac{M}{t^m},\end{aligned}$$

the only value of t for which $\psi(\tau)$ becomes infinite is a zero value. Hence $\psi(t)$ is of the form Mt^m, and consequently $\psi(\tau) = M\tau^m$. Thus

$$\psi(t)\,\psi(\tau) = M^2 t^m \tau^m = 1,$$

and therefore $$M^2 = 1.$$

We have now proved that if F is a rational integral reciprocant,

$$F(t, a, b, c, \ldots) = \pm\, t^\mu F(\tau, \alpha, \beta, \gamma, \ldots),$$

or we may say, $$= (-)^\kappa t^\mu F(\tau, \alpha, \beta, \gamma, \ldots),$$

where $\kappa = 1$ or 0 according as the reciprocant is of odd or even character.

It obviously follows that the product or quotient of any two rational integral reciprocants is itself a reciprocant; but it must be carefully observed that this is not true of their sum or difference unless certain conditions are fulfilled. For if we write

$$F_1(t, a, \ldots) = (-)^{\kappa_1} t^{\mu_1} F_1(\tau, \alpha, \ldots)$$

and $$F_2(t, a, \ldots) = (-)^{\kappa_2} t^{\mu_2} F_2(\tau, \alpha, \ldots),$$

we see that

$$pF_1(t, a, \ldots) + qF_2(t, a, \ldots) = (-)^{\kappa_1} t^{\mu_1} pF_1(\tau, \alpha, \ldots) + (-)^{\kappa_2} t^{\mu_2} qF_2(\tau, \alpha, \ldots),$$

and consequently this expression will be a reciprocant if $\kappa_1 = \kappa_2$ and $\mu_1 = \mu_2$, but not otherwise. If we call the index of t in the extraneous factor the *characteristic*, what we have proved is that no linear function of two reciprocants can be a reciprocant, unless they have the same characteristic and are of the same character In dealing with Absolute Reciprocants, since the characteristic of these is always zero, we need only attend to their character.

I propose for the present to confine myself to homogeneous and isobaric reciprocants,* *i. e.* to such as are homogeneous and isobaric when the letters $t, a, b, c, \ldots$ are considered to be each of degree 1, their respective weights being $-1, 0, 1, 2, \ldots$ The letter w will be used to denote the weight of such a reciprocant, i its degree, and j its extent, *i. e.* the weight of the most advanced letter which it contains.

Let any such reciprocant $F(t, a, b, c, \ldots)$ contain a term $At^v a^l b^m c^n \ldots$, then

$$v + l + m + n + \ldots = i$$

and $$-v + m + 2n + \ldots = w.$$

The corresponding term in $F(\tau, \alpha, \beta, \gamma, \ldots)$ will be $A\tau^v \alpha^l \beta^m \gamma^n \ldots$ where

$$\tau = \frac{1}{t}, \quad \alpha = -\frac{a}{t^3}, \quad \beta = -\frac{b}{t^4} + \ldots, \quad \gamma = -\frac{c}{t^5} + \ldots, \text{ etc.}$$

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* Here and elsewhere the word *reciprocant* is used in the sense of *rational integral reciprocant*: this will always be done when there is no danger of confusion arising from it.

Now, if no term of F contains a smaller number of the letters $a, b, c, \ldots$ than are found in the term we are considering, the first terms of β, γ, etc., may be taken instead of these quantities themselves and $A\tau^v \alpha^l \beta^m \gamma^n \ldots$ may be replaced by

$$(-)^{l+m+n+\ldots} A t^{-v-3l-4m-5n-\ldots} a^l b^m c^n \ldots = (-)^{i-v} A t^{v-3i-w} a^l b^m c^n \ldots$$

But since

$$F(t, a, b, c, \ldots) = (-)^{\kappa} t^{\mu} F(\tau, \alpha, \beta, \gamma, \ldots)$$

we must have identically

$$A t^v a^l b^m c^n \ldots = (-)^{i-v+\kappa} A t^{\mu+v-3i-w} a^l b^m c^n \ldots$$

Hence the character is even or odd according to the parity of $i - v$ (*i. e.* of the smallest number of letters different from t in any term), and the characteristic $\mu = 3i + w$.

The type of a reciprocant depends on the *character*, weight, degree and extent. As the extraneous factor is always of the form $(-)^{\kappa} t^{\mu}$, where κ is 1 or 0, we may define the type of a reciprocant by

$$1 : w : i, j \quad \text{or} \quad 0 : w : i, j,$$

according as its character is odd or even.

For Pure Reciprocants the smallest number of letters different from t in any term is (since all the letters are different from t) the same as its degree. Hence the character of a Pure Reciprocant is odd or even according to the parity of i, and for this reason the type of a Pure Reciprocant may be defined by

$$w : i, j.$$

A linear combination of reciprocants of the same type will be a reciprocant, for when the type is known both the character and characteristic are given.

LECTURE III.

Let F be any function (not necessarily homogeneous or even algebraical) of the differential derivatives which acquires a numerical multiplier M, but is otherwise unchanged when the reciprocal substitution of x for y and y for x is effected. A second reciprocation multiplies the function again by M, and thus the total effect of both substitutions is to multiply F by M^2. But since the second reciprocation reproduces the original function, we must have $M^2 = 1$. Functions of this kind are therefore unaltered by reciprocation (except it may be in sign), and for this reason are called *Absolute Reciprocants.* These, as we shall presently see, play an important part in the general theory. Like all other reciprocants, they range naturally in two distinct classes, those of odd and those of even character.

It is perhaps worthy of notice that the extraneous factor of a general reciprocant is the exponential of an absolute reciprocant of odd character. For if

$$F(t, a, b, c, \ldots) = \phi(t, a, b, c, \ldots)\, F(\tau, \alpha, \beta, \gamma, \ldots),$$

we must still have, as before,

$$\phi(t, a, b, c, \ldots)\, \phi(\tau, \alpha, \beta, \gamma, \ldots) = 1;$$

i. e. $$\log \phi(t, a, b, c, \ldots) = -\log \phi(\tau, \alpha, \beta, \gamma, \ldots);$$

or $\log \phi(t, a, b, c, \ldots)$ is an absolute reciprocant of odd character.

An absolute reciprocant may be obtained from any pair of rational integral reciprocants in the same way that an absolute invariant is found from two ordinary invariants. For let

$$F_1(t, a, b, c, \ldots) = (-)^{\varkappa_1} t^{\mu_1} F_1(\tau, \alpha, \beta, \gamma, \ldots),$$

and $$F_2(t, a, b, c, \ldots) = (-)^{\varkappa_2} t^{\mu_2} F_2(\tau, \alpha, \beta, \gamma, \ldots),$$

then $$\frac{\{F_1(t, a, b, c, \ldots)\}^{\mu_2}}{\{F_2(t, a, b, c, \ldots)\}^{\mu_1}} = (-)^{\varkappa_1\mu_2 - \varkappa_2\mu_1} \frac{\{F_1(\tau, \alpha, \beta, \gamma, \ldots)\}^{\mu_2}}{\{F_2(\tau, \alpha, \beta, \gamma, \ldots)\}^{\mu_1}};$$

or we may say that $F_1^{\mu_2} \div F_2^{\mu_1}$ is an absolute reciprocant of even or odd character according to the parity of $\varkappa_1\mu_2 - \varkappa_2\mu_1$.

Thus, for example, from

$$a = -t^3\alpha$$

and $$3ac - 5b^2 = t^6(3\alpha\gamma - 5\beta^2)$$

we form $\dfrac{(3ac - 5b^2)^3}{a^6}$, an absolute reciprocant of even character.

From a reciprocant F whose characteristic is μ we obtain an absolute reciprocant of the same character as F by dividing it by $t^{\frac{\mu}{2}}$.

For if we only remember that $\tau = \dfrac{1}{t}$, it obviously follows that

$$F(t, a, b, c, \ldots) = \pm\, t^{\mu} F(\tau, \alpha, \beta, \gamma, \ldots)$$

can be written in the form

$$\frac{F(t, a, b, c, \ldots)}{t^{\frac{\mu}{2}}} = \pm \frac{F(\tau, \alpha, \beta, \gamma, \ldots)}{\tau^{\frac{\mu}{2}}},$$

where the original character of the reciprocant F is preserved.

It may be noticed that a reciprocant of odd character cannot be divided by $\sqrt{-1}\,t^{\frac{\mu}{2}}$ so as to give an absolute reciprocant of even character; for, the reciprocal of F being $-t^{\mu}F'$, that of $F \div \sqrt{-1}\,t^{\frac{\mu}{2}}$ will still be $-F' \div \sqrt{-1}\,\tau^{\frac{\mu}{2}}$. The character of a reciprocant is thus seen to be one of its indelible attributes.

As simple examples of absolute reciprocants we may take $\dfrac{3ac-5b^2}{t^4}$, which becomes on reciprocation $\dfrac{3\alpha\gamma-5\beta^2}{\tau^4}$, and $\dfrac{a}{t^{\frac{3}{2}}}$, which reciprocates into $-\dfrac{\alpha}{\tau^{\frac{3}{2}}}$. The character of the former is even, that of the latter odd.

Observing that

$$\log t = -\log\tau \text{ and } \frac{1}{\sqrt{t}}\cdot\frac{d}{dx} = \frac{1}{\sqrt{\tau}}\cdot\frac{d}{dy},$$

we have

$$\left(\frac{1}{\sqrt{t}}\cdot\frac{d}{dx}\right)\log t = -\left(\frac{1}{\sqrt{\tau}}\cdot\frac{d}{dy}\right)\log\tau.$$

From this, in like manner, we obtain

$$\left(\frac{1}{\sqrt{t}}\cdot\frac{d}{dx}\right)^2\log t = -\left(\frac{1}{\sqrt{\tau}}\cdot\frac{d}{dy}\right)^2\log\tau;$$

and so, in general,

$$\left(\frac{1}{\sqrt{t}}\cdot\frac{d}{dx}\right)^i\log t = -\left(\frac{1}{\sqrt{\tau}}\cdot\frac{d}{dy}\right)^i\log\tau.$$

Hence $\left(\dfrac{1}{\sqrt{t}}\cdot\dfrac{d}{dx}\right)^i\log t$ is an absolute reciprocant, and of an odd character, for all positive integral values of i. We thus obtain a series of fractions with rational integral homogeneous reciprocants in their numerators and powers of $t^{\frac{1}{2}}$ in their denominators. It will be sufficient, before proceeding to the more general theory of *Eduction*, as it may be called, to examine, by way of illustration, the cases in which $i = 1$, 2 and 3.

Let $i = 1$; then

$$\left(\frac{1}{\sqrt{t}}\cdot\frac{d}{dx}\right)\log t = \frac{a}{t^{\frac{3}{2}}}.$$

So that, in the case where $i = 2$, we have

$$\left(\frac{1}{\sqrt{t}}\cdot\frac{d}{dx}\right)^2\log t = \left(\frac{1}{\sqrt{t}}\cdot\frac{d}{dx}\right)\frac{a}{t^{\frac{3}{2}}} = \frac{b}{t^2} - \frac{3}{2}\cdot\frac{a^2}{t^3}$$
$$= \frac{2bt-3a^2}{2t^3}.$$

The numerator of this fraction is the Schwarzian.

In like manner, when $i = 3$,

$$\left(\frac{1}{\sqrt{t}}\cdot\frac{d}{dx}\right)^3 \log t = \left(\frac{1}{\sqrt{t}}\cdot\frac{d}{dx}\right)\left(\frac{2bt-3a^2}{2t^3}\right) = \frac{2ct-4ab}{2t^{\frac{7}{2}}} - \frac{6abt-9a^3}{t^{\frac{9}{2}}} = \frac{2ct^2-10abt+9a^3}{2t^{\frac{9}{2}}}.$$

But here a reduction may be effected, for $\left(\frac{a}{t^{\frac{3}{2}}}\right)^3$, as well as $\frac{a}{t^{\frac{3}{2}}}$ itself, is an absolute reciprocant of the same character as the whole of the expression just found. Hence we may reject the term $\frac{9}{2}\cdot\frac{a^3}{t^{\frac{9}{2}}}$ without thereby affecting the reciprocantive property of the form, and thus obtain

$$\frac{ct-5ab}{t^{\frac{7}{2}}},$$

an absolute reciprocant of odd character. The corresponding rational integral reciprocant is

$$ct - 5ab.$$

We have found that $\frac{a}{t^{\frac{3}{2}}}$ and $\frac{2bt-3a^2}{t^3}$ are each of them reciprocants. Why, then, by parity of reasoning, is not $\frac{2bt}{t^3}$, and therefore b, a reciprocant? It is because $\frac{a^2}{t^3}$, the square of $\frac{a}{t^{\frac{3}{2}}}$, is of even character, while $\frac{2bt-3a^2}{t^3}$ is of an odd character, so that no linear combination of the two would be *legitimate.*

If we differentiate any absolute reciprocant with respect to x, we shall obtain another reciprocant of the same character. For let R be any absolute reciprocant and R' its transform, then

$$R = \pm R';$$

and since $\frac{d}{dx} = t\frac{d}{dy}$ may be written in the equivalent but more symmetrical form

$$\frac{1}{\sqrt{t}}\cdot\frac{d}{dx} = \frac{1}{\sqrt{\tau}}\cdot\frac{d}{dy},$$

we have

$$\left(\frac{1}{\sqrt{t}}\cdot\frac{d}{dx}\right)R = \pm\left(\frac{1}{\sqrt{\tau}}\cdot\frac{d}{dy}\right)R'.$$

On one side of this identical equation is a function of the differential derivatives of y with respect to x; on the other, a precisely similar function of those of x with respect to y. Hence $\frac{1}{\sqrt{t}}\cdot\frac{dR}{dx}$ is an absolute reciprocant, and therefore $\frac{dR}{dx}$ is a reciprocant, the character of each being the same as that of R.

I will avail myself of the conclusion just obtained, which is the cardinal property of absolute reciprocants, to give a general method of generating from any given Rational Integral Reciprocant an infinity of others—rational integral educts of it, we may say. Let F be such a reciprocant, and μ its characteristic; then $\frac{F}{t^{\frac{\mu}{2}}}$ is an absolute reciprocant, and consequently $\frac{d}{dx}\left(\frac{F}{t^{\frac{\mu}{2}}}\right)$ is a reciprocant, both of them of the same character as F; *i. e.*

$$\frac{t\frac{dF}{dx} - \frac{\mu}{2}\cdot aF}{t^{\frac{\mu}{2}+1}};$$

or we may say

$$2t\frac{dF}{dx} - \mu aF$$

is a reciprocant of the same character as F.

This is even true for non-homogeneous reciprocants, for the only assumption made at present as to the nature of F is that it is a rational integral reciprocant. But if we further assume that it is homogeneous and isobaric,* we know that

$$\mu = 3i + w.$$

Now, Euler's equation gives

$$3i = 3(t\partial_t + a\partial_a + b\partial_b + c\partial_c + \ldots .),$$

and from the similar equation for isobaric functions (remembering that the weights of $t, a, b, c, \ldots .$ are $-1, 0, 1, 2, \ldots .$) we obtain

$$w = -t\partial_t + b\partial_b + 2c\partial_c \ldots .,$$

so that

$$\mu = 2t\partial_t + 3a\partial_a + 4b\partial_b + 5c\partial_c + \ldots .$$

And since

$$\frac{d}{dx} = a\partial_t + b\partial_a + c\partial_b + d\partial_c + \ldots .,$$

we may in $\left(2t\frac{d}{dx} - \mu a\right)F$ replace $2t\frac{d}{dx} - \mu a$ by

$$2t(a\partial_t + b\partial_a + c\partial_b + d\partial_c + \ldots .)$$
$$- a(2t\partial_t + 3a\partial_a + 4b\partial_b + 5c\partial_c + \ldots .),$$

or by its equivalent

$$(2bt - 3a^2)\partial_a + (2ct - 4ab)\partial_b + (2dt - 5ac)\partial_c + \ldots .$$

The conclusion arrived at is that when F is a rational integral homogeneous reciprocant,

$$\{(2bt - 3a^2)\partial_a + (2ct - 4ab)\partial_b + (2dt - 5ac)\partial_c + \ldots .\}F$$

is another, and that both are of the same character.

It will be convenient to use the letter G to denote the operator just found and to speak of it as the generator for mixed reciprocants. By the repeated

* It will subsequently be proved that every rational integral reciprocant which is homogeneous is also isobaric.

operation of this generator on a we may obtain the series Ga, G^2a, G^3a, ..., whose terms will be mixed reciprocants, since each operation increases the highest power of t by unity. The forms thus obtained will, in general, not be irreducible. It is, in fact, easy to see that a reduction must always take place at every second step. Observing that GF only expresses the numerator of the absolute reciprocant $\frac{1}{\sqrt{t}}\cdot\frac{d}{dx}\left(\frac{F}{t^{\frac{\mu}{2}}}\right)$ in a convenient form, and that G^2F is equivalent to the numerator of $\left(\frac{1}{\sqrt{t}}\cdot\frac{d}{dx}\right)^2\left(\frac{F}{t^{\frac{\mu}{2}}}\right)$, we have

$$\frac{1}{\sqrt{t}}\cdot\frac{d}{dx}\left(\frac{F}{t^{\frac{\mu}{2}}}\right)=\frac{t\frac{dF}{dx}-\frac{\mu}{2}\cdot aF}{t^{\frac{\mu+3}{2}}};$$

so that

$$\left(\frac{1}{\sqrt{t}}\cdot\frac{d}{dx}\right)^2\left(\frac{F}{t^{\frac{\mu}{2}}}\right)=\frac{1}{\sqrt{t}}\cdot\frac{d}{dx}\left(\frac{t\frac{dF}{dx}-\frac{\mu}{2}\cdot aF}{t^{\frac{\mu+3}{2}}}\right)$$

$$=\frac{t\frac{d}{dx}\left(t\frac{dF}{dx}-\frac{\mu}{2}\cdot aF\right)-\frac{\mu+3}{2}\cdot a\left(t\frac{dF}{dx}-\frac{\mu}{2}\cdot aF\right).}{t^{\frac{\mu}{2}+3}}$$

The whole of this fraction is an absolute reciprocant of the same character as F; so also is $\frac{a^2F}{t^{\frac{\mu}{2}+3}}$ (the product of the *even* absolute reciprocant $\frac{a^2}{t^3}$ by $\frac{F}{t^{\frac{\mu}{2}}}$). We may therefore reject the term $\frac{\mu}{2}\cdot\frac{\mu+3}{2}\cdot a^2F$ from the numerator, and the remaining fraction

$$\frac{\frac{d}{dx}\left(t\frac{dF}{dx}-\frac{\mu}{2}\cdot aF\right)-\frac{\mu+3}{2}\cdot a\frac{dF}{dx}}{t^{\frac{\mu}{2}+2}}$$

will still be an absolute reciprocant of the same character as F. Its numerator, which is one degree lower than G^2F, may be written in the form

$$t\frac{d^2F}{dx^2}-(\mu+\tfrac{1}{2})a\frac{dF}{dx}-\frac{\mu}{2}bF.$$

This, it may be noticed, is a reciprocant of the same character as F, even when F is non-homogeneous.

Starting with a, we have

$Ga = 2bt - 3a^2$ (the Schwarzian),

$G^2a = G(2bt-3a^2) = -6a(2bt-3a^2) + 2t(2ct-4ab) = 4ct^2 - 20abt + 18a^3.$

But, for the reason previously given, $18a^3$ may be removed, so that rejecting this term and dividing out by $4t$ we obtain the form

$$ct - 5ab,$$

which may be called the Post-Schwarzian.

The next form is obtained by operating on the Post-Schwarzian with G; thus, we have to calculate the value of $G(ct - 5ab)$, where

$$G = (2bt - 3a^2)\partial_a + (2ct - 4ab)\partial_b + (2dt - 5ac)\partial_c.$$

The working may be arranged as follows:

	dt^2	act	b^2t	a^2b	
$t(2dt - 5ac) =$	2	−5	.	.	from $(2dt - 5ac)\partial_c$
$-5a(2ct - 4ab) =$	.	−10	.	20	" $(2ct - 4ab)\partial_b$
$-5b(2bt - 3a^2) =$	.	.	−10	15	" $(2bt - 3a^2)\partial_a$
	2	−15	−10	35	

The result should be read thus:

$$2dt^2 - 15act - 10b^2t + 35a^2b.$$

To obtain the next of this series of reciprocants, we have to operate on this with G and at the same time to take account of the reduction that has to be made at each alternate step. The arrangement of the work is similar to that of the former case.

	et^3	adt^2	bct^2	a^2ct	ab^2t	a^3b	
$2t^2(2et - 6ad) =$	4	−12	.	.	.	.	from $(2et - 6ad)\partial_d$
$-15at(2dt - 5ac) =$	.	−30	.	75	.	.	" $(2dt - 5ac)\partial_c$
$(35a^2 - 20bt)(2ct - 4ab) =$	.	.	−40	70	80	−140	" $(2ct - 4ab)\partial_b$
$(70ab - 15ct)(2bt - 3a^2) =$	.	.	−30	45	140	−210	" $(2bt - 3a^2)\partial_a$
	4	−42	−70	190	220	−350	
$-70a^2(ct - 5ab) =$	.	.	.	−70	.	+350	
	4	−42	−70	120	220	.	

This divides by $2t$, giving the reduced value

$$2et^2 - 21adt - 35bct + 60a^2c + 110ab^2.$$

The next obtained by this process will be seen by the following work to be

$$4ft^3 - 56aet^2 - 112bdt^2 - 70c^2t^2 + 309a^2dt + 995abct + 220b^3t - 660a^3c - 1210a^2b^2.$$

	ft^3	aet^2	bdt^2	c^2t^2	a^2dt	$abct$	b^3t	a^3c	a^2b^2	
$2t^2(2ft-7ae)=$	4	−14	.	.	.	.	.	.	.	from $(2ft-7ae)\partial_e$
$-21at(2et-6ad)=$	.	−42	.	.	126	.	.	.	.	" $(2et-6ad)\partial_d$
$(-35bt+60a^2)(2dt-5ac)=$	.	.	−70	.	120	175	.	−300	.	" $(2dt-5ac)\partial_c$
$(-35ct+220ab)(2ct-4ab)=$	.	.	.	−70	.	580	.	.	−880	" $(2ct-4ab)\partial_b$
$(-21dt+120ac+110b^2)(2bt-3a^2)=$	.	.	−42	.	63	240	220	−360	−330	" $(2bt-3a^2)\partial_a$
	4	−56	−112	−70	309	995	220	−660	−1210	

This cannot be reduced in the same manner as the preceding form; but it must not be supposed that the forms thus obtained are in general irreducible.

Having regard to the circumstance that the forms of the series a, Ga, G^2a, ... occur in the numerators of the successive values of $\left(\frac{1}{\sqrt{t}}\cdot\frac{d}{dx}\right)^n \log t$, they may be called the successive *educts*, and the reduced forms given above may be called the *reduced educts* and denoted by E_1, E_2, E_3..... Thus,

$$E_1 = a,$$
$$E_2 = 2bt - 3a^2,$$
$$E_3 = ct - 5ab,$$
$$E_4 = 2dt^2 - 15act - 10b^2t + 35a^2b,$$
$$E_5 = 2et^2 - 21adt - 35bct + 60a^2c + 110ab^2,$$
$$E_6 = 4ft^3 - 56aet^2 - 112bdt^2 - 70c^2t^2 + 309a^2dt + 995abct + 220b^3t - 660a^3c - 1210a^2b^2.$$

LECTURE IV.

We have seen that when F is a rational integral homogeneous and isobaric reciprocant, GF is another of the same character. It will now appear that the condition of isobarism is implied in that of homogeneity; for let F be a rational integral homogeneous reciprocant, μ its characteristic and i its degree in the letters $t, a, b, c, \ldots.$, then, in the identical equation

$$F(t, a, b, c, \ldots.) = \pm t^\mu F(\tau, \alpha, \beta, \gamma, \ldots.)$$

both members are homogeneous and of the same degree in the letters $t, a, b, c, \ldots.$; *i. e.* if $At^k a^l b^m c^n \ldots.$ be any term of $F(t, a, b, c, \ldots.)$, its degree must be the same as that of $t^\mu A\tau^k\alpha^l\beta^m\gamma^n \ldots.$ when $\tau, \alpha, \beta, \gamma, \ldots.$ are expressed in terms of $t, a, b, c, \ldots.$ But

$$\tau = \frac{1}{t}, \; \alpha = -\frac{a}{t^3}, \; \beta = -\frac{b}{t^4} + \ldots., \; \gamma = -\frac{c}{t^5} + \ldots.,$$

and so on. The degrees of $\tau, \alpha, \beta, \gamma, \ldots.$ are therefore $-1, -2, -3, -4, \ldots.$ respectively. Hence

$$k + l + m + n + \ldots. = \mu - k - 2l - 3m - 4n - \ldots.,$$

or

$$\mu = 2k + 3l + 4m + 5n + \ldots.$$

And by hypothesis $$i = k + l + m + n + \ldots.,$$

so that $$\mu - 3i = -k + m + 2n + \ldots.$$

Neither μ nor i is dependent for its value on the selection of a particular term of F, for all terms of $F(\tau, \alpha, \beta, \gamma, \ldots.)$ are multiplied by the same extraneous factor $\pm t^\mu$, and all terms of $F(t, a, b, c, \ldots.)$ are of the same degree i. Hence $-k + m + 2n + \ldots.$ must also be the same for each term of F; or, attributing the weights $-1, 0, 1, 2, \ldots.$ to the letters $t, a, b, c, \ldots.$, the function F is isobaric.

Next, suppose F to be fractional, and let it be the ratio of the two rational integral homogeneous reciprocants F_1 and F_2. The operation of G on F will, in this case also, generate another reciprocant of the same character as F. For, since G is linear in the differential operative symbols $\partial_a, \partial_b, \partial_c, \ldots.$, its operation will be precisely analogous to that of differentiation, so that, operating with G on

$$F = \frac{F_1}{F_2},$$

we have $$GF = \frac{F_2 GF_1 - F_1 GF_2}{F_2^2}.$$

In order to prove that this is a reciprocant, we have to show that the character and characteristic are the same for both terms of the numerator. But GF_1 is a reciprocant of the same character as F_1, and GF_2 is one of the same character as F_2; thus the two terms of the numerator are of the same character as F_1F_2. As regards the characteristic, it should be noticed that G [*i. e.* the operator $(2bt - 3a^2)\partial_a + (2ct - 4ab)\partial_b + \ldots.$] increases the degree by unity, but does not alter the weight, so that it increases the characteristic of any rational integral homogeneous reciprocant by 3. Thus the characteristic of each term in the numerator exceeds by 3 that of F_1F_2. Hence GF is a reciprocant, and, taking account of its denominator as well as its numerator, we see that the operation of G on a rational homogeneous reciprocant, whether fractional or integral, produces another in which the original character is preserved while the characteristic is increased by three units.

More generally, let F_1, F_2, F_3, ... be any rational homogeneous reciprocants whose extraneous factors are $(-)^{\varkappa_1}t^{\mu_1}$, $(-)^{\varkappa_2}t^{\mu_2}$, $(-)^{\varkappa_3}t^{\mu_3}$, ... respectively; and suppose Φ to consist of a series of terms of the form $AF_1^{\lambda_1}F_2^{\lambda_2}F_3^{\lambda_3}\ldots.$, such that the extraneous factor for each term is $(-)^{\varkappa}t^{\mu}$. Then Φ is a reciprocant, but not necessarily a rational one; for the indices λ_1, λ_2, λ_3, ... may be supposed fractional, provided only that they satisfy the conditions $\varkappa_1\lambda_1 + \varkappa_2\lambda_2 + \varkappa_3\lambda_3 + \ldots. - \varkappa =$ a positive or negative *even* integer, and

$$\mu_1\lambda_1 + \mu_2\lambda_2 + \mu_3\lambda_3 + \ldots. - \mu = 0.$$

We proceed to show that $G\Phi$ is also a reciprocant, and that its extraneous factor is $(-)^{\varkappa}t^{\mu+3}$. Since

$$G\Phi = \frac{d\Phi}{dF_1}\cdot GF_1 + \frac{d\Phi}{dF_2}\cdot GF_2 + \frac{d\Phi}{dF_3}\cdot GF_3 + \ldots.,$$

we have to prove not only that each term of this expression is a reciprocant, but also that all of them have the same extraneous factor; otherwise their sum would not be a reciprocant.

Now, in $$\Phi = \Sigma AF_1^{\lambda_1}F_2^{\lambda_2}F_3^{\lambda_3}\ldots.,$$
the extraneous factor for each term is by hypothesis $(-)^{\varkappa}t^{\mu}$, so that the extraneous factor for each term of

$$\frac{d\Phi}{dF_1} = \Sigma A\lambda_1F_1^{\lambda_1-1}F_2^{\lambda_2}F_3^{\lambda_3}\ldots.$$

is $(-)^{\varkappa-\varkappa_1}t^{\mu-\mu_1}$, and therefore $\frac{d\Phi}{dF_1}$ is a reciprocant. Also, GF_1 is a reciprocant whose extraneous factor is $(-)^{\varkappa_1}t^{\mu_1+3}$. Hence $\frac{d\Phi}{dF_1}\cdot GF_1$ is a reciprocant having $(-)^{\varkappa}t^{\mu+3}$ for extraneous factor, and in exactly the same way we see that every other term of $G\Phi$ is also a reciprocant with the same extraneous factor.

Thus G, operating on *any* homogeneous reciprocant whose extraneous factor is $(-)^{\varkappa}t^{\mu}$, generates another whose extraneous factor is $(-)^{\varkappa}t^{\mu+3}$.

If, in the generator for mixed reciprocants,

$$G = (2bt - 3a^2)\partial_a + (2ct - 4ab)\partial_b + (2dt - 5ac)\partial_c + \ldots.,$$

we write

$$a = 1.2.a_0,\ b = 1.2.3.a_1,\ c = 1.2.3.4.a_2, \ldots.$$

(*i. e.* if we use the system of modified letters previously mentioned), its expression assumes a more elegant form. Substituting for $a, b, c, \ldots.$ their values in terms of the modified letters, we have

$$2bt - 3a^2 = 2.1.2.3a_1t - 3(1.2)^2a_0^2 = 1.2^2.3(a_1t - a_0^2)$$

and
$$\partial_a = \frac{1}{1.2} \cdot \partial_{a_0};$$

so that
$$(2bt - 3a^2)\partial_a = 1.2.3(a_1t - a_0^2)\partial_{a_0}.$$

Again,
$$(2ct - 4ab) = 1.2^2.3.4(a_2t - a_0a_1)$$

and
$$\partial_b = \frac{1}{1.2.3}\partial_{a_1};$$

so that
$$(2ct - 4ab)\partial_b = 1.2.4(a_2t - a_0a_1)\partial_{a_1}.$$

Similarly,
$$(2dt - 5ac)\partial_c = 1.2.5(a_3t - a_0a_2)\partial_{a_2}.$$

Thus the modified generator for mixed reciprocants is

$$1.2.3(a_1t - a_0^2)\partial_{a_0} + 1.2.4(a_2t - a_0a_1)\partial_{a_1} + 1.2.5(a_3t - a_0a_2)\partial_{a_2} + \ldots.,$$

in which the general term is

$$1.2(n+3)(a_{n+1}t - a_0a_n)\partial_{a_n}.$$

The factor 1.2 may, of course, be rejected, and our modified generator may be written in the simple form

$$3(a_1t - a_0^2)\partial_{a_0} + 4(a_2t - a_0a_1)\partial_{a_1} + 5(a_3t - a_0a_2)\partial_{a_2} + \ldots.$$

Operating with this on the homogeneous reciprocant $F(t, a_0, a_1, a_2, \ldots.)$, the result will be another homogeneous reciprocant of the same character as F. When we start with a_0 and make the reductions which, as we have seen, occur at every second step, we find a system of reduced educts corresponding in every particular with those formerly given, but expressed in terms of the modified letters $a_0, a_1, a_2, \ldots.$ instead of $a, b, c, \ldots.$ These are as follows:

a_0,

* $a_1t - a_0^2$,

$2a_2t - 5a_0a_1$,

* $2a_3t^2 - 6a_0a_2t - 3a_1^2t + 7a_0^2a_1$,

$2a_4t^2 - 7a_0a_3t - 7a_1a_2t + 8a_0^2a_2 + 11a_0a_1^2$,

* $14a_5t^3 - 56a_0a_4t_2 - 56a_1a_3t^2 - 28a_2^2t^2 + 103a_0^2a_3t + 199a_0a_1a_2t$
$+ 33a_1^3t - 88a_0^3a_2 - 121a_0^2a_1^2$.

. . . .

* It will be observed that in the unreduced forms, marked with an asterisk, the sum of the numerical coefficients is zero. This is a direct consequence, as may be easily seen, of the form of the modified generator, in which the sum of the numerical coefficients in each term is also zero.

It will be found on trial that these modified educts are obtained with greater ease and with less liability to error by a direct application of the generator

$$3(a_1t - a_0^2)\partial_{a_0} + 4(a_2t - a_0a_1)\partial_{a_1} + 5(a_3t - a_0a_2)\partial_{a_2} + \dots\,,$$

than by making the substitution of $1.2.a_0$, $1.2.3.a_1$, $1.2.3.4.a_2, \dots$ for $a, b, c, \dots$ in the system of educts already given. For this reason the working by the former method is here performed, instead of being merely indicated.

From a_0 we obtain immediately

$$a_1t - a_0^2.$$

Operating on this with the generator, there results

$$4t(a_2t - a_0a_1) - 6a_0(a_1t - a_0^2) = 4a_2t^2 - 10a_0a_1t + 6a_0^3.$$

This, when reduced by removing its last term and dividing the others by $2t$, gives

$$2a_2t - 5a_0a_1.$$

The next form is found from this by a simple operation, without subsequent reduction, and is therefore

$$10t(a_3t - a_0a_2) - 20a_0(a_2t - a_0a_1) - 15a_1(a_1t - a_0^2).$$

Or, collecting the terms and rejecting the numerical factor 5,

$$2a_3t^2 - 6a_0a_2t - 3a_1^2t + 7a_0^2a_1.$$

The operation of the generator on this gives

$$12t^2(a_4t - a_0a_3) - 30a_0t(a_3t - a_0a_2) + 4(7a_0^2 - 6a_1t)(a_2t - a_0a_1) + 3(14a_0a_1 - 6a_2t)(a_1t - a_0^2).$$

The collection of terms and subsequent reduction is shown below:

	a_4t^3	$a_0a_3t^2$	$a_1a_2t^2$	$a_0^2a_2t$	$a_0a_1^2t$	$a_0^3a_1$
	12	—12	.	.	.	.
	.	—30	.	30	.	.
	.	.	—24	28	24	—28
	.	.	—18	18	42	—42
	12	—42	—42	76	66	—70
$-14a_0^2(2a^2t - 5a_0a_1) =$	.	.	.	—28	.	+70
	12	—42	—42	48	66	.

Removing the factor $6t$, the reduced form is

$$2a_4t^2 - 7a_0a_3t - 7a_1a_2t + 8a_0^2a_2 + 11a_0a_1^2.$$

Operating on this with the generator, we have

$$14t^2(a_5t - a_0a_4) - 42a_0t(a_4t - a_0a_3) + 5(8a_0^2 - 7a_1t)(a_3t - a_0a_2)$$
$$+ 4(22a_0a_1 - 7a_2t)(a_2t - a_0a_1) + 3(11a_1^2 + 16a_0a_2 - 7a_3t)(a_1t - a_0^2)$$
$$= 14a_5t^3 - 56a_0a_4t^2 - 56a_1a_3t^2 - 28a_2^2t^2 + 103a_0^2a_3t + 199a_0a_1a_2t$$
$$+ 33a_1^3t - 88a_0^3a_2 - 121a_0^2a_1^2,$$

which cannot be reduced in the same manner as the preceding form.

To obtain a generator for passing from pure to pure reciprocants a process is employed similar to that which gave the generator for mixed reciprocants which we have just been using. I state the results before giving the proof, and then proceed to speak of generators in the theory of Invariants. The generator for pure reciprocants is

$$(3ac - 4b^2)\partial_b + (3ad - 5bc)\partial_c + (3ae - 6bd)\partial_d + \ldots\ldots ;$$

or, expressed in terms of the modified letters,

$$4(a_0a_2 - a_1^2)\partial_{a_1} + 5(a_0a_3 - a_1a_2)\partial_{a_2} + 6(a_0a_4 - a_1a_3)\partial_{a_3} + \ldots\ldots$$

By operating with this on any pure reciprocant R, we generate another pure reciprocant of opposite character to that of R.

The connection between the two theories of Reciprocants and Invariants is so close, and these brother-and-sister theories throw so much light upon each other, that I began to inquire whether, in the latter, there did not exist a theory of Generators parallel to that of the former.

Fortunately, Mr. Hammond was able to recall a correspondence in which Prof. Cayley had given such a theory, which he regarded, and justly, as an important invention. Its substance has been subsequently incorporated in the *Quarterly Journal* (Vol. XX, p. 212). It offers itself spontaneously in the Reciprocantive Theory; in the Invariantive one it calls for a distinct act of invention. Prof. Cayley has discovered two generators similar in form with those for reciprocants, and one of them strikingly so; in a letter to me he calls these P and Q. As given by him,

$$P = ab\partial_a + ac\partial_b + ad\partial_c + \ldots\ldots - ib,$$
$$Q = \qquad ac\partial_b + 2ad\partial_c + \ldots\ldots - 2wb,$$

where i is the degree and w the weight, the weights of $a, b, c, d, \ldots\ldots$ being taken to be $0, 1, 2, 3, \ldots\ldots$ (I supply the a which Cayley turns into unity.) As an example he takes the "Invariant" $a^2d - 3abc + 2b^3 = I$, suppose. We have then

$$\begin{aligned}PI &= (ab\partial_a + ac\partial_b + ad\partial_c + ae\partial_d - 3b)I \\ &= ab(2ad - 3bc) + ac(-3ac + 6b^2) - 3a^2bd + a^3e - 3b(a^2d - 3abc + 2b^3) \\ &= a^3e - 4a^2bd - 3a^2c^2 + 12ab^2c - 6b^4 \\ &= a^2(ae - 4bd + 3c^2) - 6(ac - b^2)^2,\end{aligned}$$

and $$QI = (ac\partial_b + 2ad\partial_c + 3ae\partial_d - 6b)\,I$$
$$= ac(-3ac + 6b^2) - 6a^2bd + 3a^2e - 6b(a^2d - 3abc + 2b^3)$$
$$= 3a^3e - 12a^2bd - 3a^2c^2 + 24ab^2c - 12b^4$$
$$= 3a^2(ae - 4bd + 3c^2) - 12(ac - b^2)^2.$$

P and Q may be transformed by means of Euler's equation and the similar one for isobaric functions, which enable us to write

$$i = a\partial_a + b\partial_b + c\partial_c + d\partial_d + \ldots.$$

and $$w = \quad b\partial_b + 2c\partial_c + 3d\partial_d + \ldots.;$$

P thus becomes $$ab\partial_a + ac\partial_b + ad\partial_c + ae\partial_d + \ldots.$$
$$- ab\partial_a - b^2\partial_b - bc\partial_c - bd\partial_d - \ldots.$$
$$= (ac - b^2)\partial_b + (ad - bc)\partial_c + (ae - bd)\partial_d + \ldots.,$$

the same in *form* as either of our generators, except that the arithmetical coefficients are all made units; $a, b, c, \ldots.$ taking the place of the $t, a, b, \ldots.$ of the generator for mixed reciprocants.

In like manner, Q becomes

$$(ac - 2b^2)\partial_b + 2(ad - 2bc)\partial_c + 3(ae - 2bd)\partial_d + \ldots.,$$

where the arithmetical series 1, 2, 3, ... takes the place of 3, 4, 5, ... or of 4, 5, 6, ... in the two Reciprocant Generators.

The effect of P and of Q is obviously to raise the degree and the weight of the operand I each by one unit. But if we take $R = \frac{1}{a}(2wP - iQ)$, the terms in Cayley's original formulae containing b cancel, so that $2wP - iQ$ divides out by a and the weight is raised one unit without the degree being affected. This is mentioned in the *Quarterly Journal* (*loc. cit.*); but it may also be remarked that when I is a *satisfied invariant*, it is annihilated by the operation of R; when the *invariant* is *unsatisfied*, each of the three operators P, Q and R increases its extent by an unit, *i. e.* introduces an additional letter. For let j denote the extent, then, writing $a_0, a_1, a_2, \ldots. a_j$ for $a, b, c, \ldots.$, we have

$$P = a_0a_1\partial_{a_0} + a_0a_2\partial_{a_1} + \ldots. + a_0a_{j+1}\partial_{a_j} - ia_1,$$
$$Q = \quad a_0a_2\partial_{a_1} + 2a_0a_3\partial_{a_2} + \ldots. + ja_0a_{j+1}\partial_{a_j} - 2wa_1;$$

whence we find

$$R = \frac{1}{a_0}(2wP - iQ)$$
$$= 2wa_1\partial_{a_0} + (2w - i)a_2\partial_{a_1} + \ldots. + (2w - ij + i)a_j\partial_{a_{j-1}} + (2w - ij)a_{j+1}\partial_{a_j}.$$

But for a *satisfied invariant* $\qquad 2w = ij;$

and substituting this value for $2w$ in the above expression for R, it becomes

$$i\{ja_1\partial_{a_0} + (j-1)a_2\partial_{a_1} + \ldots. + a_j\partial_{a_{j-1}}\},$$

which, as is well known, annihilates any satisfied invariant.

LECTURE V.

It will be desirable to fill up some of the previous investigations by discussing some points in them that have not yet received our consideration.

There may be some to whom it may appear tedious to watch the complete exposition of the algebraical part of the Theory and who are impatient to rush on to its applications. But it is my duty to consider what may be expected to be most useful to the great majority of the class, and for that purpose to make the ground sure under our feet as I proceed. To the greater number it will, I think, be of advantage to have their memories refreshed on the kindred subject of invariants, and probably made acquainted with some important points of that theory which are new to them.

I confess that, to myself, the contemplation of this relationship—the spectacle of a new continent rising from the waters, resembling yet different from the old, familiar one—is a principal source of interest arising out of the new theory. I do not regard Mathematics as a science purely of calculation, but one of ideas, and as the embodiment of a Philosophy. An eminent colleague of mine, in a public lecture in this University, magnifying the importance of classical over mathematical studies, referred to a great mathematician as one who might possibly know every foot of distance between the earth and the moon; and when I was a member, at Woolwich, of the Government Committee of Inventions, one of my colleagues, appealing to me to answer some question as to the number of cubic inches in a pipe, expressed his surprise that I was not prepared with an immediate answer, and said he had supposed that I had all the tables of weights and measures at my fingers' ends.

I hope that in any class which I may have the pleasure of conducting in this University, other ideas will prevail as to the true scope of mathematical science as a branch of liberal learning; and it will be my endeavor to regulate the pace in a manner which seems to me most conducive to real progress in the order of ideas and philosophical contemplation, thus bringing our noble science into harmony and in a line with the prevailing tone and studies of this University. So long as we are content to be regarded as mere calculators (Faraday, at the end of his experimental lectures, was accustomed to say—I have myself heard him do so—"We will now leave that to the calculators"), we shall be the Pariahs of the University, living here on sufferance, instead of being regarded, as is our right and privilege, as the real leaders and pioneers of thought in it.

That Cayley's two operators, which have been called P and Q, are in fact generators, may be proved as follows:*

Let $\Omega = a\partial_b + 2b\partial_c + 3c\partial_d + 4d\partial_e + \ldots$

and $\Theta = a(\lambda b\partial_a + \mu c\partial_b + \nu d\partial + \ldots) - xb$,

where $x, \lambda, \mu, \nu, \ldots$ are numbers.

When x is the degree of the operand, and $\lambda = \mu = \nu = \ldots = 1$, the operator Θ is identical with P; but Θ is identical with Q when x is twice the weight of the operand and $\lambda = 0$, $\mu = 1$, $\nu = 2, \ldots$

If now we use $*$ to signify the act of pure differential operation, it is obvious that

$$\Omega\Theta = (\Omega \times \Theta) + (\Omega * \Theta),$$

$$\Theta\Omega = (\Omega \times \Theta) + (\Theta * \Omega),$$

so that $\Omega\Theta - \Theta\Omega = (\Omega * \Theta) - (\Theta * \Omega)$.

But since $\Omega a = 0$, $\Omega b = a$, $\Omega c = 2b, \ldots,$

we have $\Omega * \Theta = a(\lambda a\partial_a + 2\mu b\partial_b + 3\nu c\partial_c + \ldots - x)$

and $\Theta * \Omega = a(\lambda b\partial_b + 2\mu c\partial_c + 3\nu d\partial_d + \ldots)$.

Hence $\Omega\Theta - \Theta\Omega = a\{\lambda a\partial_a + (2\mu - \lambda)b\partial_b + (3\nu - 2\mu)c\partial_c + \ldots - x\}$;

or if the operand I be any *invariant* (satisfied or unsatisfied), we have $\Omega I = 0$, and therefore $\Theta\Omega I = 0$; so that we find

$$\Omega\Theta I = a\{\lambda a\partial_a + (2\mu - \lambda)b\partial_b + (3\nu - 2\mu)c\partial_c + \ldots - x\} I.$$

If in this we write $\lambda = \mu = \nu = \ldots = 1$, and $x = i$, where i is the degree of the operand, Θ becomes P and we have

$$\Omega PI = a(a\partial_a + b\partial_b + c\partial_c + \ldots - i) I.$$

But, by Euler's theorem, the right-hand side of this vanishes, and therefore

$$\Omega PI = 0.$$

Similarly, by means of the corresponding theorem for isobaric functions, we may prove that $\Omega QI = 0$.

For if, in the general formula, we write $\lambda = 0$, $\mu = 1$, $\nu = 2, \ldots$ and $x = 2w$, where w is the weight of the operand, we find

$$\Omega QI = a(2b\partial_b + 4c\partial_c + 6d\partial_d + \ldots - 2w) I = 0.$$

Thus, when Θ stands either for P or for Q, it is either an annihilator or a

* In the *Quarterly Journal* (Vol. XX, p. 212) Prof. Cayley only considers a special example, and has not given the proof of the general theorem.

generator (*i. e.* ΘI is either identically zero or else an invariant). But if l be the most advanced, or say the *radical letter* of I, no term of $m\partial_l I$ can cancel with any other term of ΘI; and since, for this reason, ΘI cannot vanish identically, it must be an invariant, and the operators P and Q must be generators.

The generators previously given for reciprocants also possess this property of introducing a fresh radical letter at each step. The radical letter, on its first introduction, enters in the first degree only, and in the case of the educts of $\log t$, whose values have been calculated, its multiplier is seen to be a power of t. The form of the generator for mixed reciprocants

$$3(a_1 t - a_0^2)\partial_{a_0} + 4(a_2 t - a_0 a_1)\partial_{a_1} + \ldots . + (n+3)(a_{n+1}t - a_0 a_n)\partial_{a_n}$$

shows this, or it may be seen by considering the successive values of

$$\left(\frac{1}{\sqrt{t}}\cdot\frac{d}{dx}\right)^i \log t.$$

For let $\dfrac{F(t, a_0, a_1, a_2, \ldots .)}{t^\lambda}$ denote this expression, and let its radical letter be a_n; then, on differentiating again with respect to x, the new letter introduced arises solely from a term in the numerator

$$\frac{d}{da_n} F(t, a_0, a_1, a_2, \ldots . a_n)\cdot\frac{da_n}{dx}.$$

But $\quad a_n = \dfrac{d^n y}{dx^n} \div 2.3 \ldots . n+2$; so that $\dfrac{da_n}{dx} = (n+3)a_{n+1}$.

Hence, if when a_n is the radical letter, it occurs in the first degree only and multiplied by a power of t, it follows that, since $\dfrac{dF}{da_n}$ will be a power of t, the derived expression which contains the radical letter a_{n+1} will contain it in the first degree only and multiplied by a power of t. And since this is true for the case $i = 1$, when $\dfrac{1}{\sqrt{t}}\cdot\dfrac{d}{dx}\log t = \dfrac{a_0}{t^{\frac{1}{2}}}$, it is true universally.

Observe that for $i = 1, 2, 3, \ldots .$ the radical letter is $a_0, a_1, a_2, \ldots .$ respectively.

It will be remembered that $\left(\dfrac{1}{\sqrt{t}}\cdot\dfrac{d}{dx}\right)^i \log t$ is an absolute reciprocant. It may be called the i^{th} absolute educt, to distinguish it from the rational integral educts $E_1, E_2, E_3, \ldots .$ whose values have already been calculated.

Let $R(t, a_0, a_1, a_2, \ldots . a_n)$ be any homogeneous rational integral reciprocant, and let the educts be $A_0, A_1, A_2, \ldots . A_n$; then obviously

a_n may be expressed rationally in terms of A_n and $a_{n-1}, a_{n-2}, \ldots . a_0, t$,
a_{n-1} " " " " " A_{n-1} and $a_{n-2}, \ldots . a_0, t$,
. . . .
a_1 " " " " " A_1, a_0 and t,
a_0 " " " " " A_0 and t,

where observe that the denominators in these expressions are all powers of t. Hence, by successive substitutions, $R(t, a_0, a_1, \ldots . a_n)$ may be expressed rationally in terms of $A_n, \ldots . A_1, A_0$, and t. Thus any rational integral homogeneous reciprocant is a rational function of educts, and is of the form $\frac{E}{t^\rho}$, where E is a rational *integral* function of the educts.

Does not this prove too much, it may be asked, viz.: that any function F of the letters is a rational function of the educts, which are themselves reciprocants, and will therefore be a reciprocant? But this is not so; for observe that although F will be expressed as a sum of products of educts, such products will not in general be all of the same character, and their linear combination will be an illicit one, such as is seen in the illicit combination of a_0^2 with the Schwarzian $(a_1 t - a_0^2)$.

We have seen that by differentiating an absolute reciprocant, or by the use of a generator, we obtain a fresh reciprocant. But there are other methods of finding reciprocants; as, for example, if the transform of $\phi(t, a, b, c, \ldots .)$ is $\psi(\tau, \alpha, \beta, \gamma, \ldots .)$, *i. e.* if

$$\phi(t, a, b, c, \ldots .) = \psi(\tau, \alpha, \beta, \gamma, \ldots .),$$

then

$$\psi(t, a, b, c, \ldots .) = \phi(\tau, \alpha, \beta, \gamma, \ldots .).$$

Whence, by multiplication,

$$\phi(t, a, b, c, \ldots .)\, \psi(t, a, b, c, \ldots .) = \phi(\tau, \alpha, \beta, \gamma, \ldots .)\, \psi(\tau, \alpha, \beta, \gamma, \ldots .).$$

Thus $\phi . \psi$ is a reciprocant, and, moreover, an absolute one of even character, although neither ϕ, which is a perfectly arbitrary function, nor ψ, its transform, is a reciprocant.

Herein a mixed reciprocant differs from an invariant, which cannot be resolved into non-invariantive factors. It is worth while to give a proof of this proposition; but first I prove its converse, that if $p, q, r, \ldots .$ are all invariants, their product must be so too. This is an immediate consequence of the well-known theorem that

$$\Omega I = 0$$

is the necessary and sufficient condition that I may be an invariant where, as usual, Ω is the operator

$$a\partial_b + 2b\partial_c + 3c\partial_d + \ldots . ,$$

and the word invariant has been used in the same extended sense as formerly.

For $$\Omega(pqrs \ldots .) = \left(\frac{\Omega p}{p} + \frac{\Omega q}{q} + \frac{\Omega r}{r} + \ldots .\right) pqrs \ldots .$$

But since $p, q, r, \ldots$ are all invariants, we have $\Omega p = 0$, $\Omega q = 0$, $\Omega r = 0, \ldots,$ and therefore

$$\Omega(pqrs \ldots .) = 0.$$

Next, suppose that $$I = P_1 Q_1,$$
where I is but Q_1 is not an invariant.

To meet the case in which P_1 and Q_1 are not prime to one another, Q_1, if resolved into its factors, must contain one Q^i where Q is not an invariant.

Suppose that P_1 contains Q^j, and let $i + j = k$; then we may write

$$I = PQ^k,$$

where P is prime to Q. But since I is an invariant by hypothesis,

$$\Omega I = 0,$$

and therefore,

$$Q^k \Omega P + kPQ^{k-1}\Omega Q = 0;$$

or,
$$\frac{Q}{P} = -k\frac{\Omega Q}{\Omega P}.$$

Now P is prime to Q, so that the fraction $\frac{Q}{P}$ is in its lowest terms; therefore ΩQ contains Q; but this is impossible, for the weight of ΩQ is less than that of Q. Hence I cannot contain any non-invariantive factor Q_1.

All this will be equally true for a general function J annihilated by any operator Ω which is *linear* in the differential operators $\partial_a, \partial_b, \partial_c, \ldots$ no matter what its degree in the letters $a, b, c, \ldots$ themselves; *i. e.* we shall still have

$$J = PQ^k$$

and
$$\frac{Q}{P} = -k\frac{\Omega Q}{\Omega P},$$

where P and Q are prime to each other, and, as before, ΩQ will contain Q as a factor. But if Ω is an operator which diminishes either the degree or the weight, ΩQ is either of lower degree or of lower weight than Q, and so cannot contain it as a factor. Hence J cannot contain a factor Q not subject to annihilation by Ω.

If, however, Ω does not diminish either the degree or the weight, it may be objected that ΩQ might conceivably contain the factor Q; and were it so, there would be nothing to show the impossibility, in this case, of a function J subject to annihilation by Ω containing a factor Q, which is not so. But *quaere:* Is it possible, when J is a general homogeneous and isobaric function of $a, b, c, \ldots,$ for ΩJ to contain J and at the same time the quotient to be other than a number?* *Valde dubitor.* But I reserve the point. Setting aside this doubtful case, and considering only such *linear* partial differential operators as *diminish* either the degree or the weight of the operand, we see that there cannot exist any universal operator of this kind whose effect in annihilating a form is the necessary and sufficient condition of that form being a reciprocant. But this does not preclude the possibility of the existence of such annihilators for special classes of reciprocants, and in fact (as we have already stated and shall hereafter prove) Pure Reciprocants are definable by means of the Partial Differential Annihilator

$$V = 4 \cdot \frac{a_0^2}{2}\partial_{a_1} + 5a_0a_1\partial_{a_2} + 6\left(a_0a_2 + \frac{a_1^2}{2}\right)\partial_{a_3} + \ldots,$$

which is *linear* in the differential operators, and *diminishes* the weight.

The generator for mixed reciprocants, which we have called G, will not assist us in obtaining pure reciprocants, but generates a mixed reciprocant in every case, even when the one we start with is pure. Thus, starting with the pure reciprocant R, our formula

$$GR = \{3(a_1t - a_0^2)\partial_{a_0} + 4(a_2t - a_0a_1)\partial_{a_1} + 5(a_3t - a_0a_2)\partial_{a_2} + \ldots\}R$$

may be written thus

$$GR = t(3a_1\partial_{a_0} + 4a_2\partial_{a_1} + 5a_3\partial_{a_2} + \ldots)R$$
$$- a_0(3a_0\partial_{a_0} + 4a_1\partial_{a_1} + 5a_2\partial_{a_2} + \ldots)R.$$

Here R being *pure*, *i. e.* a function of $a_0, a_1, a_2, \ldots$ (without t), we see that

$$(3a_0\partial_{a_0} + 4a_1\partial_{a_1} + 5a_2\partial_{a_2} + 6a_3\partial_{a_3} + \ldots)R$$
$$= 3(a_0\partial_{a_0} + a_1\partial_{a_1} + a_2\partial_{a_2} + \ldots)R$$
$$+ (a_1\partial_{a_1} + 2a_2\partial_{a_2} + 3a_3\partial_{a_3} + \ldots)R$$
$$= (3i + w)R,$$

where i is the degree and w the weight of R. Hence

$$GR = t(3a_1\partial_{a_0} + 4a_2\partial_{a_1} + 5a_3\partial_{a_2} + \ldots)R - (3i + w)a_0R,$$

where it should be noticed that a_0R is of opposite character to R (for a_0 is of odd character), while GR has been proved to be of the same character as R. Thus we cannot infer that $t(3a_1\partial_{a_0} + 4a_2\partial_{a_1} + 5a_3\partial_{a_2} + \ldots)R$ is a reciprocant. The mixed reciprocant GR cannot therefore be resolved into the sum of two terms, one of which is a pure reciprocant and the other a pure reciprocant multiplied by t.

* If $\Omega = pa\partial_a + qb\partial_b + rc\partial_c + \ldots$, where $p, q, r, \ldots$ are in Arithmetical Progression, $\frac{\Omega J}{J}$ is a number; but then Ω could not be an annihilator.

LECTURE VI.

Before proceeding to prove that, as was stated in anticipation in Lecture IV, the operator

$$(3ac - 4b^2)\,\partial_b + (3ad - 5bc)\,\partial_c + (3ae - 6bd)\,\partial_d + \ldots ,$$

or, when the modified letters are used,

$$4\,(a_0a_2 - a_1^2)\,\partial_{a_1} + 5\,(a_0a_3 - a_1a_2)\,\partial_{a_2} + 6\,(a_0a_4 - a_1a_3)\,\partial_{a_3} + \ldots ,$$

will serve to generate a pure reciprocant from a pure one, it may be useful to briefly recapitulate what has been said concerning the character and characteristic of reciprocants. It will be remembered that the extraneous factor of any rational integral reciprocant is of the form $(-)^\kappa t^\mu$, that the character is determined by the parity (oddness or evenness) of κ, and that μ is what has been called the characteristic.

For homogeneous reciprocants it has been proved that $\mu = 3i + w$, where i is the degree of the reciprocant and w its weight, the weights of the letters $t, a, b, c, \ldots$ being taken to be $-1, 0, 1, 2, \ldots$ respectively. The character is odd or even according as the number of letters other than t in the principal term or terms is odd or even. By a principal term is to be understood one in which t is contained the greatest number of times. So that, in other words, the character is governed by the parity of the smallest number of non-t letters that can be found in any term. For pure reciprocants, there being no t in any term, the character is determined by the parity of the number of letters in any one term.

Let R be any pure reciprocant, and suppose its characteristic to be μ; then $\frac{R}{t^{\frac{\mu}{2}}}$ is an absolute reciprocant. If, however, we differentiate this with respect to x, and thus obtain another reciprocant, the resulting form will not be pure, for its numerator will be identical with the form obtained by the direct operation on R of the generator for mixed reciprocants, and its denominator will be a power of t. But, remembering that $\frac{a}{t^{\frac{3}{2}}}$, and therefore $\frac{a^{\frac{\mu}{3}}}{t^{\frac{\mu}{2}}}$, is an absolute reciprocant, we see that $\frac{R}{a^{\frac{\mu}{3}}}$, which is the quotient of the two absolute reciprocants $\frac{R}{t^{\frac{\mu}{2}}}$ and $\frac{a^{\frac{\mu}{3}}}{t^{\frac{\mu}{2}}}$, is so also. Hence $\frac{d}{dx}\left(\frac{R}{a^{\frac{\mu}{3}}}\right)$ is a reciprocant, and, since it no longer contains t, a pure one. Now,

$$\frac{d}{dx}\left(\frac{R}{a^{\frac{\mu}{3}}}\right) = \frac{a\,\frac{dR}{dx} - \frac{\mu}{3}\cdot bR}{a^{\frac{\mu}{3}+1}}$$

remains a reciprocant when multiplied by any power of the reciprocant a. Hence the numerator of this expression, or

$$\left(3a\frac{d}{dx}-\mu b\right)R,$$

is a reciprocant. The general value of $\frac{d}{dx}$ has been seen to be

$$a\partial_t + b\partial_a + c\partial_b + d\partial_c + \ldots,$$

but, since R is supposed to be *pure*, $\partial_t R = 0$.

We may therefore, in $3a\frac{d}{dx}-\mu b$, replace $\frac{d}{dx}$ by

$$b\partial_a + c\partial_b + d\partial_c + e\partial_d + \ldots.$$

Now, remembering that $\mu = 3i + w$, and that by Euler's theorem and the similar one for isobaric functions

$$i = a\partial_a + b\partial_b + c\partial_c + d\partial_d + \ldots.$$

and

$$w = \quad b\partial_b + 2c\partial_c + 3d\partial_d + \ldots,$$

we see that μ is equivalent to

$$3a\partial_a + 4b\partial_b + 5c\partial_c + 6d\partial_d + \ldots.$$

Hence,

$$\begin{aligned} 3a\frac{d}{dx}-\mu b &= 3a(b\partial_a + c\partial_b + d\partial_c + e\partial_d + \ldots) \\ &\quad - b(3a\partial_a + 4b\partial_b + 5c\partial_c + 6d\partial_d + \ldots) \\ &= (3ac-4b^2)\partial_b + (3ad-5bc)\partial_c + (3ae-6bd)\partial_d + \ldots. \end{aligned}$$

Thus, if R be any *pure* reciprocant,

$$\{(3ac-4b^2)\partial_b + (3ad-5bc)\partial_c + (3ae-6bd)\partial_d + \ldots\}R$$

is also a pure reciprocant. If the type of R be w; i, j, that of the form derived from it will clearly be $w+1$; $i+1$, $j+1$. Its character (which, for pure reciprocants, depends solely on the degree) will therefore be opposite to that of R, and its characteristic will be $\mu+4$, that of R being μ.

Beginning with the form $3ac-5b^2$, which was given as an example in Lecture II, a series of pure "educts" may be obtained by the repeated use of the above generator; and it will be noticed that the successive educts thus formed are alternately of even and odd character, whereas those previously given, viz. a, $2bt-3a^2$. . . . , were all negative. A reduction similar to that which formerly took place when the generator for mixed reciprocants was used, may be effected at each second step in the present case. For, since the characteristic of $\left(3a\frac{d}{dx}-\mu b\right)R$ is $\mu+4$, the next operation will give

$$\left(3a\frac{d}{dx}-(\mu+4)b\right)\left(3a\frac{d}{dx}-\mu b\right)R.$$

Performing the indicated differentiations, this becomes

$$3a\frac{d}{dx}\left(3a\frac{dR}{dx}-\mu bR\right)-3(\mu+4)ab\frac{dR}{dx}+\mu(\mu+4)b^2R$$

$$=9a^2\frac{d^2R}{dx^2}+9ab\frac{dR}{dx}-3\mu ab\frac{dR}{dx}-3\mu acR-3(\mu+4)ab\frac{dR}{dx}+\mu(\mu+4)b^2R$$

$$=9a^2\frac{d^2R}{dx^2}-3(2\mu+1)ab\frac{dR}{dx}-3\mu acR+\mu(\mu+4)b^2R.$$

Adding $\mu(\mu+4)(3ac-5b^2)R$ to 5 times the above expression, we obtain

$$45a^2\frac{d^2R}{dx^2}-15(2\mu+1)ab\frac{dR}{dx}+3\mu(\mu-1)acR,$$

which, when divided by $3a$, gives the pure reciprocant

$$15a\frac{d^2R}{dx^2}-5(2\mu+1)b\frac{dR}{dx}+\mu(\mu-1)cR.$$

This form is one degree lower than the second educt from R, the depression of degree being due to the removal of a factor a by division.

When the modified letters $a_0, a_1, a_2, a_3, \ldots.$ are used, the generator

$$(3ac-4b^2)\partial_b+(3ad-5bc)\partial_c+(3ae-6bd)\partial_d+\ldots. \qquad (1)$$

is easily transformed by writing in it

$$a=2a_0,\ b=2.3.a_1,\ c=2.3.4.a_2,\ d=2.3.4.5.a_3\ldots.,$$

and consequently

$$\partial_b=\frac{\partial_{a_1}}{2.3},\ \partial_c=\frac{\partial_{a_2}}{2.3.4},\ \partial_d=\frac{\partial_{a_3}}{2.3.4.5}\ldots.,$$

when it becomes

$$\frac{2^2.3^2.4}{2.3}(a_0a_2-a_1^2)\partial_{a_1}+\frac{2^2.3^2.4.5}{2.3.4}(a_0a_3-a_1a_2)\partial_{a_2}+\frac{2^2.3^2.4.5.6}{2.3.4.5}(a_0a_4-a_1a_3)\partial_{a_3}+\ldots.$$

Dividing each term of this by 2.3, and writing the numerical coefficients in their simplest form, we have

$$4(a_0a_2-a_1^2)\partial_{a_1}+5(a_0a_3-a_1a_2)\partial_{a_2}+6(a_0a_4-a_1a_3)\partial_{a_3}+\ldots., \qquad (2)$$

which is the modified generator previously mentioned.

The generators formerly used in the theory of mixed reciprocants were

$$(2tb-3a^2)\partial_a+(2tc-4ab)\partial_b+(2td-5ac)\partial_c+\ldots. \qquad (3)$$

and

$$3(ta_1-a_0^2)\partial_{a_0}+4(ta_2-a_0a_1)\partial_{a_1}+5(ta_3-a_0a_2)\partial_{a_2}+\ldots.. \qquad (4)$$

The memory will be assisted in retaining these formulae if we observe that (1) is obtainable from (3), or (2) from (4), by increasing at the same time each numerical coefficient and the weight of each letter by unity.

It will, I think, be instructive to see how the form $3ac - 5b^2$ was found originally by combining mixed reciprocants. The degree alone of a pure reciprocant suffices, as we have seen, to determine its character; but when we are dealing with mixed reciprocants their character does not depend either on the degree or the weight, so that we require a notation to discriminate between forms of the same degree-weight, but of opposite character. In what follows, (+) placed before any form signifies that it is a reciprocant of *even* character, while (—) signifies that its character is odd.

I have previously given the three *odd* reciprocants

$$(-)\quad a, \tag{A}$$

$$(-)\quad 2bt - 3a^2, \tag{B}$$

$$(-)\quad ct - 5ab. \tag{C}$$

From these we obtain *even* reciprocants; thus the product of (A) and (C) is

$$(+)\quad act - 5a^2b, \tag{D}$$

and the square of (B) is $\quad (+)\quad 4b^2t^2 - 12a^2bt + 9a^4.$

After subtracting the *even* reciprocant $9a^4$ from this, we may remove the factor $4t$ from the remainder without thereby affecting its character. These reductions give $\quad (+)\quad b^2t - 3a^2b,$

which may be combined with the *even* reciprocant (D) in such a manner that the combination contains a factor t. In fact,

$$3(act - 5a^2b) - 5(b^2t - 3a^2b) = (3ac - 5b^2)t,$$

so that a *legitimate* combination of mixed reciprocants can be made to give the pure one $\quad 3ac - 5b^2.$

Similarly we might find the known form

$$9a^2d - 45abc + 40b^3,$$

which equated to zero expresses Sextactic Contact at a point x, y. But it is more readily obtained by operating with the generator on $3ac - 5b^2$; thus,

$$\{(3ac - 4b^2)\partial_b + (3ad - 5bc)\partial_c\}(3ac - 5b^2) = -10b(3ac - 4b^2) + 3a(3ad - 5bc)$$
$$= 9a^2d - 45abc + 40b^3.$$

An *orthogonal reciprocant* may be defined as a mixed reciprocant whose form remains invariable (save as to the acquisition of an extraneous factor when the reciprocant is not absolute) when any orthogonal substitution is impressed on the variables x and y. Concerning such reciprocants, we have the very beautiful theorem: *If* R *and* $\frac{dR}{dt}$ *are both of them reciprocants, then* R *is an orthogonal reciprocant.*

First suppose R to be an absolute reciprocant; *i. e.* let

$$R = qR' \ (q = \pm 1),$$

where R is a function of $t, a, b, c, \ldots.$ and R' the same function of $\tau, \alpha, \beta, \gamma, \ldots.$; then, denoting by ΔR the variation of R due to the variation of y by εx, and by DR the variation of R due to the variation of x by $-\varepsilon y$, we have

$$\Delta R = \varepsilon \frac{dR}{dt}.$$

For the variation of t is ε and the variations of $a, b, c, \ldots.$ vanish. Similarly

$$DR' = -\varepsilon \frac{dR'}{d\tau}.$$

Now, since

$$R = qR',$$

$$DR = qDR' = -\varepsilon q \frac{dR'}{d\tau},$$

therefore

$$DR + \Delta R = \varepsilon \left(\frac{dR}{dt} - q \frac{dR'}{d\tau} \right);$$

i. e. the total variation of R (due to the change of x into $x - \varepsilon y$ and of y into $y + \varepsilon x$) vanishes if

$$\frac{dR}{dt} = q \frac{dR'}{d\tau}.$$

Hence, if R be an absolute orthogonal reciprocant, $\frac{dR}{dt}$ is also an absolute reciprocant (though it is not orthogonal) of the same character as R.

If R be not absolute, suppose its characteristic to be μ; then it can be made absolute by dividing it by $a^{\frac{\mu}{3}}$. The application of the foregoing method of variations will now prove that $\frac{d}{dt}\left(\frac{R}{a^{\frac{\mu}{3}}}\right)$ is an absolute reciprocant of the same character as $\frac{R}{a^{\frac{\mu}{3}}}$. But $\frac{d}{dt}\left(\frac{R}{a^{\frac{\mu}{3}}}\right) = \frac{1}{a^{\frac{\mu}{3}}}\frac{dR}{dt}$. Hence $\frac{dR}{dt}$ is a reciprocant whose characteristic is μ, and character the same as that of R.

The simplest Orthogonal Reciprocant is the form

$$(1 + t^2)\, b - 3a^2 t,$$

which occurs on p. 19 of Boole's Differential Equations. When equated to zero it is the general differential equation of a circle. It is noticeable that although Boole obtains this form by equating to zero the differential of the radius of curvature

$$\frac{(1 + t^2)^{\frac{3}{2}}}{a},$$

he does not recognize the fact that it vanishes at points of maximum or minimum curvature of any plane curve, but says that the "geometrical property which this equation expresses is the invariability of the radius of curvature."

Taking this form as an example of our general theorem, let

$$R = (1 + t^2)\, b - 3a^2 t;$$

then

$$\frac{dR}{dt} = 2bt - 3a^2,$$

which is the familiar Schwarzian. Observe that

$$(1 + t^2)\, b - 3a^2 t = -t^6 \{(1 + \tau^2)\,\beta - 3\alpha^2 \tau\}$$

and

$$2bt - 3a^2 = -t^6 (2\beta\tau - 3\alpha^2),$$

so that the characteristic and character arè the same for both these forms.

The form $ct - 5ab$, which we have called the Post-Schwarzian, when multiplied by 2 and integrated with respect to t, gives

$$ct^2 - 10abt + \phi(a, b, \ldots.).$$

In order that this may be a reciprocant, we must have

$$\phi(a, b, \ldots.) = c + 15a^3.$$

In this way the Orthogonal Reciprocant

$$(1 + t^2)c - 10abt + 15a^3$$

was obtained originally.

It will be easy to verify that this is a reciprocant by means of the identical relations

$$t = \frac{1}{\tau},$$

$$a = -\frac{\alpha}{\tau^3},$$

$$b = -\frac{\beta\tau - 3\alpha^2}{\tau^5},$$

$$c = -\frac{\gamma\tau^2 - 10\alpha\beta\tau + 15\alpha^3}{\tau^7}.$$

We shall find that

$$(1 + t^2)c - 10abt + 15a^3 = -t^7 \{(1 + \tau^2)\gamma - 10\alpha\beta\tau + 15\alpha^3\},$$

and comparing this with

$$ct - 5ab = -t^7 (\gamma\tau - 5\alpha\beta),$$

it will be noticed that both forms have the same character and the same characteristic.

The complete primitive of the differential equation

$$c(1 + t^2) - 10abt + 15a^3 = 0$$

has been found by Mr. Hammond and Prof. Greenhill. The solution may be written in the following forms:

$$\left.\begin{aligned} x &= \int \frac{dt}{\sqrt{\kappa(1-15t^2+15t^4-t^6)+\lambda(6t-20t^3+6t^5)}} + \mu \\ y &= \int \frac{tdt}{\sqrt{\kappa(1-15t^2+15t^4-t^6)+\lambda(6t-20t^3+6t^5)}} + \nu \end{aligned}\right\},$$

$$\left.\begin{aligned} x &= \int \frac{\cos(\theta-A)\,d\theta}{\sqrt{B\cos 6(\theta-A)}} + \text{const.} \\ y &= \int \frac{\sin(\theta-A)\,d\theta}{\sqrt{B\cos 6(\theta-A)}} + \text{const.} \end{aligned}\right\}.$$

$$k^{12}\mathrm{tn}^2(X, k) = k^2\mathrm{tn}^2(Y, k'),$$

where $\qquad k = \sin 15^\circ,\ k' = \sin 75^\circ,$

and $\qquad X = lx + my + n_1,$

$$Y = mx - ly + n_2,$$

l, m, n_1, n_2 being arbitrary constants.

The last two forms of solution are due to Prof. Greenhill.

LECTURE VII.

I have frequently referred to, and occasionally dilated on, the analogy between pure reciprocants and invariants. A new bond of connection between the two theories has been established by Capt. MacMahon, which I will now explain. Let me, by way of preface, so far anticipate what I shall have to say on the Theorem of Aggregation in Invariants (*i. e.* the theorem concerning the number of linearly independent invariants of a given type) as to remark that the proof of this theorem, first given by me in Crelle's Journal and subsequently in the Phil. Mag. for March, 1878, depends on the fact that if we take two operators, viz. the Annihilator, say

$$\Omega = a_0\partial_{a_1} + 2a_1\partial_{a_2} + 3a_2\partial_{a_3} + \ldots . + ja_{j-1}\partial_{a_j}$$

and its opposite, say

$$O = a_j\partial_{a_{j-1}} + 2a_{j-1}\partial_{a_{j-2}} + 3a_{j-2}\partial_{a_{j-3}} + \ldots . + ja_1\partial_{a_0},$$

then $(\Omega O - O\Omega) I$ is a multiple of I.

Thus, if I stands for any invariant (*i. e.* if $\Omega I = 0$), it follows immediately that $\Omega O I$ is a multiple of I, and consequently $\Omega^m O^m I$ is also a multiple of I. We may call Ω and O, which are exact opposites to each other, reversing operators.

Now, MacMahon has found out the reversor to V, the Annihilator of pure reciprocants. His reversing operator is no longer of a similar, though opposite, form to V, as O is to Ω, but is simply $\frac{d}{dx}$; nor is the effect of operating with $V\frac{d}{dx}$ on any pure reciprocant R equivalent to multiplication by a merely numerical factor, as was the case with $\Omega O I$, but $\left(V\frac{d}{dx}\right)R$ is a numerical multiple of aR, and as a consequence of this $\left(V^m\frac{d^m}{dx^m}\right)R$ is a numerical multiple of $a^m R$. Thus the parallelism is like that between the two sexes, the same with a difference, as is usually the case in comparing the two theories.

This remarkable relation between the operators V and $\frac{d}{dx}$ may be seen *a priori* if we assume that (as we shall hereafter prove) to each pure reciprocant R there is an annihilator V of the form

$$3a^2\partial_b + (\ldots)\partial_c + (\ldots)\partial_d + (\ldots)\partial_e + \ldots$$

not containing ∂_a and linear in the remaining differential operators $\partial_b, \partial_c, \partial_d, \ldots$ For if we call the characteristic μ, by differentiating the absolute pure reciprocant $\frac{R}{a^{\frac{\mu}{3}}}$ with respect to x we obtain, as was shown in the last lecture, the pure reciprocant

$$3a\frac{dR}{dx} - \mu bR.$$

Since this is annihilated by V, we have

$$3a\left(V\frac{d}{dx}\right)R - \mu R Vb - \mu b VR = 0.$$

But, since R is a pure reciprocant, $VR = 0$; and from the assumed form of V it follows that

$$Vb = 3a^2.$$

Hence

$$3a\left(V\frac{d}{dx}\right)R - 3\mu a^2 R = 0,$$

or

$$\left(V\frac{d}{dx}\right)R = \mu a R.$$

Thus the operation of $V\frac{d}{dx}$ is equivalent to multiplication by μa, so that (barring the introduction of a) V restores to $\frac{dR}{dx}$ the form it had antecedent to the operation of $\frac{d}{dx}$, and may be called a qualified reversor to $\frac{d}{dx}$.

For example, suppose that

$$R = 3ac - 5b^2.$$

Since we are using *natural* letters for the derivatives of y with respect to x, we have

$$\frac{d}{dx} = b\partial_a + c\partial_b + d\partial_c + \ldots\ldots,$$

and, as we shall presently see,

$$V = 3a^2\partial_b + 10ab\partial_c + (15ac + 10b^2)\partial_d + \ldots\ldots$$

Now, $\frac{dR}{dx} = (b\partial_a + c\partial_b + d\partial_c)(3ac - 5b^2) = 3bc - 10bc + 3ad = 3ad - 7bc.$

Operating on this with V, we find

$$V\frac{dR}{dx} = V(3ad - 7bc) = -21a^2c - 70ab^2 + 3a(15ac + 10b^2) = 24a^2c - 40ab^2;$$

i. e. $\quad V\frac{d}{dx}(3ac - 5b^2) = 8a(3ac - 5b^2).$

Let us now inquire whether it is possible so to determine an operator V that the relation

$$\left(V\frac{d}{dx} - \frac{d}{dx}V\right)F = (3i + w)\,aF$$

may be satisfied identically when F is any homogeneous isobaric function of the letters $a, b, c, \ldots\ldots$ of degree i and weight w. If so, we must be able to satisfy each of the equations

$$\left(V\frac{d}{dx} - \frac{d}{dx}V\right)a = 3a^2,$$
$$\left(V\frac{d}{dx} - \frac{d}{dx}V\right)b = 4ab,$$
$$\left(V\frac{d}{dx} - \frac{d}{dx}V\right)c = 5ac,$$
$$\left(V\frac{d}{dx} - \frac{d}{dx}V\right)d = 6ad,$$
$$\ldots\ldots\ldots\ldots\ldots$$

which are found by writing $a, b, c, d, \ldots\ldots$ successively in the place of F.

Now $\frac{da}{dx} = b$, $\frac{db}{dx} = c$, $\frac{dc}{dx} = d$, ...so that the above equations may be written

$$Vb = 3a^2 + \frac{d}{dx}(Va),$$

$$Vc = 4ab + \frac{d}{dx}(Vb),$$

$$Vd = 5ac + \frac{d}{dx}(Vc),$$

$$Ve = 6ad + \frac{d}{dx}(Vd),$$

$$\cdots\cdots\cdots$$

These equations are sufficient to completely determine V on the supposition previously made that it is linear in the differential operators and does not contain ∂_a; for, since V is linear, it must be of the form

$$(Va)\partial_a + (Vb)\partial_b + (Vc)\partial_c + \ldots.,$$

and, since it does not contain ∂_a, we must have $Va = 0$, and therefore

$$Vb = 3a^2,$$

$$Vc = 4ab + \frac{d}{dx}(3a^2) = 4ab + 6ab = 10ab,$$

$$Vd = 5ac + \frac{d}{dx}(10ab) = 5ac + 10b^2 + 10ac = 15ac + 10b^2,$$

$$Ve = 6ad + \frac{d}{dx}(15ac + 10b^2) = 6ad + 15bc + 20bc + 15ad = 21ad + 35bc,$$

$$\cdots\cdots\cdots\cdots\cdots\cdots$$

Hence $V = 3a^2\partial_b + 10ab\partial_c + (15ac + 10b^2)\partial_d + (21ad + 35bc)\partial_e + \ldots..$

When the modified letters $a_0, a_1, a_2, \ldots.$ are used, we shall have, in consequence of the change of notation, $\left(V\frac{d}{dx}\right)R = 2\mu a_0 R$ (instead of $\mu a R$). If, as before, we seek to satisfy the equation

$$\left(V\frac{d}{dx} - \frac{d}{dx}V\right)F = 2(3i + w)a_0 F, \tag{1}$$

we shall find, on writing a_n in the place of F,

$$\left(V\frac{d}{dx} - \frac{d}{dx}V\right)a_n = 2(3 + n)a_0 a_n. \tag{2}$$

This condition will be sufficient, as well as necessary, for the satisfaction of (1) when V is linear; for then

$$V\frac{d}{dx} - \frac{d}{dx}V$$

will also be linear, its general term being

$$\left(V\frac{da_n}{dx} - \frac{d}{dx}\,Va_n\right)\partial_{a_n};$$

which is equal to $2(3+n)a_0a_n\partial_{a_n}$ by equation (2). Hence

$$\left(V\frac{d}{dx} - \frac{d}{dx}V\right)F = \text{a sum of terms of the form } 2(3+n)a_0a_n\partial_{a_n}F$$
$$= 2a_0(3a_0\partial_{a_0} + 3a_1\partial_{a_1} + 3a_2\partial_{a_2} + \ldots)F$$
$$+ 2a_0(a_1\partial_{a_1} + 2a_2\partial_{a_2} + \ldots)F;$$

i. e. equation (1) is satisfied whenever (2) is. Writing in (2)

$$\frac{da_n}{dx} = (n+3)a_{n+1},$$

we obtain $$(n+3)\,Va_{n+1} = 2(n+3)a_0a_n + \frac{d}{dx}(Va_n), \tag{3}$$

from which the values of Va_n may be successively determined.

When $Va_0 = 0$, the value of Va_n, which satisfies (3), is

$$Va_n = \frac{n+3}{2}(a_0a_{n-1} + a_1a_{n-2} + \ldots + a_{n-2}a_1 + a_{n-1}a_0);$$

thus $$Va_1 = \frac{4}{2}\cdot a_0^2,\ Va_2 = 5a_0a_1,\ Va_3 = 6a_0a_2 + 3a_1^2,\ \ldots$$

and the value of V is therefore

$$\frac{4}{2}\cdot a_0^2\partial_{a_1} + 5a_0a_1\partial_{a_2} + 6\left(a_0a_2 + \frac{1}{2}a_1^2\right)\partial_{a_3} + 7(a_0a_3 + a_1a_2)\partial_{a_4} + \ldots\ldots$$

Now that we are on the subject of parallelism between the old and new worlds of Algebraical Form, I feel tempted to point out yet another very interesting bond of connection between them. There is a theorem concerning Invariants which I am not aware that any one but myself has noticed, or at all events I do not remember ever seeing it in print,* which is this: If we take any "invariant" and regard its most advanced letter as a variable, or say rather as the ratio of two variables $u:v$, by multiplying by a proper power of v we obtain a new Quantic in u, v; or, if we take any number of such invariants with the same most advanced letter (or, as we may call it in a double sense, the same radical letter) in common, we shall have a system of binary Quantics in u, v. My theorem is, or was, that an Invariant of any one or more of such Quantics is an Invariant of the original Quantic. I recently found a similar proposition to be true for Reciprocants, viz. forming as before a system of *pure*

* The theorem is, however, given in Vol. XI, p. 98 of the Bulletin de la Société Mathematique de France, in a paper by M. Perrin, which has only recently come under the lecturer's notice.

Reciprocants into Quantics in u, v, any "Invariant" of such system is itself a Reciprocant.

The two theorems may be stated symbolically thus:

$$\left.\begin{array}{l} II' = I'' \\ IR = R' \end{array}\right\}.$$

On mentioning this to Mr. L. J. Rogers, he sent me next day a proof which, although only stated as applicable to Reciprocants, is equally so, *mutatis mutandis*, to Invariants. Although given for a single invariant, it applies equally to a system.

I give Mr. Rogers' proof that any invariant of a *pure* reciprocant (the proof will not hold for impure ones) is a pure reciprocant; or rather I use his method to prove the analogous theorem that any invariant of an invariant is itself an invariant. It will be seen hereafter that this same proof applies to *pure* reciprocants with only trifling changes; but the proof as given by Mr. Rogers requires some further considerations to be gone into for which we are not yet ripe.

Consider, for the sake of simplicity, the binary Quintic

$$(a, b, c, d, e, f \between x, y)^5,$$

and let I be any invariant of it (satisfied or unsatisfied); then

$$I = a_0 f^n + a_1 f^{n-1} + a_2 f^{n-2} + \ldots . + a_n,$$

where a_0, a_1, a_2, ... a_n do not contain f, but are functions of a, b, c, d, e alone.

Let the Protomorphs for our Quintic be denoted by A, B, C, D, E, F; then

$$F = a^2 f - 5abe + 2acd + 8b^2 d + 6bc^2.$$

Eliminating f from I by means of this equation, we have

$$Ia^{2n} = A_0 F^n + A_1 F^{n-1} + A_2 F^{n-2} + \ldots . + A_n,$$

where A_0, A_1, A_2, ... A_n are all of them invariants (not necessarily integral forms, but this is immaterial to the proof, for Ω annihilates fractional and integral invariants alike). For

$$\Omega (Ia^{2n}) = \Omega (A_0 F^n + A_1 F^{n-1} + \ldots . + A_n),$$

and, in consequence of Ia^{2n} and F being invariants, so that, as regards Ω, F may be treated as if it were a constant, this becomes

$$0 = F^n \Omega A_0 + F^{n-1} \Omega A_1 + F^{n-2} \Omega A_2 + \ldots . + \Omega A_n,$$

in which the coefficients of the several powers of F must be separately equated to zero. In other words, $A_0, A_1, A_2, \ldots . A_n$ are all of them invariants. Now, any invariant of

$$A_0F^n + A_1F^{n-1} + A_2F^{n-2} + \ldots . + A_n$$

is a function of $A_0, A_1, A_2, \ldots . A_n$, and therefore an invariant.

(N. B.—We cannot assume that any function of general reciprocants is itself a reciprocant.)

Again, since

$$A_0F^n + \ldots . + A_n, \text{ and } a_0f^n + \ldots . + a_n$$

are connected by the substitution

$$F = a^2f - 5abe + \ldots . ,$$

which is *linear* in respect to the letters F and f, any invariant of

$$A_0F^n + \ldots . + A_n$$

is (to a factor *près*, that factor being a power of a which is itself an invariant) equal to the corresponding invariant of

$$a_0f^n + \ldots . + a_n .$$

But every invariant of the former has been shown to be an invariant of the original quantic, and therefore every invariant of the latter is so also.

I add some examples in illustration of this theorem:

Ex. 1. Take the invariant of the Quintic

$$a^2f^2 - 10abef + 4acdf + 16b^2df - 12bc^2f + 16ace^2 + 9b^2e^2 - 12ad^2e - 76bcde + 48c^3e + 48bd^3 - 32c^2d^2.$$

The discriminant of this, considered as a quadratic in f, is

$$a^2(16ace^2 + 9b^2e^2 - 12ad^2e - 76bcde + 48c^3e + 48bd^3 - 32c^2d^2) - (5abe - 2acd = 8b^2d + 6bc^2)^2$$
$$= 16a^3ce^2 - 16a^2b^2e^2 - 12a^3d^2e - 56a^2bcde + 48a^2c^3e + 80ab^3de - 60ab^2c^2e + 48a^2bd^3 - 36a^2c^2d^2 - 32ab^2cd^2 - 64b^4d^2 + 24abc^3d + 96b^3c^2d - 36b^2c^4.$$

It will be found on trial that this is divisible by the invariant

$$4(ae - 4bd + 3c^2),$$

the quotient being

$$4a^2ce - 4ab^2e - 3a^2d^2 + 2abcd + 4b^3d - 3b^2c^2$$
$$= 3a(ace - b^2e - ad^2 + 2bcd - c^3) + (ac - b^2)(ae - 4bd + 3c^2).$$

Thus the discriminant of the quadratic in f, *i. e.* of the invariant

$$a^2f^2 - 2f(5abe - 2acd + 8b^2d - 6bc^2) + \ldots . ,$$

is shown to be an invariant. It will further illustrate the proof of the theorem if we remark that precisely the same invariant is obtained by eliminating f between the above form and the protomorph

$$a^2f - 5abe + 2acd + 8b^2d - 6bc^2.$$

Ex. 2. If we take the pure reciprocant

$$45a^3d^2 - 450a^2bcd + 400ab^3d + 192a^2c^3 + 165ab^2c^2 - 400b^4c,$$

which, from its similarity to the Discriminant of the Cubic, I have called the Quasi-Discriminant, and form *its* discriminant, when regarded as a quadratic in d, we find

$$45a^3(192a^2c^3 + 165ab^2c^2 - 400b^4c) - (225a^2bc - 200ab^3)^2.$$

If, in this expression, we write $P = 3ac - 5b^2$, so that $3ac = P + 5b^2$, it becomes

$$5.64a^2(P + 5b^2)^3 + 5.165a^2b^2(P + 5b^2)^2 - 15.400a^2b^4(P + 5b^2) - 625a^2b^2(3P + 7b^2)^2.$$

On performing the calculation it will be found that all the terms involving b will disappear from this result, and there will remain the single term $320a^2P^3$, *i. e.* $320a^2(3ac - 5b^2)^3$, which is a reciprocant.

LECTURE VIII.

In my last lecture the complete expression, both in terms of the modified and unmodified letters, was obtained for V, the annihilator for pure reciprocants assuming its existence and its form. These assumptions I shall now make good by proving, from first principles, the fundamental theorem that the satisfaction of the equation

$$VR = 0$$

is a necessary and sufficient condition in order that R may be a pure reciprocant.

It will be advantageous to use the modified system of letters, in which

$$t,\ a_0,\ a_1,\ a_2,\ \ldots \text{ stand for } \frac{dy}{dx},\ \frac{1}{1.2}\cdot\frac{d^2y}{dx^2},\ \frac{1}{1.2.3}\cdot\frac{d^3y}{dx^3},\ \frac{1}{1.2.3.4}\cdot\frac{d^4y}{dx^4},\ \ldots$$

and

$$\alpha_0,\ \alpha_1,\ \alpha_2,\ \ldots \text{ for } \frac{1}{1.2}\cdot\frac{d^2x}{dy^2},\ \frac{1}{1.2.3}\cdot\frac{d^3x}{dy^3},\ \frac{1}{1.2.3.4}\cdot\frac{d^4x}{dy^4},\ \ldots$$

respectively. Let the variation due to the change of x into $x + \varepsilon y$, where ε is an infinitesimal number, be denoted by Δ. Obviously this change leaves the value of each of the quantities $\alpha_0, \alpha_1, \alpha_2, \ldots$ unaltered, and therefore

$$\Delta R(\alpha_0, \alpha_1, \alpha_2, \ldots) = 0,$$

whatever the nature of R may be. But when R is a pure reciprocant,

$$R(a_0, a_1, a_2, \ldots.) = \pm t^{\mu} R(\alpha_0, \alpha_1, \alpha_2, \ldots.),$$

whence it immediately follows that

$$\Delta t^{-\mu} R(a_0, a_1, a_2, \ldots.) = 0.^*$$

Before proceeding to determine the values of

$$\Delta t,\ \Delta a_0,\ \Delta a_1,\ \Delta a_2, \ldots.$$

it will be useful to remark that since

$$\frac{dy}{dx} = t,\ \frac{d^2y}{dx^2} = 1.2.a_0,\ \frac{d^3y}{dx^3} = 1.2.3.a_1, \ldots.$$

we have $$\frac{dt}{dx} = 2a_0,\ \frac{da_0}{dx} = 3a_1, \ldots.$$

and generally $$\frac{da_n}{dx} = (n+3)\,a_{n+1}.$$

Now let $[t]$ denote the augmented value of t, and in general let $[\ ]$ be used to signify that the augmented value of the quantity enclosed in it is to be taken. Then

$$[t] = \frac{dy}{d[x]} = \frac{dy}{d(x+\varepsilon y)} = \frac{dy}{dx\left(1+\varepsilon\dfrac{dy}{dx}\right)} = \frac{t}{1+\varepsilon t}$$

$$= t - \varepsilon t^2;$$

so also $$2[a_0] = [2a_0] = \frac{d[t]}{d[x]} = \frac{d[t]}{d(x+\varepsilon y)} = \frac{d[t]}{dx(1+\varepsilon t)} = (1-\varepsilon t)\frac{d[t]}{dx}$$

$$= (1-\varepsilon t)\frac{d}{dx}(t - \varepsilon t^2) = (1-\varepsilon t)(2a_0 - 4\varepsilon t a_0)$$

$$= 2a_0 - 6\varepsilon t a_0;$$

i. e. $$[a_0] = a_0 - 3\varepsilon t a_0.$$

Reasoning precisely similar to that which gave

$$2[a_0] = (1-\varepsilon t)\frac{d}{dx}[t]$$

leads to the formula

$$(n+3)[a_{n+1}] = (1-\varepsilon t)\frac{d}{dx}[a_n],$$

from which the augmented values of $a_1, a_2, a_3, \ldots.$ may be found by giving to n the values $0, 1, 2, \ldots.$ in succession. Thus, writing $n = 0$, we have

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* It has been suggested by Mr. J. Chevallier that the proof might be simplified by considering the variation $\Delta a_0^{-\frac{\mu}{3}} R(a_0, a_1, a_2, \ldots.)$ instead of $\Delta t^{-\mu} R(a_0, a_1, a_2, \ldots.)$.

$$3\,[a_1] = (1-\varepsilon t)\frac{d}{dx}\,[a_0] = (1-\varepsilon t)\frac{d}{dx}\,(a_0 - 3\varepsilon t a_0)$$
$$= (1-\varepsilon t)(3a_1 - 9\varepsilon t a_1 - 6\varepsilon a_0^2) = 3a_1 - \varepsilon\,(12 t a_1 + 6a_0^2),$$

or $\qquad [a_1] = a_1 - \varepsilon\,(4ta_1 + 2a_0^2).$

Similarly, when $n = 1$,

$$4\,[a_2] = (1-\varepsilon t)\frac{d}{dx}\,[a_1] = (1-\varepsilon t)\frac{d}{dx}\,(a_1 - 4\varepsilon t a_1 - 2\varepsilon a_0^2)$$
$$= (1-\varepsilon t)(4a_2 - 16\varepsilon t a_2 - 20\varepsilon a_0 a_1)$$
$$= 4a_2 - 20\varepsilon t a_2 - 20\varepsilon a_0 a_1,$$

and $\qquad [a_2] = a_2 - 5\varepsilon\,(ta_2 + a_0 a_1).$

Again, $\qquad 5\,[a_3] = (1-\varepsilon t)\frac{d}{dx}\,[a_2] = (1-\varepsilon t)\frac{d}{dx}\,(a_2 - 5\varepsilon t a_2 - 5\varepsilon a_0 a_1)$

$$= (1-\varepsilon t)(5a_3 - 25\varepsilon t a_3 - 30\varepsilon a_0 a_2 - 15\varepsilon a_1^2)$$
$$= 5a_3 - 30\varepsilon t a_3 - 30\varepsilon a_0 a_2 - 15\varepsilon a_1^2,$$

so that $\qquad [a_3] = a_3 - \varepsilon\,(6ta_3 + 6a_0 a_2 + 3a_1^2).$

In like manner we shall find

$$[a_4] = a_4 - 7\varepsilon\,(ta_4 + a_0 a_3 + a_1 a_2).$$

These results may be written in a more symmetrical form; thus:

$$2\,[t] \;\; = 2t - 2\varepsilon t^3,$$
$$2\,[a_0] = 2a_0 - 3\varepsilon\,(ta_0 + a_0 t),$$
$$2\,[a_1] = 2a_1 - 4\varepsilon\,(ta_1 + a_0^2 + a_1 t),$$
$$2\,[a_2] = 2a_2 - 5\varepsilon\,(ta_2 + a_0 a_1 + a_1 a_0 + a_2 t),$$
$$2\,[a_3] = 2a_3 - 6\varepsilon\,(ta_3 + a_0 a_2 + a_1^2 + a_2 a_0 + a_3 t),$$
$$2\,[a_4] = 2a_4 - 7\varepsilon\,(ta_4 + a_0 a_3 + a_1 a_2 + a_2 a_1 + a_3 a_0 + a_4 t).$$

The general law

$$2\,[a_n] = 2a_n - (n+3)\,\varepsilon\,(ta_n + a_0 a_{n-1} + \ldots + a_{n-1} a_0 + a_n t),$$

or, as it may also be written,

$$\Delta a_n = -\frac{n+3}{2}\,\varepsilon\,(ta_n + a_0 a_{n-1} + \ldots + a_{n-1} a_0 + a_n t),$$

admits of an easy inductive proof.

Assuming the truth of the theorem for $[a_n]$, and writing for brevity in what follows,

$$S_n = ta_n + a_0 a_{n-1} + a_1 a_{n-2} + \ldots + a_{n-2} a_1 + a_{n-1} a_0 + a_n t,$$

we have $\quad [a_n] = a_n - \dfrac{n+3}{2}\,\varepsilon S_n.$

Now,

$$\begin{aligned}\frac{dS_n}{dx} &= (n+3)\,ta_{n+1} + 2a_0a_n \\ &\quad + (n+2)\,a_0a_n + 3a_1a_{n-1} \\ &\quad + (n+1)\,a_1a_{n-1} + 4a_2a_{n-2} \\ &\quad + \dots\dots\dots + \dots\dots \\ &\quad + 4a_{n-2}a_2 + (n+1)\,a_{n-1}a_1 \\ &\quad + 3a_{n-1}a_1 + (n+2)\,a_na_0 \\ &\quad + 2a_na_0 + (n+3)\,a_{n+1}t \\ &= (n+4)(ta_{n+1} + a_0a_n + a_1a_{n-1} + \dots + a_{n-1}a_1 + a_na_0 + a_{n+1}t) - 2ta_{n+1} \\ &= (n+4)\,S_{n+1} - 2ta_{n+1}.\end{aligned}$$

Hence $\quad \frac{d}{dx}[a_n] = (n+3)\,a_{n+1} - \frac{n+3}{2}\,\varepsilon\{(n+4)\,S_{n+1} - 2ta_{n+1}\}.$

But, as we have already seen,

$$(n+3)[a_{n+1}] = (1-\varepsilon t)\frac{d}{dx}[a_n];$$

consequently,

$$[a_{n+1}] = (1-\varepsilon t)\,a_{n+1} - \frac{n+4}{2}\,\varepsilon S_{n+1} + \varepsilon ta_{n+1} = a_{n+1} - \frac{n+4}{2}\,\varepsilon S_{n+1};$$

i. e. the theorem holds for $[a_{n+1}]$ when it holds for $[a_n]$. But we know that it is true for the cases $n = 0, 1, 2, 3, 4$, and therefore it is true universally.

Resuming the proof of the main theorem, it has been shown that

$$\Delta t^{-\mu} R(a_0, a_1, a_2, \dots) = 0;$$

i. e. $\quad -\mu t^{-1}\Delta t + R^{-1}\Delta R = 0,$

or $\quad -\mu R t^{-1}\Delta t + \frac{dR}{da_0}\Delta a_0 + \frac{dR}{da_1}\Delta a_1 + \frac{dR}{da_2}\Delta a_2 + \dots = 0.$

But

$$\begin{aligned}\Delta t &= -\varepsilon t^2, \\ \Delta a_0 &= -3\varepsilon ta_0, \\ \Delta a_1 &= -\varepsilon(4ta_1 + 2a_0^2), \\ \Delta a_2 &= -\varepsilon(5ta_2 + 5a_0a_1), \\ \Delta a_3 &= -\varepsilon(6ta_3 + 6a_0a_2 + 3a_1^2), \\ \Delta a_4 &= -\varepsilon(7ta_4 + 7a_0a_3 + 7a_1a_2), \\ &\dots\dots\dots\dots\dots\dots\end{aligned}$$

and consequently

$$t(\mu - 3a_0\partial_{a_0} - 4a_1\partial_{a_1} - 5a_2\partial_{a_2} - 6a_3\partial_{a_3} - 7a_4\partial_{a_4} - \dots)\,R$$
$$-\left\{4\left(\frac{a_0^2}{2}\right)\partial_{a_1} + 5(a_0a_1)\partial_{a_2} + 6\left(a_0a_2 + \frac{a_1^2}{2}\right)\partial_{a_3} + 7(a_0a_3 + a_1a_2)\partial_{a_4} + \dots\right\}R = 0.$$

This is equivalent to the two conditions

$$(3a_0\partial_{a_0} + 4a_1\partial_{a_1} + 5a_2\partial_{a_2} + \ldots) R = \mu R$$

and

$$VR = 0,$$

where

$$V = 4\left(\frac{a_0^2}{2}\right)\partial_{a_1} + 5(a_0a_1)\partial_{a_2} + 6\left(a_0a_2 + \frac{a_1^2}{2}\right)\partial_{a_3} + 7(a_0a_3 + a_1a_2)\partial_{a_4} + \ldots$$

For greater simplicity I confine what I have to say to the only essential case, to which every other may be reduced, of a *homogeneous* pure reciprocant. The equation

$$(3a_0\partial_{a_0} + 4a_1\partial_{a_1} + 5a_2\partial_{a_2} + \ldots) R = \mu R$$

shows that for every term $w + 3i$ is constant; *i. e.* w is constant and therefore the function R is isobaric. This is also immediately deducible from the form of the relations between $a_0, a_1, a_2, \ldots$; $\alpha_0, \alpha_1, \alpha_2, \ldots$, and, what is important to notice, for future purposes,

$$F(a_0, a_1, a_2, \ldots) - t^\mu F(\alpha_0, \alpha_1, \alpha_2, \ldots),$$

when F is a homogeneous isobaric function, and $\mu = w + 3i$ is itself a homogeneous function of $(a_0, a_1, a_2, \ldots)$, whose degree is the same as that of F.

The only condition affecting R, a function of $a_0, a_1, a_2, \ldots$, supposed homogeneous and isobaric, is

$$VR = 0.$$

I shall now prove the converse, that if $R = F(a_0, a_1, a_2, \ldots)$ (being homogeneous and isobaric) has V for its annihilator, then R is a pure reciprocant. Let D be the value of $F(a_0, a_1, a_2, \ldots) - t^\mu F(\alpha_0, \alpha_1, \alpha_2, \ldots)$ expressed as a function of $a_0, a_1, a_2, \ldots$ alone. Then D will be a function of the same type as $F(a_0, a_1, a_2, \ldots)$.

Suppose that

$$\Delta D = 0;$$

i. e. that the variation of D due to the change of x into $x + \varepsilon y$ vanishes in virtue of the equation $VR = 0$.

Let D become D' when y receives an arbitrary variation $y + \eta u$, where η is an infinitesimal constant and u an arbitrary function of x; then the variation of D' will vanish when x is changed into $x + \varepsilon y + \varepsilon\eta u$, and consequently when x is changed into $x + \varepsilon y$ the variation of D' will also vanish. Hence

$$\Delta D' = 0,$$

and if we take the difference of the variations of D and D', we shall find

$$\Delta\left(u'' \frac{d}{da_0} D + u''' \frac{d}{da_1} D + u^{\text{IV}} \frac{d}{da_2} D + \ldots\right) = 0.$$

Now, the arbitrary nature of the function u shows that we must have

$$\Delta \frac{d}{da_0} D = 0 \,, \; \Delta \frac{d}{da_1} D = 0 \,, \; \Delta \frac{d}{da_2} D = 0 \,, \ldots .$$

and if we reason on $\frac{d}{da_0} D$, $\frac{d}{da_1} D$, ... in the same way as we have on D, we see that the variation Δ of each of the second differential derivatives of D will also vanish; and, pursuing the same argument further, it will be evident that the Δ of any derivative of D, of any order whatever, with respect to $a_0, a_1, a_2, \ldots .$ will vanish. Hence

$$D = 0 \,;$$

for if this is not so we may, supposing D to be a function of degree i in the letters $a_0, a_1, a_2, \ldots$, take the Δ of each of the differential derivatives of D of the order $i-1$; each of these variations would vanish by what precedes; *i. e.* the variation due to the change of x into $x + \varepsilon y$ of each of the letters $a_0, a_1, a_2, \ldots .$ contained in D would be identically zero, which is absurd. We see, therefore, that when $\Delta D = 0$ (*i. e.* when R is annihilated by V), $D = 0$, or

$$F(a_0, a_1, a_2, \ldots .) = t^{\mu} F(a_0, a_1, a_2, \ldots .),$$

which proves the converse proposition.

It will not fail to be noticed how much language, and as a consequence algebraical thought (for words are the tools of thought), is facilitated by the use of the concept of annihilation in lieu of that of equality as expressed by a partial differential equation.

It is somewhat to the point that in the recent two grand determinations of the order of precedence among the so-called fixed stars relative to our planet, as approximately represented by the intensities of the light from them which reaches the eye, the one is directed by the principle of annihilation, the other by that of equality. Prof. Pritchard's method essentially consists in determining what relative thicknesses of an interposed glass screen, effected by means of a sliding wedge of glass, will serve to extinguish the light of a star; that employed by Prof. Pickering depends on finding what degree of rotation of an interposed prism of Iceland spar (a Nicol Prism) will serve to bring to an equality the ordinary image of one star with the extra-ordinary one of another. As these intensities depend on the squared sines and cosines of this angle of rotation measured from the position of non-visibility of one of them, it follows that the tangent squared of the twist measures the relative intensities by this method.

Hereafter it will be shown that if F is a homogeneous isobaric function of

$$y, y', y'', y''', \ldots .$$

whose weights are reckoned as

$$-2, -1, 0, 1, \dots .$$

then, when x becomes $x + hy$, where h is any constant quantity, F becomes

$$(1 + ht)^{-\mu} e^{-\frac{hV_1}{1+ht}} F,$$

where $$t = y', \; V_1 = -t^2\partial_t + V, \text{ and } \mu = 3i + w,$$
i being the degree and w the weight of F.

From this, by an obvious course of reasoning, could be deduced as a particular case the condition of $F(a_0, a_1, a_2, \dots .)$ remaining a factor of its altered self when *any* linear substitutions are impressed on x and y; viz. the necessary and sufficient condition is that F has V for its annihilator.

LECTURE IX.

The prerogative of a Pure Reciprocant is that it continues a factor of its altered self when the variables x and y are subjected to any linear substitution. Its form, like that of any other reciprocant, is of course persistent when the variables are interchanged; *i. e.* when in the general substitution, in which y is changed into $$fy + gx + h$$
and x into $$f'y + g'x + h',$$
we give the particular values $h = 0$, $h' = 0$, $f = 0$, $g' = 0$, $f' = 1$, $g = 1$, to the constants. Stated geometrically, the theorem is that the evanescence of any pure reciprocant R indicates a property independent of transformation of axes in a plane. We suppose R to be homogeneous and isobaric in $a, b, c, \dots .$ (If it were not, the theorem could not hold, for either the change of y into κy or that of x into λx would destroy the form.)

The persistence, under any linear substitution, of the form of pure reciprocants may be easily established as follows:

By a *semi-substitution* understand one where one of the variables remains unaltered. There are two such semi-substitutions, viz. where x remains unaltered, and where y does.

1°. Let x remain unaltered and y become $fy + gx + h$; then $a, b, c, \ldots$ become $fa, fb, fc, \ldots$ respectively; and therefore

$$R(a, b, c, \ldots) \text{ becomes } f^i R(a, b, c, \ldots),$$

where i is the degree of R.

2°. Let y remain unchanged and x become $f'y + g'x + h'$. Then, instead of R, I look to its equal

$$qt^{\mu} R(\alpha, \beta, \gamma, \ldots) \; (q = \pm 1);$$

i. e. to $\qquad q\tau^{-\mu} R(\alpha, \beta, \gamma, \ldots),$

which becomes $\qquad q(f' + g'\tau)^{-\mu} g'^i R(\alpha, \beta, \gamma, \ldots).$

Since R is a reciprocant, this is equal to

$$\frac{\tau^{\mu}}{(f' + g'\tau)^{\mu}} g'^i R(a, b, c, \ldots),$$

or, replacing τ by its equivalent $\frac{1}{t}$,

$$(f't + g')^{-\mu} g'^i R(a, b, c, \ldots).$$

Thus we see that the proposition is true for a semi-substitution of either kind. Consider now the complete substitution made by changing y into

$$fy + gx + h$$

and x into $\qquad Fy + Gx + H.$

If $f = 0$ and $G = 0$, then $\frac{d^2y}{dx^2}, \frac{d^3y}{dx^3}, \ldots$ become $\frac{g}{F^3} \cdot \frac{d^2x}{dy^2}, \frac{g}{F^4} \cdot \frac{d^3x}{dy^3}, \ldots$; so that $R(a, b, c, \ldots)$ becomes $\frac{g^i}{F^{3i+w}} \cdot R(\alpha, \beta, \gamma, \ldots)$; and since this is equal to

$$\frac{g^i}{F^{3i+w}} \cdot qt^{-\mu} R(a, b, c, \ldots),$$

the proposition is true.

But if either of the two letters f, G (say f) is not zero, we may combine two semi-substitutions so as to obtain the complete substitution, in which y changes into $\qquad fy + gx + h$

and x changes into $\qquad Fy + Gx + H.$

1°. Substitute $\quad y_1 (= fy + gx + h)$ for y, and $x_1 (= x)$ for x.

2°. Then substitute $y_2 (= y_1)$ for y_1, and $x_2 (= f'y_1 + g'x_1 + h')$ for x_1.

By the first of these semi-substitutions

$$R(a, b, c, \ldots)$$

takes up an extraneous factor f^ι. By the second it acquires the factor

$$\left(f'\frac{dy_1}{dx_1}+g'\right)^{-\mu}g'^\iota, \text{ where } \frac{dy_1}{dx_1}=f\frac{dy}{dx}+g=ft+g.$$

Hence we see that the extraneous factor is a negative power of a linear function of t, which we shall presently particularize, though it is not essential to the present demonstration to do so.

It only remains to show how the combination of these two semi-substitutions can be made to give the complete one in question. We have

$$y_2=y_1=fy+gx+h$$

and

$$\begin{aligned} x_2 &= f'y_1+g'x_1+h'=f'(fy+gx+h)+g'x+h' \\ &= ff'y+(f'g+g')x+(f'h+h'). \end{aligned}$$

In order that this may be equal to $Fy+Gx+H$, we must be able to satisfy the equations

$$f'=\frac{F}{f},\ g'=G-\frac{gF}{f},\ h'=H-\frac{hF}{f},$$

which is always possible, since by hypothesis f is not zero. Similarly it may be shown that when f vanishes, but G does not, by substituting

1°. $x_1\,(=Fy+Gx+H)$ for x, and $y_1\,(=y)$ for y,
2°. $x_2\,(=x_1)$ for x_1, and $y_2\,(=f''y_1+g''x_1+h'')$ for y_1,

we may so determine f'', g'', h'' as to get the complete substitution as before.

In every case, therefore, any linear substitution impressed upon the variables x and y will leave $R\,(a, b, c, \ldots.)$ unaltered, barring the acquisition of an extraneous factor which is a negative power of a linear function of t.

Now, the first semi-substitution introduces, as we have seen, the constant factor

$$f^\iota;$$

the second introduces the factor

$$\left(f'\frac{dy_1}{dx_1}+g'\right)^{-\mu}g'^\iota,$$

where

$$\frac{dy_1}{dx_1}=ft+g.$$

The complete extraneous factor is the product of these two, and is therefore

$$f^\iota g'^\iota(ff't+f'g+g')^{-\mu}.$$

To express f' and g' in terms of the constants of the complete substitution we have

$$f'=\frac{F}{f},\ g'=G-\frac{gF}{f}.$$

Writing these values for f' and g' in the expression just found, we obtain

$$(fG - gF)^i (Ft + G)^{-\mu},$$

which is the extraneous factor acquired by R when the complete substitution is made. For example, if x becomes

$$Fy + Gx + H$$

and y becomes

$$fy + gx + h,$$

the altered value of a $\left(i.\ e.\ \text{of } \frac{d^2y}{dx^2}\right)$ is

$$(fG - gF)(Ft + G)^{-3}a.$$

Corresponding to the simple interchange of the variables, we have

$$F = 1,\ G = 0,\ H = 0;\ f = 0,\ g = 1,\ h = 0,$$

so that

$$fG - gF = -1,$$

and the altered value of a is $\frac{d^2x}{dy^2}$,

or

$$a = -\frac{a}{t^3},$$

which is right. In this case the general value of the acquired extraneous factor

$$(fG - gF)^i (Ft + G)^{-\mu} \text{ becomes } (-)^i t^{-\mu},$$

thus showing, what we have already proved from other considerations, that the character of a pure reciprocant is odd or even according as its degree is odd or even.

We saw in the last lecture that *every* pure reciprocant necessarily satisfied the two conditions

$$(3a_0\partial_{a_0} + 4a_1\partial_{a_1} + 5a_2\partial_{a_2} + \ldots .)\, R = \mu R$$

(where μ is the characteristic), and

$$VR = 0.$$

We also saw that $VR = 0$ was a sufficient as well as necessary condition that *any homogeneous function* R of $a_0, a_1, a_2, \ldots .$ should be a pure reciprocant. It will now be shown that every pure reciprocant is either homogeneous and isobaric, or else resoluble into a sum of homogeneous and isobaric reciprocants. Non-homogeneous mixed ones, it may be observed, are not so resoluble, so that the theorem only holds for pure reciprocants.

1°. Let us suppose that R (a pure reciprocant) is homogeneous in $a_0, a_1, a_2, \ldots .$; then it must be isobaric also. For, if i is the degree of R, Euler's theorem shows that

$$(3a_0\partial_{a_0} + 3a_1\partial_{a_1} + 3a_2\partial_{a_2} + 3a_3\partial_{a_3} + \ldots .)\, R = 3iR;$$

and since R is a pure reciprocant, the condition

$$(3a_0\partial_{a_0} + 4a_1\partial_{a_1} + 5a_2\partial_{a_2} + 6a_3\partial_{a_3} + \ldots\ldots) R = \mu R$$

is necessarily satisfied. Hence

$$(a_1\partial_{a_1} + 2a_2\partial_{a_2} + 3a_3\partial_{a_3} + \ldots\ldots) R = (\mu - 3i)R = \text{a constant multiple of } R,$$

which is the distinctive property of isobaric functions.

And, *vice versâ*, if R is homogeneous and isobaric of weight w and degree i, then

$$(3a_0\partial_{a_0} + 4a_1\partial_{a_1} + 5a_2\partial_{a_2} + \ldots\ldots) R = (w + 3i) R = \mu R.$$

Thus homogeneous pure reciprocants are also isobaric and their characteristic is $3i + w$. (This property is also true for mixed reciprocants, as we have previously shown.)

2°. Suppose that R is not homogeneous, but made up of the homogeneous parts

$$R_{,},\ R_{,,},\ R_{,,,},\ \ldots\ldots$$

Then, since

$$V(R_{,} + R_{,,} + R_{,,,} + \ldots\ldots) = 0$$

is satisfied identically, it is obvious that

$$VR_{,} + VR_{,,} + VR_{,,,} + \ldots\ldots = 0$$

must also be satisfied identically.

But since all the terms are of different degrees, the only way in which this can happen is by making $VR_{,}$, $VR_{,,}$, $VR_{,,,}$, ... separately vanish. Now, $R_{,}$, $R_{,,}$, $R_{,,,}$, ... are by hypothesis *homogeneous* functions of a_0, a_1, a_2, ..., and it has just been shown that each of them is annihilated by V, which has been shown to be a sufficient condition that any homogeneous function of a_0, a_1, a_2, ... may be a pure reciprocant. Thus each part $R_{,}$, $R_{,,}$, $R_{,,,}$, ... of R is a pure reciprocant.

Also, the condition

$$(3a_0\partial_{a_0} + 4a_1\partial_{a_1} + 5a_2\partial_{a_2} + \ldots\ldots)R = \mu R$$

shows that if i_1, w_1; i_2, w_2; i_3, w_3; .... are the deg. weights of $R_{,}$, $R_{,,}$, $R_{,,,}$, ..., we must have

$$3i_1 + w_1 = \mu,\ 3i_2 + w_2 = \mu,\ 3i_3 + w_3 = \mu,\ \ldots\ldots$$

Thus non-homogeneous pure reciprocants are severable into parts each of which is a homogeneous and isobaric pure reciprocant, the characteristic of each part being equal to the same quantity μ, which is the characteristic of the whole.

I will now explain what information concerning the number of pure reciprocants of a given type is afforded by the equation $VR = 0$. Let

$$Aa_0^{\lambda_0}a_1^{\lambda_1}a_2^{\lambda_2}\ \ldots\ldots\ a_j^{\lambda_j}$$

be a term of a homogeneous isobaric function (with its full number of terms) of $a_0, a_1, a_2, \ldots . a_j$, whose degree is i, extent j, and weight w, and which we will call R.

Then in the entire function there are as many terms as there are solutions in integers of the equations

$$\lambda_0 + \lambda_1 + \lambda_2 + \lambda_3 + \ldots . + \lambda_j = i,$$
$$\lambda_1 + 2\lambda_2 + 3\lambda_3 + \ldots . + j\lambda_j = w.$$

In other words, the number of terms in R is equal to the number of ways in which w can be made up of i or fewer parts, none greater than j. This number will be denoted by

$$(w; i, j).$$

Since the function R is the sum of every possible term of the form

$$Aa_0^{\lambda_0}a_1^{\lambda_1} \ldots . a_j^{\lambda_j},$$

each multiplied by an arbitrary constant, the number of these arbitrary constants is also

$$(w; i, j).$$

Now, suppose R to be a reciprocant; this imposes the condition

$$VR = 0.$$

Consider the effect produced by the operation of any term of

$$V = 4\left(\frac{a_0^3}{2}\right)\partial_{a_1} + 5a_0a_1\partial_{a_2} + 6\left(a_0a_2 + \frac{a_1^2}{2}\right)\partial_{a_3} + \ldots .,$$

say $\left(a_0a_2 + \frac{a_1^2}{2}\right)\partial_{a_3}$ (rejecting the numerical coefficient 6).

Operating on R with ∂_{a_3} decreases its weight by 3 and its degree by 1 unit. The subsequent multiplication by $a_0a_2 + \frac{a_1^2}{2}$, on the other hand, increases the weight by 2 and the degree by 2 units. Hence the total effect of $\left(a_0a_2 + \frac{a_1^2}{2}\right)\partial_{a_3}$ is to increase the degree by 1 and to diminish the weight by 1 unit. The same is evidently true for any other term of V. Thus the total effect of V operating on the general homogeneous isobaric function R of weight w, degree i, extent j, is to change it into another homogeneous isobaric function whose weight, degree and extent are respectively $w - 1, i + 1, j$. Observe that the extent is not altered by the operation of V.

It is easily seen that the coefficients of VR are linear functions of the coefficients of R; *e. g.* if

$$R = Aa_0^2a_3 + Ba_0a_1a_2 + Ca_1^3,$$
$$VR = a_0^3a_2(6A + 2B) + a_0^2a_1^2(3A + 5B + 6C).$$

Hence the condition $VR = 0$ gives us $(w-1;\, i+1,\, j)$ *linear* equations between the $(w;\, i,\, j)$ coefficients of R; so that, *assuming that these equations of condition are all independent*, after they have been satisfied the number of arbitrary constants remaining in R (*i. e.* the number of linearly independent reciprocants of the type $w;\, i,\, j$) is equal to

$$(w;\, i,\, j) - (w-1;\, i+1,\, j)$$

when this difference is positive; but when it is zero or negative there are no reciprocants of the given type.

If, however, any r of the $(w-1;\, i+1,\, j)$ equations of condition should not be independent of the rest, these equations would be equivalent to $(w-1;\, i+1,\, j) - r$ *independent* conditions, and therefore the number of linearly independent reciprocants of the type $w;\, i,\, j$ would be

$$(w;\, i,\, j) - (w-1;\, i+1,\, j) + r.$$

It is therefore certain that this number *cannot be less* than

$$(w;\, i,\, j) - (w-1;\, i+1,\, j).$$

We shall assume provisionally that $r = 0$, or in other words that the above partition formula is exact, instead of merely giving an inferior limit. Though it would be unsafe to rely on its accuracy, no positive grounds for doubting its exactitude have been revealed by calculation.

Such attempts as I have hitherto made to demonstrate the theorem have proved infructuous, but it must be remembered that more than a quarter of a century elapsed between the promulgation of Cayley's analogous theorem and its final establishment by myself on a secure basis of demonstration.

LECTURE X.

I will commence this lecture with a proof of Capt. MacMahon's theorem that if R is any pure reciprocant and μ its characteristic (*i. e.* its weight added to three times its degree),

$$\left(V^m \frac{d^m}{dx^m}\right) R = 1.2.3 \ldots . m \{\mu(\mu+2)(\mu+4) \ldots . (\mu+2m-2)\} (y'')^m R,$$

where y'' may be replaced by either $2a_0$ or a, according as the modified or unmodified system of letters is employed.

Instead of a pure reciprocant, let us consider any homogeneous isobaric function F of degree i and weight w; and $\left(\text{for the sake of simplicity writing } \partial_x \text{ for } \frac{d}{dx}\right)$ instead of the operator $V^m\partial_x^m$ let us consider $V^m\partial_x^n - \partial_x^n V^m$. We have identically

$$\begin{aligned}(V^m\partial_x^n - \partial_x^n V^m)F = {} & V^{m-1}(V\partial_x - \partial_x V)\partial_x^{n-1}F \\ & + V^{m-2}(V\partial_x - \partial_x V)V\partial_x^{n-1}F \\ & + V^{m-3}(V\partial_x - \partial_x V)V^2\partial_x^{n-1}F \\ & + \dots\dots\dots\dots\dots\dots \\ & + V(V\partial_x - \partial_x V)V^{m-2}\partial_x^{n-1}F \\ & + \quad (V\partial_x - \partial_x V)V^{m-1}\partial_x^{n-1}F \\ & + \partial_x(V^m\partial_x^{n-1} - \partial_x^{n-1}V^m)F.\end{aligned}$$

Now, the operation of $(V\partial_x - \partial_x V)$ on any homogeneous isobaric function whose characteristic is μ_1 is equivalent, as we have seen in Lecture VII, to multiplication by $\mu_1 y''$; so that if the characteristics of

$$\partial_x^{n-1}F, \quad V\partial_x^{n-1}F, \quad V^2\partial_x^{n-1}F, \dots . \; V^{m-1}\partial_x^{n-1}F$$

are $\qquad \mu_1, \quad \mu_2, \quad \mu_3, \dots . \quad \mu_m,$

it follows that

$$(V^m\partial_x^n - \partial_x^n V^m) = (\mu F_1 + \mu_2 + \mu_3 + \dots . + \mu_m)y''V^{m-1}\partial_x^{n-1}F + \partial_x(V^m\partial_x^{n-1} - \partial_x^{n-1}V^m)F.$$

Observe that

$$V^{m-1}(V\partial_x - \partial_x V)\partial_x^{n-1}F = V^{m-1}\mu_1 y''\partial_x^{n-1}F = \mu_1 y'' V^{m-1}\partial_x^{n-1}F,$$

where the transposition of the y'' is permissible because V does not act on it; but if y'' were preceded by ∂_x it could not be similarly transposed.

The numbers $\mu_1, \mu_2, \mu_3, \dots .$ form an arithmetical progression, for each operation of V increases the degree by unity and diminishes the weight by unity, so that

$$\mu_1 = 3i_1 + w_1 \text{ become } \mu_2 = 3(i_1 + 1) + (w_1 - 1) = \mu_1 + 2.$$

Similarly $\quad \mu_3 = \mu_1 + 4, \; \mu_4 = \mu_1 + 6, \dots . \; \mu_m = \mu_1 + 2m - 2.$

The characteristic of F being

$$\mu = 3i + w, \text{ that of } \partial_x^{n-1}F \text{ is } \mu_1 = \mu + n - 1;$$

for each operation of ∂_x leaves the degree unaltered, but adds an unit to the weight; hence

$$\mu_1 + \mu_2 + \mu_3 + \dots . + \mu_m = m(\mu + m + n - 2);$$

so that

$$(V^m\partial_x^n - \partial_x^n V^m)F = m(\mu + m + n - 2)y''V^{m-1}\partial_x^{n-1}F + \partial_x(V^m\partial_x^{n-1} - \partial_x^{n-1}V^m)F. \quad (1)$$

When $F=R$, a pure reciprocant, so that $VR=0$, our formula becomes

$$V^m\partial_x^n R = m(\mu+m+n-2)y''V^{m-1}\partial_x^{n-1}R + \partial_x V^m \partial_x^{n-1}R. \qquad (2)$$

Suppose that in (2) $m>n$, then $V^m\partial_x^n R=0$. This is obviously true when $n=0$, and when $n=1$. When $n=2$ we find

$$\begin{aligned} V^m\partial_x^2 R &= m(\mu+m)y''V^{m-1}\partial_x R + \partial_x V^m \partial_x R \\ &= 0 \text{ if } m>2. \end{aligned}$$

Similarly the case $n=3$, $m>3$ can be made to depend on $n=2$, $m>2$, and in general each case depends on the one immediately preceding it. Next let $n=m$ in (2); then, remembering that $V^m\partial_x^{m-1}R=0$, we have

$$V^m\partial_x^m R = m(\mu+2m-2)y''V^{m-1}\partial_x^{m-1}R,$$

from which MacMahon's theorem that

$$V^m\partial_x^m R = 1.2.3\ldots\ldots m\,\{\mu(\mu+2)(\mu+4)\ldots\ldots(\mu+2m-2)\}(y'')^m R$$

is an immediate consequence.

Another special case of Formula (1) is worthy of notice, viz. that in which we take $n=1$, when we obtain the simple formula

$$(V^m\partial_x - \partial_x V^m)F = m(\mu+m-1)y''V^{m-1}F. \qquad (3)$$

If in this we write a_n in the place of F, and (the modified system of letters being used) $2a_0$ for y'', μ becomes $3+n$, and we have

$$(V^m\partial_x - \partial_x V^m)a_n = 2m(m+n+2)a_0V^{m-1}a_n,$$

or, as it may also be written,

$$\frac{V^m\partial_x a_n}{1.2.3\ldots\ldots m} = \frac{\partial_x V^m a_n}{1.2.3\ldots\ldots m} + \frac{2(m+n+2)a_0V^{m-1}a_n}{1.2.3\ldots\ldots(m-1)}. \qquad (4)$$

Mr. Hammond remarks that this last formula may be used to prove the theorem

$$a_n = -t^{-n-3}\left(e^{-\frac{V}{t}}\right)a_n,$$

which was given without proof in Lecture II. Assuming that

$$a_n = -t^{-n-3}a_n + t^{-n-4}Va_n - t^{-n-5}\frac{V^2a_n}{1.2} + \ldots\ldots,$$

we have to prove that the theorem is also true when n is increased by unity. Differentiating both sides of the assumed identity with respect to x, we find

$$\begin{aligned} \partial_x a_n &= \partial_x\left(-t^{-n-3}a_n + t^{-n-4}Va_n - t^{-n-5}\frac{V^2a_n}{1.2} + \ldots\ldots\right) \\ &= -t^{-n-3}\partial_x a_n + t^{-n-4}\{\partial_x Va_n + 2(n+3)a_0a_n\} \\ &\qquad - t^{-n-5}\left\{\frac{\partial_x V^2 a_n}{1.2} + 2(n+4)a_0Va_n\right\} \\ &\qquad + \ldots\ldots\ldots\ldots\ldots\ldots\ldots\ldots\ldots\ldots\ldots; \end{aligned}$$

the general term being

$$(-)^{m+1}t^{-n-m-3}\left\{\frac{\partial_x V^m a_n}{1.2.3\ldots.m}+\frac{2(m+n+2)a_0V^{m-1}a_n}{1.2.3\ldots.(m-1)}\right\}$$

which, by means of (4), reduces to

$$(-)^{m+1}t^{-n-m-3}\frac{V^m\partial_x a_n}{1.2.3\ldots.m}.$$

Hence

$$\partial_x a_n=-t^{-n-3}\partial_x a_n+t^{-n-4}V\partial_x a_n-t^{-n-5}\frac{V^2\partial_x a_n}{1.2}+\ldots.,$$

or, more concisely, $\partial_x a_n=-t^{-n-3}\left(e^{-\frac{V}{t}}\right)\partial_x a_n.$

But $\partial_x a_n=(n+3)a_{n+1}$, and $\partial_x a_n=t\partial_y a_n=(n+3)ta_{n+1}$,

and therefore $(n+3)ta_{n+1}=-(n+3)t^{-n-3}\left(e^{-\frac{V}{t}}\right)a_{n+1}$,

or $a_{n+1}=-t^{-n-4}\left(e^{-\frac{V}{t}}\right)a_{n+1}.$

The theorem is easily seen to be true, for $n=0, 1, 2$, and is thus proved to be true universally.

I will now return to the point at which I left off in my previous lecture. We saw that the exactitude of the formula

$$(w;i,j)-(w-1;i+1,j)$$

for the number of pure reciprocants of the type $w;i,j$ could not be inferred with certainty unless we were able to prove that the $(w-1;i+1,j)$ linear equations between the coefficients of R, found by equating VR to zero, were all of them independent. A similar difficulty presents itself in the proof of the corresponding formula $(w;i,j)-(w-1;i,j)$ in the invariantive theory; but in that case I succeeded in making out a proof of the independence of the equations of condition founded on the fact that $\Omega^m O^m I$ is a numerical multiple of I, where I is any invariant, and Ω, O are the well-known operators

$$a_0\partial_{a_1}+2a_1\partial_{a_2}+3a_2\partial_{a_3}+\ldots.+ja_{j-1}\partial_{a_j}$$
$$a_j\partial_{a_{j-1}}+2a_{j-1}\partial_{a_{j-2}}+3a_{j-2}\partial_{a_{j-3}}+\ldots.+ja_1\partial_{a_0}.$$

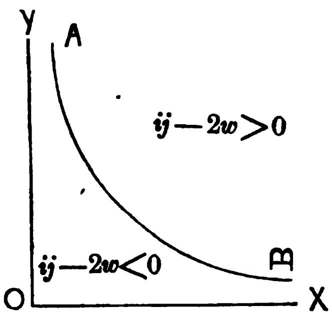
I have since discovered a second proof of the theorem for invariants which, though very interesting, is less simple than my first; but neither of these methods can be extended to the case of reciprocants.

It was suggested by Capt. MacMahon that the fact that $V^m\partial_x^m R$ is a numerical multiple of $a^m R$ ought to lead to a proof of the theorem for reciprocants similar to that obtained for invariants by my first method, alluded to above, but this I find is not the case; and indeed it is capable of being shown *a priori* that it cannot lead to a proof. One great distinction between the two theories, which is fatal to the success of the proposed method, is well worthy of notice.

If $(w; i, j) - (w - 1; i, j) => 0$ (I shall sometimes call this positive), then
$$(w'; i, j) - (w' - 1; i, j) => 0 \text{ for all values of } w' \text{ less than } w;$$
the condition that this difference, say $\Delta(w; i, j)$ shall be positive being simply that $ij - 2w$ is positive (*i. e.* $ij - 2w => 0$). This is not the case with the difference $(w; i, j) - (w - 1; i + 1, j)$, say $E(w; i, j)$; it by no means follows that if this is positive for a given value of w (i, j being kept constant), it will be so for any inferior value of w.

We may illustrate geometrically the condition $ij - 2w => 0$, which holds when $\Delta(w; i, j)$ is non-negative.

Let (i, j) be co-ordinates of a point in a plane and draw the positive branch of the rectangular hyperbola
$$ij - 2w = 0.$$



Then, $ij - 2w < 0$ for all points in the area $YOXBA$ between the curve and its asymptotes; but for points on the curve AB,
$$ij - 2w = 0,$$
and for all points of the infinite area on the side of AB remote from the origin,
$$ij - 2w > 0.$$
Thus, for all points which lie either on or beyond the curve AB,
$$\Delta(w; i, j) \text{ is non-negative,}$$
and for all points between the curve and the asymptotes
$$\Delta(w; i, j) \text{ is non-positive.}$$

We have here considered w as constant and i, j as variable, but in the case where all three are variable we should have to consider the hyperbolic paraboloid

$$ij - 2w = 0,$$

of which the curve AB is a section, by the plane $w =$ const.; and the condition of $\Delta(w; i, j)$ being non-negative or non-positive depends on the variable point (i, j, w) lying in the one case on or beyond the surface, and in the other between the surface and the planes of reference.

The function of i, j, w, whose positive or negative sign determines in like manner that of $E(w; i, j)$, cannot be linear in w. What its form is, or whether it is an Algebraical or Transcendental function, no one at present can say. Indeed, except for the light shed on the subject by the Algebraical Theory of Invariants, it would have been exceedingly difficult (as I know from vain efforts made by myself and others in Baltimore) to prove the much simpler theorem that $\Delta(w; i, j)$ is positive (*i. e.* non-negative) when $ij - 2w$ is so. It amounts to the assertion that the coefficient of $a^i x^w$ in the expansion of

$$\frac{1-x}{(1-a)(1-ax)(1-ax^2)\ldots\ldots(1-ax^j)}$$

is always non-negative, provided that $ij - 2w$ is non-negative.

This is a theorem of great importance in the ordinary Theory of Invariants, and may be seen to be a consequence of the fact, which I have proved, that (using $[w; i, j]$ to denote a function of the type $w; i, j$ having its full number of arbitrary coefficients) there are no linear connections between the coefficients of $\Omega[w; i, j]$ when $ij - 2w => 0$; but no one, as far as I know, has ever found a *direct* proof of it.

Viewing the connection between the two theories of Invariants and Reciprocants, I think it desirable to recapitulate with some improvements the proof, given in the Phil. Mag. for March, 1878, of the theorem that the number of linearly independent invariants of the type $w; i, j$ is exactly $\Delta(w; i, j)$ when this quantity is positive, and exactly zero when it is 0 or negative.

As regards reciprocants, at present we can only say that the number of linearly independent ones of the type $w; i, j$ is never less than $E(w; i, j)$, leaving to some gifted member of the class to prove or disprove that the first is always exactly equal to the second. The *exact* theorem to be proved in the theory of invariants is as follows:

If $ij - 2w = > 0$, the number of linearly independent invariants of the type $w; i, j$ is
$$\Delta(w; i, j).$$
If $ij - 2w < 0$, the number of such invariants is zero; *i. e.* there are none. The proof is made to depend on the properties of
$$\Omega = a_0\partial_{a_1} + 2a_1\partial_{a_2} + 3a_2\partial_{a_3} + \ldots + ja_{j-1}\partial_{a_j}$$
and of
$$O = a_j\partial_{a_{j-1}} + 2a_{j-1}\partial_{a_{j-2}} + 3a_{j-2}\partial_{a_{j-3}} + \ldots + ja_1\partial_{a_0}.$$
If U be any homogeneous isobaric function of degree i and weight w in the letters $a_0, a_1, a_2, \ldots a_j$, it is easy to prove that
$$(\Omega O - O\Omega)\, U = (ij - 2w)\, U,$$
and consequently, if U is an invariant I, so that $\Omega I = 0$,
$$\Omega O I = (ij - 2w)\, I.$$
I call $ij - 2w$ the *excess* and denote it by η, and shall first show that if η is negative $I = 0$; *i. e.* there exists no invariant with a negative excess. This will prove that when $\Delta(w; i, j)$ is negative, *i. e.* when $(w - 1; i, j) > (w; i, j)$, the number of independent functions of the coefficients of $[w; i, j]$ which appear in $\Omega\,[w; i, j]$ is exactly equal to $(w; i, j)$, which is the number of the coefficients themselves. Clearly it cannot be greater; for, no matter what the number of linear functions of n quantities may be, only n at the utmost can be independent; there might be fewer, there cannot possibly be more. The complete theorem is that the number of independent coefficients in $\Omega\,[w; i, j]$ is the *subdominant* of two numbers: one the number of terms of the type $w; i, j$, the other the number of terms of the type $w - 1; i, j$.

N. B.—That one of two numbers which is not greater than the other is called the subdominant.

The Binomial Equation $x^p - 1 = 0$.

By Charlotte Angas Scott, *Bryn Mawr College.*

Quartisection and Quinquisection.

The equation for Quartisection may be written in a form somewhat simpler than that given by Professor Cayley in the Proceedings of the London Mathematical Society, Vol. XI (1879), pp. 11–14, viz.:

$$\begin{aligned}&\eta^4 + \eta^3 - \eta^2\{\tfrac{1}{4}(p-1) + l + m\}\\&\quad + \eta\{\tfrac{1}{4}(f+h)(p-1) - \tfrac{1}{2}k(p+1) + 2(g-k)(l+m)\}\\&\quad + lm - \tfrac{1}{4}(l-m)^2(p-1) = 0,\end{aligned}$$

where

$$\begin{aligned}X^2 &= (a,\ b,\ c,\ d)(X,\ Y,\ Z,\ W),\\XY &= (f,\ g,\ h,\ k)(X,\ Y,\ Z,\ W),\\XZ &= (l,\ m,\ l,\ m)(X,\ Y,\ Z,\ W),\end{aligned}$$

and $l + m = \tfrac{1}{8}(p-1)$ or $\tfrac{1}{8}(3p+1)$

according as $p \equiv 1$ or 5 (mod. 8),

$$f + g + h + k = \tfrac{1}{4}(p-1), \tag{1}$$

$$a + b + c + d = -1 - 2(f+g+h+k) - 2(l+m). \tag{2}$$

The simplification is in the coefficient of η, which may be written

$$\tfrac{1}{2}\{p(l-m) - (l+m)\};$$

and then we have for the equation for Quartisection

$$\begin{aligned}&\eta^4 + \eta^3 - \{\tfrac{1}{4}(p-1) + l + m\}\eta^2\\&\quad + \tfrac{1}{2}\{p(l-m) - (l+m)\}\eta\\&\quad - \tfrac{1}{4}\{p(l-m)^2 - (l+m)^2\} = 0,\end{aligned}$$

where $l - m$ is the only quantity whose value in terms of p is unknown.

To find the coefficient of η, viz. $-\Sigma XYZ$,

$$\begin{aligned}\Sigma XYZ &= YZW + ZWX + WXY + XYZ \\ &= Z.YW + W.ZX + X.WY + Y.XZ \\ &= (Y+W)XZ + (X+Z)YW \\ &= (Y+W)(l, m, l, m) + (X+Z)(m, l, m, l) \\ &= 2l(YZ + ZW + XY + WX) \\ &\quad + m(X^2 + Y^2 + Z^2 + W^2 + 2XZ + 2YW) \\ &= -2l(f+g+h+k) \\ &\quad - m(a+b+c+d+2l+2m) \\ &= -2l(f+g+h+k) \qquad \text{[by (2)} \\ &\quad + m\{1 + 2(f+g+h+k)\} \\ &= -(l-m)\frac{p-1}{2} + m \qquad \text{[by (1)} \\ &= -\tfrac{1}{2}\{p(l-m) - (l+m)\},\end{aligned}$$

$\therefore$ coefficient of $\eta = \frac{1}{2}\{p(l-m) - (l+m)\}$.

In the Proc. Lond. Math. Soc., Vol. XII, pp. 15–16, and Vol. XVI, pp. 61–63, Professor Cayley considers the theory of the Quinquisection. He gives the expression of the fifteen coefficients in X^2, XY, XZ in terms of five quantities $\alpha, \beta, \vartheta, f, k$, connected by two quadric equations and by the linear equation

$$\alpha + \beta + \vartheta = \tfrac{1}{5}(p-1).$$

Professor Cayley finds the expression of these coefficients in terms of $\alpha, \beta, \vartheta, f, k$, by induction from numerical examples, but refers to Mr. F. S. Carey's Trinity Fellowship Dissertation (1884) for proofs of equivalent formulae. From the form of Mr. Carey's results, I imagine that his proof is not the one here given. Using Professor Cayley's notation, h is the number of positive integral solutions (with regard to λ, μ, ν) of the congruence

$$\gamma^{5\lambda} + \gamma^{5\mu+1} \equiv \gamma^{5\nu+2} \ (\text{mod. } p),$$

where γ is the selected prime root, and therefore $\gamma^{\frac{1}{2}(p-1)} \equiv -1$ (mod. p); *i. e.*,

$$\gamma^{5\rho} \equiv -1 \ (\text{mod. } p) \ [p = 10\rho + 1].$$

The congruence is therefore equivalent to

$$\gamma^{5(\rho+\mu)+1} + \gamma^{5\nu+2} \equiv \gamma^{5\lambda},$$

the number of solutions of which gives the coefficient of X in the product YZ; *i. e.*, j, $\quad \therefore h = j$.

This method is applicable to the coefficients in the products of the periods, but not to the coefficients in the squares of the periods, for, since there are five periods, complementary terms come in the same period. Thus the only systems of equalities that we get are

$$h=j=l;\quad i=n=o.$$

Write $\frac{1}{5}(p-1)=\lambda$. We have then (see Prof. Cayley's paper)

$$\left.\begin{array}{l}\Sigma f=\lambda.\qquad \Sigma k=\lambda.\qquad \Sigma a=-1-4\lambda.\\ \alpha+\beta+\vartheta=\lambda.\end{array}\right\}. \tag{3}$$

For the equation for Quinquisection we have

$$\text{coefficient of } \eta^4=-\Sigma X=1;$$
$$\text{coefficient of } \eta^3=\Sigma(XY+XZ)=-\Sigma f-\Sigma k=-2\lambda.$$

The products of the periods in threes are such as ZWT or else such as YWT.

$$\begin{aligned}\Sigma ZWT &= X.YZ+Y.ZW+Z.WT+W.TX+T.XY\\ &= X(j,f,g,h,i)+Y(i,j,f,g,h)+Z(h,i,j,f,g)+W(g,h,j,i,f)\\ &\qquad +T(f,g,h,i,j)\\ &= \quad f(XY+YZ+ZW+WT+TX)\\ &\quad +g(XZ+YW+ZT+WX+TY)\\ &\quad +h(XW+YT+ZX+WY+TZ)\\ &\quad +i(XT+YX+ZY+WZ+TW)\\ &\quad +j(X^2+Y^2+Z^2+W^2+T^2)\\ &= (f+i)\Sigma f.\Sigma X+(g+h)\Sigma k.\Sigma X+j\Sigma a.\Sigma X\\ &= -\lambda(f+g+h+i)+j(1+4\lambda) \qquad \text{[by (3)}\\ &= -\lambda(f+g+h+i+j)+j(1+5\lambda)\\ &= -\lambda^2+pj.\end{aligned}$$

Similarly,

$$\begin{aligned}\Sigma YWT &= X.ZW+Y.WT+Z.TX+W.XY+T.YZ\\ &= -\lambda(f+g+h+j)+i(1+4\lambda)\\ &= -\lambda^2+pi.\end{aligned}$$

$$\begin{aligned}\therefore \text{ coefficient of } \eta^2 &= -\Sigma(ZWT+YWT)\\ &= 2\lambda^2-p(i+j)\\ &= 2\lambda^2-p(\alpha+\beta).\end{aligned}$$

Write $\qquad s_1$ for $fg+gh+hi+ij+jf$

and $\qquad s_2$ for $fh+gi+hj+if+jg$,

then
$$\Sigma YZWT = \Sigma(j, f, g, h, i)(h, i, j, f, g)$$
$$= s_2 \Sigma X^2 + (s_1 + s_2)\Sigma XY + (s_1 + \Sigma f^2)\Sigma XZ$$
$$= -\Sigma f^2 . \Sigma k - s_1(\Sigma f + \Sigma k) - s_2(\Sigma a + \Sigma f)$$
$$= -\lambda \Sigma f^2 - 2\lambda s_1 + (1 + 3\lambda)s_2 \qquad \text{[by (3)}$$
$$= -\lambda\{\Sigma f^2 + 2s_1 + 2s_2\} + ps_2 .$$
Now
$$\Sigma f^2 + 2s_1 + 2s_2 = [\Sigma f]^2 = \lambda^2, \qquad \text{[by (3)}$$
and
$$s_2 = fh + gi + hj + if + jg$$
$$= f\alpha + g\beta + \alpha^2 + \beta f + \alpha g$$
$$= (\alpha + \beta)(f + g) + \alpha^2$$
$$= (\alpha + \beta)(\vartheta + \alpha) + \alpha^2$$
$$= \vartheta(\alpha + \beta) - \alpha\beta ,$$
$\therefore$ coefficient of $\eta = -\lambda^3 + p\{\vartheta(\alpha + \beta) - \alpha\beta\}$.

The remaining term I have not succeeded in finding. The equation for Quinquisection is therefore
$$\eta^5 + \eta^4 - 2\frac{p-1}{5}\eta^3 + \left\{2\left(\frac{p-1}{5}\right)^2 - p(\alpha + \beta)\right\}\eta^2$$
$$- \left\{\left(\frac{p-1}{5}\right)^3 - p\,[\vartheta(\alpha + \beta) - \alpha\beta]\right\}\eta + [?] = 0,$$
where $\vartheta + \alpha + \beta = \frac{1}{5}(p-1)$.

A Contribution to the Theory of the General Equation of the Sixth Degree.

By F. N. Cole.

The subject of the present article was suggested to me by Prof. Klein, when I was a student in his Seminar at Leipzig, and I wish here to acknowledge my great indebtedness to him for valuable advice and suggestion, which have been of the greatest use to me. The fundamental idea of the entire treatment of the subject is due to him, as I have indicated below, and he might claim many of the particular methods involved as his own, if he should consider them worthy of such recognition.

The modern theory of the equation of the fifth degree may be said to date from the demonstration by Abel of the impossibility of the algebraic solution of the general equation of the fifth degree,* and the establishment by the same mathematician of the algebraic equations for the division of the elliptic functions,† equations of which the general equation of degree five is essentially a particular case. With this latter discovery of Abel, two important steps were made. The actual solution of the general equation of the fifth degree was obtained, and its theory identified with that of certain elliptic functions whose nature was fast becoming known through the work of Abel and Jacobi, and, what was more important, the *formal constitution* of the equation was ascertained, and thereby an important impetus given to a theory then just struggling into existence, but now become one of the most powerful of mathematical instruments—the Theory of Substitutions. Fifty years before Abel, Lagrange‡ had succeeded in unifying the theories of the equations of the first four degrees, which, before his time, had been a set of detached facts rather than a theory.

* Crelle's Journal, Bd. I.

† Abel: Oeuvres complètes. Edition of Helmboe, 1881.

‡ Mem. d. Berliner Akademie 1770–1. Oeuvres, T. III.

Encouraged by his success in this direction, he proposed a similar theory for the algebraic solution of the general equation of any degree. This theory, although it necessarily failed, contained the germs from which, later, the Theory of Substitutions developed. It was Evariste Galois* who understood how to reject what was erroneous in Lagrange's theory, and to adapt the remainder to the increased mathematical knowledge of his own day. He had succeeded in advancing the theory so far that he was able to make a direct application of it to this new discovery of Abel's. On this double foundation of the theory of the elliptic functions and the Theory of Substitutions, the theory of the equation of the fifth degree is based. Had Galois lived a few years longer, he would undoubtedly have constructed a tolerably complete theory of this and other equations, but, as it is, this theory has been one of slow growth. It may now be regarded as complete, and further investigation naturally takes a turn in the direction of the theory of the higher equations.

To Hermite† and Kronecker‡ we owe the greater part of our knowledge of the general equation of the fifth degree. The former started from the equation of the sixth degree, which Abel deduced for the division of the elliptic functions by 5, and obtained the general equation of the fifth degree as a resolvent of this. The latter operated with Jacobi's multiplier equation, which is also a resolvent of the equation of the fifth degree. For the further history of this subject, I refer to Prof. Klein's§ work on the Ikosahedron, which I shall frequently have occasion to cite, and which abounds in historical information, and is particularly rich in footnotes. To this work, too, I refer for the elegant geometrical developments which in late years Klein has given to this theory, connecting it with the modern geometrical methods, and giving it a completeness which was wanting in the purely analytical theory.

While the theory of the equation of the fifth degree is thus complete in all directions, that of the sixth degree is only just begun. Two methods of solution have been proposed, both analogous to the method for the fifth degree, but essentially different from each other in their points of view. Camille Jordan‖

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* **Evariste Galois**: Oeuvres mathematiques. Edited by Liouville in T. XIV of Liouville's Journal.

† **Hermite**: Sur la résolution de l'équation du cinquième degré. C. R., t. 46 and 62.

‡ **Kronecker**: Monatsberichte der Berliner Akademie, 1861. Crelle, Bd. 59.

§ **Klein**: Vorlesungen über das Ikosaeder u. die Auflösung der Gleichungen vom fünften Grade. Teubner: Leipzig, 1884.

‖ Traité des Substitutions, p. 380.

has shown that any algebraic equation of degree n may be solved by the aid of the formulae for the bisection of the periods of the corresponding hyperelliptic functions. This method would require the establishment of these formulae, and the reduction of the equation of the sixth degree to the form of a resolvent of them. This, so far as I know, has not yet been done. The second method was proposed by Klein,* and it is this method with which the present article has to deal. On account of the analogy of the method to that of the equations of lower degree, I prefix a short, and necessarily incomplete, account of the latter, referring to the original articles cited in the footnotes for complete information.

The central feature of the theories of the equations of degree one to five is the important part played by certain groups of linear transformations of a single variable. In fact, the theory of these groups is, formally, the theory of the corresponding equations, and appears, under one metamorphosis or another, in all the various phases in which their theory can be presented. The fourth degree is the first for which the group of linear transformation is of prominence. Here it is composed of the six linear transformations of the anharmonic ratio of the roots. The four roots have six anharmonic ratios, and, if one of them be denoted by λ, the others, as is well known, are:† $\frac{1}{\lambda}$, $1-\lambda$, $\frac{1}{1-\lambda}$, $\frac{\lambda}{1-\lambda}$, $\frac{\lambda-1}{\lambda}$, all of which proceed from any one by linear transformation. These six anharmonic ratios satisfy an equation of the sixth degree, which, of course, is not the general equation, but is characterized by the fact that every root is a rational linear function of every other root. This is the equation of the sixth degree which appears in the ordinary theory of the solution of the equation of the fourth degree; it reduces at once to a cubic which may then be solved, and from these solutions those of the biquadratic are at once obtained.

For the equation of the fifth degree, the matter is more complicated. Here the group of linear transformations contains sixty operations, which are represented as follows: If we denote by z a number congruent to $0, 1, 2, 3, 4$, or ∞ mod. 5, and from $z' \equiv \frac{\alpha z+\beta}{\gamma z+\delta}$ mod. 5, then, by proper choice of α, β, γ, δ, z' will be an integer, whichever quantity we take z congruent to. It is found that there are 60 sets of values, α, β, γ, δ, which satisfy these conditions.‡ If

* Ikosaeder, p. 126, footnote. Math. Ann. XV.

† See, say, Clebsch: Geometrie, p. 39.

‡ Netto: Substitutionentheorie, p. 141.

we select any one of these sets, and put z successively congruent to $0, 1, \ldots . 4, \infty$ mod. 5, we shall get for z' values congruent to $0 \ldots . \infty$, but in a different order from the original. Now, a rational function of the roots of the equation of the fifth degree can be found,* which, when these roots undergo any *even* permutation, is linearly transformed by the formula $\phi' = \frac{\alpha\varphi + \beta}{\gamma\varphi + \delta}$. This function ϕ then plays to a certain extent the same part in the theory of the equation of the fifth degree which the anharmonic ratio λ plays in that of the fourth degree. It satisfies an equation of degree 60, of which the coefficients involve no further irrationality than the square root of the discriminant of the given equation. There is, however, the important difference that the equation in λ is simpler than the general equation of the fourth degree, while that in ϕ is precisely the equivalent of the general equation of the fifth degree, and is neither more difficult nor simpler. In the language of the theory of substitutions, λ belongs to the exceptional self-conjugate† sub-group of the general group of 24 permutations of the 4 roots, the result being that the group of the λ equation is reduced to 6 permutations, while the ϕ equation has still the group of 60 subst. which the general equation of the fifth degree has after the adjunction of the square root of the discriminant.

If the z's above be taken as suffixes of six quantities x, every permutation of the z's will give a corresponding permutation of the x's. The resulting group of 60 substitutions could easily be identified with the monodromic‡ group of the modular equation of Abel and Hermite for the division of the elliptic functions by 5, but this would be beyond the scope of the present article. For the same reason I can only make passing mention here of the identification of the group of linear substitutions of the ϕ's with the group of rotations of the ikosahedron, and the connection with the line geometry of the surfaces of the second order.§

Of the greatest importance, on the other hand, for the present purpose, is the theory of the linear differential equations of the second and third orders given in the third chapter of Klein's work. We have seen that the quantities

* Klein: Ikosaeder, p. 187, where λ and μ are such functions.

† I have ventured to translate Klein's "ausgezeichnete Untergruppe" by "self-conjugate sub-group," and use this nomenclature in preference to the "groupe permutable" of Jordan. For the theory of this group see in particular Klein's book, pp. 88-9.

‡ Camille Jordan: Traité, p. 277.

§ Cf. throughout Klein's book.

ϕ above satisfy an equation of degree 60 with coefficients which are rational after the adjunction of the square root of the discriminant. These coefficients may all be expressed as rational functions of a single parameter (Z of Klein's work). The roots of the equation are all obtainable from any one of these by linear transformation, $\phi' = \frac{\alpha\varphi + \beta}{\gamma\varphi + \delta}$, or $\zeta = \frac{\alpha\eta + \beta}{\gamma\eta + \delta}$, to follow Klein's notation. If this last equation be differentiated three times with respect to Z, we may eliminate α, β, γ and δ from the four resulting equations, and obtain the differential equation $\frac{\zeta'''}{\zeta'} - \frac{3}{2}\left(\frac{\zeta''}{\zeta'}\right)^2 = \frac{\eta'''}{\eta'} - \frac{3}{2}\left(\frac{\eta''}{\eta'}\right)^2$.* Either side of this equation is therefore an expression unchanged by any even permutation of the roots, and is therefore a rational function of Z. We have thus a differential equation which the quantities ϕ satisfy, and may study the solutions of this equation, from which those of the general equation of the fifth degree are rationally compounded.

The method proposed by Klein for the solution of the general equation of any degree is perfectly analogous to this. We have to seek a group of linear substitutions which shall be isomorphic† with the group of $n!$ permutations belonging to the equation. Functions of the roots must then be found which undergo these linear transformations when the roots are permuted; and finally, corresponding differential equations must be obtained and their solutions studied. In the present article, I have begun the determination of these differential equations for the case of the general equation of the sixth degree.

First of all, a group of linear transformations must be found which shall be isomorphic with the entire group of $6! = 720$ permutations of six things, or with the corresponding alternating group of 360 permutations. Here a short digression on the composition of the group of 720 permutations of six things, in which, as before, I merely indicate the sources of information, may not be out of place. First of all, this group contains the self-conjugate alternating sub-group of 360 permutations, for which the square root of the discriminant is the corresponding function. Next to this come, according to the law demonstrated by Bertrand and Serret,‡ the sub-groups of 120 permutations. By

* Either side of this equation is the "Schwarzian Derivative" of Cayley. See Schwarz: Crelle, Bd. 75. Also Klein: Ikosaeder, p. 74, footnote.

† Jordan: Traité, p. 56.

‡ *Ib.* p. 67. Serret: Algèbre supérieur, Vol. II, Chap. 3.

exception there are two of these, the ordinary intransitive group, affecting only five of the elements, and the extraordinary group containing only permutations of four, five and six elements.* As a result of the presence of this exceptional group, all equations of the sixth degree are connected in pairs, the roots of the two equations of each pair belonging respectively to the ordinary and the extraordinary groups of 120 permutations. There will be six of these groups of each kind. Half the permutations of each group will be odd and half even. The intransitive and imprimitive groups of lower orders are easily determined. The former are the system of sub-groups belonging to the general group of 5 elements. The latter are of orders which are divisors: (1) of 72, corresponding to the division of the six elements into two sets of three each; (2) of 48, for two sets, one of two, the other of four elements; and (3) also of 48, for three sets of two each. The primitive groups can contain no permutations which affect less than four elements.† They are to be determined according to the known methods.‡

If now we attempt to find a group of linear transformations which shall be isomorphic with the group of 720 or 360 permutations of six elements, we meet at the start with a certain difficulty. There is no such finite group of linear transformations of a single variable. The presence of such a linear group is a characteristic feature of the equations of the first five degrees, and seems to group these equations by themselves, as Klein has shown.§ Two ways offer themselves in which we can overcome the apparent difficulty arising from the absence of the linear group. On the one hand, we may look for infinite groups of linear transformations of a single variable; on the other, we may introduce additional variables and consider finite groups.‖ The latter method is that with which we are to deal. We inquire, then, What is the smallest number of variables for which a group of linear substitutions isomorphic with the 720 permutations of six elements exists? There is no such group for one or for two variables. There is, however, such a group for three variables, or, if we write our linear transformation in homogeneous form, for four variables, of which the ratio of three to the fourth will then be transformed by a non-homogeneous linear transformation. This group of transformations is best known under the geometrical form in which its theory has been treated in connection with the

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* Serret: Alg. sup., Vol. II, Chap. 3. Netto: Substitutionentheorie, pp. 124–5.

† Netto, Chap. V.

‡ *Ib.* p. 125.

§ Klein, pp. 130–1.

‖ *Ib.* Chap. V throughout.

remarkable surface of the fourth order and class known as Kummer's surface,* and with theory of the complexes of the second order of Plücker's Line Geometry.†

As is well known, the general equation of the second order in Plücker's line co-ordinates denotes a geometrical configuration composed of ∞^3 straight lines in space. These are so arranged that in every plane lie ∞^1 lines enveloping a conic section and through every point in space pass ∞^1 lines, forming a cone of the second order. One condition being necessary in order that a conic section should break up into two points, or that a cone of the second order should break up into two planes, there will be ∞^2 planes for which the conic is a pair of points and ∞^2 points for which the cone is a pair of planes. The surface enveloped by those ∞^2 planes and that traced out by the ∞^2 points are identical, and this surface, of which class and order are evidently equal, is Kummer's surface.‡ Written in the ordinary line co-ordinates, the equation of the general complex of the second order is $a_{12}p_{12}^2 + a_{13}p_{13}^2 + \text{etc.} + \beta_{1231}p_{12}p_{31} + \text{etc.} = 0$, to which is conjoined the quadratic identity $p_{12}p_{34} + p_{13}p_{42} + p_{14}p_{23} = 0$. If we introduce the canonical co-ordinates of Klein into these equations, they become $\sum\limits_{\iota=1}^{\iota=6} \varkappa_\iota x_\iota^2 = 0$ and $\Sigma x_\iota^2 = 0$, the cross-terms disappearing in both equations. The formulae connecting the two sets of equations may be taken as follows:

$$x_1 = p_{12} + p_{34},\ x_2 = -i(p_{12} - p_{34}),\ x_3 = p_{13} + p_{42},\ x_4 = -(p_{13} - p_{42}),$$
$$x_5 = p_{14} + p_{23},\ x_6 = -i(p_{14} - p_{23}).$$

If we denote the left-hand side of the equation of the complex by Ω, a linear tangential complex for the line y_ι of the complex will be $\Sigma\left(\frac{d\Omega}{dyi}\right)x_\iota = 0$ or $\Sigma \varkappa_\iota x_\iota y_\iota = 0$. In order that the line y_ι should be a singular line of the complex, we must have $\Sigma \varkappa_\iota^2 y_\iota^2 = 0$; *i. e.* the linear tangential complex must be a special one. The line y_ι is then tangent to the Kummer surface, and the equation $\Sigma \varkappa_\iota x_\iota y_\iota = 0$ shows that every tangent to the surface at the point of contact of y_ι will also cut $\varkappa_\iota y_\iota$. The bundle of tangents at this point is therefore represented by $\rho x_\iota = (\varkappa_\iota + \lambda)\, y_\iota$, where λ is an arbitrary variable parameter. These lines satisfy the equations $\Sigma x_\iota^2 = 0$, $\Sigma \varkappa_\iota y_\iota = 0$, $\Sigma y_\iota^2 = 0$; or, putting for y its value in terms of x, $\Sigma x_\iota^2 = 0$, $\Sigma \frac{x_\iota^2}{\varkappa_\iota + \lambda} = 0$, $\Sigma \frac{x_\iota^2}{(\varkappa_\iota + \lambda)^2} = 0$.

* Kummer: Monatsberichte der Berliner Akademie, 1864.

† Plücker: Neue Geometrie des Raumes, etc. Edited by Clebsch and Klein.

‡ For the geometry of Kummer's surface cf. throughout Rohn's article in Bd. XV and XVIII of the Math. Ann.

The first of these equations shows that x_i is a line; the second, that it belongs to a complex of the second order; the third, that it is a singular line of this complex. The Kummer surface belongs therefore to ∞' different complexes of the second order, corresponding to the ∞' different values of λ.

The six linear complexes $x_i = 0$ are called by Klein the fundamental complexes. A congruence of any two of them will consist of all lines which cut two fixed straight lines, the directrices of the congruence. If the six fundamental complexes be divided into three congruences, the six directrices will cut each other in pairs so as to form a tetrahedron. The 15 tetrahedra obtainable in this way are called the fundamental tetrahedra.

If, in the equation of the complex, we put for any x_i its negative, the equation will be unchanged. There are 32 possible arrangements of the signs, and these give rise to 32 transformations of space, which leave the Kummer surface unchanged. Sixteen of these are collineations, the other 16 being dualistic transformations. These convert in particular the system of 16 double points and 16 double tangent planes of the surface into the same system in a transposed order. Again, if we have used any system of co-ordinates based on any one of the 15 fundamental tetrahedra, these will be transformed if we take any other tetrahedron. Such a transformation amounts to a permutation of the fundamental complexes, *i. e.* of the x_i's. These permutations evidently leave Σx_i^2 unchanged. Every straight line therefore transforms into a straight line; that is, these transformations are either collineations or dualistic transformations. There are as many of them as there are permutations of the x_i's, *i. e.* 720. If we combine these with the 32 transformations above, we have a group of 32.720 linear transformations of space which is isomorphic with the group of the 720 permutations of six elements. This is the geometrical linear group to which I had reference above.*

We have next to determine a function of six arbitrary elements which is linearly transformed by this group of substitutions. To obtain such a function we observe that a permutation of the x's in this equation of our complex $\Sigma \kappa_i x_i^2 = 0$ amounts to a permutation of the κ's. Now, the point co-ordinates of a double point of the Kummer surface, referred to any one of the 16 fundamental tetrahedra, will be 16-valued functions of $\kappa_1 \ldots . \kappa_6$. It appears, therefore, that the point co-ordinates of any double point of the Kummer surface referred to a

* Cf. here Jordan: Traité, p. 317.

fundamental tetrahedron are functions which undergo precisely the $\frac{32.720}{2}$ linear substitutions corresponding to the $\frac{32.720}{2}$ collineations of our group. These co-ordinates are the moduli whose theory was given by Borchardt in his article on the Arithmetic-Geometric Mean of Four Quantities.*

The moduli are 4 in number, corresponding to the 4 co-ordinates of the double point. They are given by the equations:

$$\xi^2+\eta^2+\zeta^2+\vartheta^2=\sqrt{135.246},\quad \xi^2-\eta^2-\zeta^2+\vartheta^2=\sqrt{136.245},$$
$$\xi^2-\eta^2+\zeta^2-\vartheta^2=\sqrt{145.236},\quad \xi^2+\eta^2-\zeta^2-\vartheta^2=\sqrt{146.235},$$
$$2(\xi\zeta+\eta\vartheta)=\sqrt{124.356},\quad 2(\xi\vartheta+\eta\zeta)=\sqrt{125.346},\quad 2(\xi\eta+\zeta\vartheta)=\sqrt{134.256},$$
$$2(\xi\zeta-\eta\vartheta)=\sqrt{123.456},\quad 2(\xi\vartheta-\eta\zeta)=\sqrt{126.345},\quad 2(\xi\eta-\zeta\vartheta)=\sqrt{156.234},$$

when, *e. g.* 135.246 denote $(x_1-x_3)(x_3-x_5)(x_5-x_1)(x_2-x_4)(x_4-x_6)(x_6-x_2)$.

In regard to the isomorphism between the senary group of 360 permutations of the x's and the quaternary group of the linear transformations of the point co-ordinates of the double points (disregarding for the moment the changes of sign which arise from even changes of sign of the x's of the fundamental complexes), Prof. Klein noted in one of his lectures that this might be a holoedric or hemiedric isomorphism, since the change of sign of all the point co-ordinates leaves the line co-ordinates unchanged, so that it might be that to every operation of the senary group two operations of the quaternary group, differing only in sign, correspond. That the isomorphism is hemiedric is evident from the following consideration: For every even permutation of the x's there will be either one or two corresponding linear substitutions of the $\xi, \eta, \zeta, \vartheta$, according as the isomorphism is holoedric or hemiedric. In particular, for the identical substitutions of the x's we shall have either simply the identical substitution of the $\xi, \eta, \zeta, \vartheta$, or this and a second substitution consisting simply in the change of sign of $\xi, \eta, \zeta, \vartheta$. If, therefore, we can simultaneously change the sign of all forms of the Borchardt moduli by a series of permutations of the x's, which result in each x's returning to its original position, the isomorphism will be hemiedric, and conversely, if the latter be the case, such a set of permutations can be found. The quantities x being entirely arbitrary, we select any convenient values of x_2, x_5 and x_6 (no two of which shall coincide) and take x_1 and x_3 and x_4 very near each other. (We may conveniently represent the 6 x's by six points in a plane.) Consider now the quantity: $\sqrt{(135)(246)}$. If x_1 and x_3

* Borchardt: Abh. d. Berliner Akademie, 1878.

describe continuous paths, so that finally x_1 becomes x_2 and x_2 becomes x_1, this square root will have been multiplied by $\pm i$. A repetition of this operation will give us $-\sqrt{(135)(246)}$. The quantity $\sqrt{(136)(245)}$ will undergo the same transformation. The other two square roots $\sqrt{(145)(236)}$ and $\sqrt{(146)(235)}$ will be unchanged. Again, if we permute x_1 and x_4 in the same way, the latter two square roots will change sign, while the former two remain unchanged. By an identical permutation of the x's we can therefore change the sign of ξ^2, η^2, ζ^2 and ϑ^2 simultaneously. Then ξ, η, ζ and ϑ will each have been multiplied by $\pm i$. A repetition of the operation will change the sign in each case. To every operation of the senary group will correspond the two operations of the quaternary linear group, so that the latter group consists of 32.360 substitutions.

Camille Jordan,* by a process nearly the inverse of the one just given, but which does not employ the line geometry, has exhibited geometrically that it is possible to obtain a group of linear substitutions of four elements which shall be exactly holoedrically isomorphic with the 360 even permutations of six elements. Thus the 16 double points of the Kummer surface satisfy an equation of degree 16, of which the group is determined readily from the fact that the 16 points lie on 16 planes, 6 points in each plane, and 6 planes through each point. This group consists of 16.720 substitutions, and to it corresponds holoedrically a linear group of 16.720 collineations of space. Corresponding functions are then constructed and grouped in sets of 16, each set representing the six double points in a plane, these six sets being permuted among each other. There must be 720 of these permutations, since any interchange of the planes interchanges the double points, and conversely, to every one of these 720 permutations of the six points in a plane will correspond one, and only one, of the 720 collineations. The two groups are therefore referred holoedrically to each other. It follows also that the two alternating sub-groups are holoedrically related. I have introduced this result of Jordan's because it has an immediate bearing on the construction of functions which shall undergo the operations of our linear group. Klein† has shown that if we have two linear groups, one affecting n variables, and containing N substitutions, the other affecting μ variables, and containing $\frac{N}{\nu}$ substitutions, then, if isomorphism exists between these groups, we can construct rational, homogeneous, integral functions of the x's

* Camille Jordan: Traité, etc., p. 313.

† Klein: Ueber die Auflösung gewisser Gleichungen, etc., Bd. XV of the Math. Ann.

which undergo the substitutions of the y's. As a particular case, the linear substitutions may be merely the permutations of the x's. It appears, therefore, that in the present case we can obtain rational, homogeneous, integral functions of the x's which shall undergo one group of linear substitutions when the x's are permuted. Klein shows in the same article how actually to construct such functions, but I am not aware that this construction has actually been carried through for the present case.

The Borchardt moduli are therefore, in two respects, by no means the simplest functions which could be used. They are irrational to a considerable degree, and their group is $32 - 1$-edric isomorphic with the alternating group of the x's, and even after the adjunction of the geometry of the Kummer surface we have a meriedric isomorphism. They are, however, well-known functions which have a bearing on various mathematical theories, and they are in a very compact form. Moreover, a set of transformations will readily be found for passing from the formulae which may be obtained from them to any other system of quantities which may afterwards be obtained.

Taking, then, the Borchardt moduli as the fundamental functions, we can immediately obtain a system of linear differential equations with coefficients rational in the coefficients of the given equation and the $\sqrt{\Delta}$, and satisfied by these moduli. These are at once formally given by the following method:

We construct the determinant in which the derivatives are taken with respect to any coefficient of the given equation of the sixth degree:

$$\begin{vmatrix} \omega & \xi & \eta & \zeta & \vartheta \\ \omega' & \xi' & \eta' & \zeta' & \vartheta' \\ \omega'' & \xi'' & \eta'' & \zeta'' & \vartheta'' \\ \omega''' & \xi''' & \eta''' & \zeta''' & \vartheta''' \\ \omega^{\text{IV}} & \xi^{\text{IV}} & \eta^{\text{IV}} & \zeta^{\text{IV}} & \vartheta^{\text{IV}} \end{vmatrix}.$$

This will evidently be 0 if ω equals ξ, η, ζ or ϑ.

The desired differential equation is got by expanding this and equating to 0:

$$\omega^{\text{IV}}\begin{vmatrix} \xi & \eta & \zeta & \vartheta \\ \xi' & \eta' & \zeta' & \vartheta' \\ \xi'' & \eta'' & \zeta'' & \vartheta'' \\ \xi''' & \eta''' & \zeta''' & \vartheta''' \end{vmatrix} + \omega'''\begin{vmatrix} \xi & \eta & \zeta & \vartheta \\ \xi' & \eta' & \zeta' & \vartheta' \\ \xi'' & \eta'' & \zeta'' & \vartheta'' \\ \xi^{\text{IV}} & \eta^{\text{IV}} & \zeta^{\text{IV}} & \vartheta^{\text{IV}} \end{vmatrix} + \omega''\begin{vmatrix} \xi & \eta & \zeta & \vartheta \\ \xi' & \eta' & \zeta' & \vartheta' \\ \xi''' & \eta''' & \zeta''' & \vartheta''' \\ \xi^{\text{IV}} & \eta^{\text{IV}} & \zeta^{\text{IV}} & \vartheta^{\text{IV}} \end{vmatrix}$$

$$+ \omega'\begin{vmatrix} \xi & \eta & \zeta & \vartheta \\ \xi'' & \eta'' & \zeta'' & \vartheta'' \\ \xi''' & \eta''' & \zeta''' & \vartheta''' \\ \xi^{\text{IV}} & \eta^{\text{IV}} & \zeta^{\text{IV}} & \vartheta^{\text{IV}} \end{vmatrix} + \omega\begin{vmatrix} \xi' & \eta' & \zeta' & \vartheta' \\ \xi'' & \eta'' & \zeta'' & \vartheta'' \\ \xi''' & \eta''' & \zeta''' & \vartheta''' \\ \xi^{\text{IV}} & \eta^{\text{IV}} & \zeta^{\text{IV}} & \vartheta^{\text{IV}} \end{vmatrix} = 0.$$

These sub-determinants will evidently only be multiplied by the same constant factor for all when the $\xi, \eta, \zeta, \vartheta$ are linearly transformed, *i. e.* when the x's are permuted an even number of times. Their ratios are therefore unchanged by any even permutations of the x's. From this we cannot, however, infer directly that these ratios are rational functions of the coefficients of the proposed equation and of the square root of the discriminant, since we are not certain that the determinant ratios are rational in the x's, while the theory of substitutions deals primarily only with such functions as are rational in the roots. Precisely the same process, however, as that of the theory of substitutions in the limited sense of the word will demonstrate that our determinant ratios are rational in the coefficient and the square root of the discriminant. Thus let $\phi_1\phi_2 \ldots . \phi_\lambda$ be the values of the determinant ratios and form

$$\phi_1 + \phi_2 \ldots . + \phi_\lambda = A_0,$$
$$\phi_1\sqrt{\Delta} \pm \phi_2\sqrt{\Delta} + \ldots . \pm \phi_\lambda\sqrt{\Delta} = A_1, \text{ etc.}$$

From these we can get ϕ_1 as a rational function of the A's and $\sqrt{\Delta}$. From which again it is apparent that $\lambda = 2$. This is a perfectly general proposition. Any two algebraic functions, whether rational in the roots or not, belonging to the same group, are rational functions of each other and of the coefficients of the proposed equation.

To solve the equation of the sixth degree, we must first remove the coefficients of the terms x^5 and x^4, form the corresponding differential equation with respect to each of the four remaining coefficients, and actually calculate the ratios of the determinants which appear as coefficients in these differential equations.

Actually to differentiate $\xi, \eta, \zeta, \vartheta$ four times, substitute the results in the 5 determinants, and reduce the result, removing all common factors from numerator and denominator, etc., would be nearly impossible, as one will readily perceive on remembering that any function has 6 first, 21 second, 56 third, and 126 fourth derivatives with respect to six quantities, and that we must first differentiate the $\xi, \eta, \zeta, \vartheta$ with respect to the x's and afterwards the x's with respect to the coefficients of the proposed equation. It is therefore necessary to have recourse to infinite series. We must expand the roots in terms of the coefficients, substitute these values in the Borchardt moduli, and obtain them also as infinite series, which are then to be substituted in the determinant ratios, and the latter obtained as rational functions of the coefficients, and of the square root of the discriminant.

Here we meet with a certain technical difficulty. The coefficients of our differential equations are, to be sure, rational functions of the coefficients of the given equation and of the $\sqrt{\Delta}$. But they are not integral functions. From the final infinite series we shall therefore be unable to separate numerator and denominator. There is no method of determining *a priori* the roots and poles of these functions, and any attempt to determine them would necessarily be a tentative process of which the success would be highly improbable. Again, if we attempt to avoid this difficulty by considering each determinant by itself, we are checked in two ways. First, the single determinants may or may not be rational, after the adjunction of the $\sqrt{\Delta}$, according as the constant multiplier is unity or not. As a matter of fact, we shall find later that it is unity in all cases. And further, we have the former difficulty, that the determinants themselves are not integral functions of the cofficients. The former difficulty is easily got over as follows: Every even permutation of the x's multiplies any one of the determinants by some constant factor, which may in particular cases be unity. All permutations of the x's, therefore, give the determinant a set of 32.720 values, all of which are constant multiples of any one of them (equality being regarded as a particular case). All these values of the determinant have, therefore, the same group of substitutions, which will therefore be a self-conjugate sub-group of the alternating group of six elements. But there are only two such sub-groups, namely, the alternating group itself and the group consisting of the identical substitution. The determinant therefore belongs to the entire group of 720 substitutions and is rational, or it belongs to the alternating group and is rational after the adjunction of the square root of the discriminant, or it belongs to the identical substitution and is 32.720-valued. That the last does not occur is evident from the consideration of the group itself, one of the substitutions of which interchanges ξ and ζ, and a second η and ϑ, the product of the two, therefore, leaving each determinant unchanged. The determinants are, therefore, of the form $\frac{S_1+S_2\sqrt{\Delta}}{S_3+S_4\sqrt{\Delta}}$, where S_1, S_2, S_3, S_4 denote integral rational functions of the coefficients of the proposed equation. To determine more completely the nature of the determinant, we must observe how it behaves when the x's undergo odd permutations. On the one hand, these will all convert $\frac{S_1+S_2\sqrt{\Delta}}{S_3+S_4\sqrt{\Delta}}$ into $\frac{S_1-S_2\sqrt{\Delta}}{S_3-S_4\sqrt{\Delta}}$. On the other hand, if we interchange, say, x_1 and x_2, ξ^2 and η^2 are unchanged, while ζ^2 and ϑ^2 are each multiplied by -1. The 10 conditions between the

$\xi, \zeta, \eta, \vartheta$ are satisfied if we put for $\xi, \eta, \zeta, \vartheta, \xi, \eta, i\zeta, i\vartheta$. The interchange of x_1 and x_2, therefore, changes the sign of each determinant. That these two results may agree, we must have $S_1 S_3 + S_x S_y \Delta = 0$. From this either $\frac{S_1 + S_2\sqrt{\Delta}}{S_3 + S_4\sqrt{\Delta}} = \frac{S_2\sqrt{\Delta}}{S_3}$, or $S_1 = S_4 = 0$, or $S_2 = S_3 = 0$. In any case we see that our determinants are of the form $\frac{S}{S'\sqrt{\Delta}}$, where the S's are rational integral functions of the coefficients.

The second difficulty would be more formidable if the nature of the Borchardt moduli themselves were not in our favor. For it will appear that for these moduli the function S' is a power of the discriminant, so that the denominator is fully determined from the start, except that the index of the power is unknown. This simplification will also occur, as we shall see, in the case of any of the systems of moduli proposed by Klein, it being a sufficient condition for rational moduli that they shall be integral. That it is the case with the Borchardt moduli appears in actually differentiating. Suppose, for instance, we consider the differential equation with respect to the coefficient of x. For the first derivative of ξ we have

$$\frac{d\xi}{df} = \frac{d\xi}{dx_1}\frac{dx_1}{df} + \frac{d\xi}{dx_2}\frac{dx_2}{df} + \frac{d\xi}{dx_3}\frac{dx_3}{df} + \frac{d\xi}{dx_4}\frac{dx_4}{df} + \frac{d\xi}{dx_5}\frac{dx_5}{df} + \frac{d\xi}{dx_6}\frac{dx_6}{df},$$

and if our equation be

$$fx = x^6 + dx^3 + ex^2 + fx + g = 0, \quad \frac{dx_i}{df} = -\frac{x_i}{f'x_i} = -\frac{x_i}{\pi(x'_j - x_i)},$$

where j takes every value from 1 to 6 except i. Again, each of the derivatives of ξ with respect to the x_i will evidently contain in the denominator only powers of ξ and products of differences of roots. The denominator of $\frac{d\xi}{df}$ will, therefore, be composed entirely of products of differences of roots and powers of ξ, and it is evident that the same will be true for all the derivatives of any order of $\xi, \eta, \zeta, \vartheta$ taken with respect to any coefficient. For future reference I give the six first derivatives of ξ with respect to the x_i:

$$\begin{aligned}\frac{d\xi}{dx_1} = \frac{1}{2\xi}\Bigg\{ & \frac{(x_3 - x_5)(x_3 - 2x_1 + x_5)(x_2 - x_4)(x_4 - x_6)(x_6 - x_2)}{2\sqrt{(x_1 - x_3)(x_3 - x_5)(x_5 - x_1)(x_2 - x_4)(x_4 - x_6)(x_6 - x_2)}} \\ & + \frac{(x_4 - x_5)(x_4 - 2x_1 + x_5)(x_2 - x_3)(x_3 - x_6)(x_6 - x_2)}{2\sqrt{(x_1 - x_4)(x_4 - x_5)(x_5 - x_1)(x_2 - x_3)(x_3 - x_6)(x_6 - x_2)}} \\ & + \frac{(x_3 - x_6)(x_3 - 2x_1 + x_6)(x_2 - x_4)(x_4 - x_5)(x_5 - x_2)}{2\sqrt{(x_1 - x_3)(x_3 - x_6)(x_6 - x_1)(x_2 - x_4)(x_4 - x_5)(x_5 - x_2)}} \\ & + \frac{(x_4 - x_6)(x_4 - 2x_1 + x_6)(x_2 - x_3)(x_3 - x_5)(x_5 - x_2)}{2\sqrt{(x_1 - x_4)(x_4 - x_6)(x_6 - x_1)(x_2 - x_3)(x_3 - x_5)(x_5 - x_2)}}\Bigg\},\end{aligned}$$

$$\frac{d\xi}{dx_3}=\frac{1}{2\xi}\left\{\frac{(x_5-x_1)(x_5-2x_3+x_1)(x_2-x_4)(x_4-x_6)(x_6-x_2)}{2\sqrt{(x_1-x_3)(x_3-x_5)(x_5-x_1)(x_2-x_4)(x_4-x_6)(x_6-x_2)}}\right.$$
$$+\frac{(x_1-x_4)(x_4-x_5)(x_5-x_1)(x_6-x_2)(x_6-2x_3+x_2)}{2\sqrt{(x_1-x_4)(x_4-x_5)(x_5-x_1)(x_2-x_3)(x_3-x_6)(x_6-x_2)}}$$
$$+\frac{(x_6-x_1)(x_6-2x_3+x_1)(x_2-x_4)(x_4-x_5)(x_5-x_1)}{2\sqrt{(x_1-x_3)(x_3-x_6)(x_6-x_1)(x_2-x_4)(x_4-x_5)(x_5-x_2)}}$$
$$\left.+\frac{(x_1-x_4)(x_4-x_6)(x_6-x_1)(x_5-x_2)(x_5-2x_3+x_2)}{2\sqrt{(x_1-x_4)(x_4-x_6)(x_6-x_1)(x_2-x_3)(x_3-x_5)(x_5-x_2)}}\right\},$$

$$\frac{d\xi}{dx_5}=\frac{1}{2\xi}\left\{\frac{(x_1-x_3)(x_1-2x_5+x_3)(x_2-x_4)(x_4-x_6)(x_6-x_2)}{2\sqrt{(x_1-x_3)(x_3-x_5)(x_5-x_1)(x_2-x_4)(x_4-x_6)(x_6-x_2)}}\right.$$
$$+\frac{(x_1-x_4)(x_1-2x_5+x_4)(x_2-x_3)(x_3-x_6)(x_6-x_2)}{2\sqrt{(x_1-x_4)(x_4-x_5)(x_5-x_1)(x_2-x_3)(x_3-x_6)(x_6-x_2)}}$$
$$+\frac{(x_2-x_4)(x_2-2x_5+x_4)(x_1-x_3)(x_3-x_6)(x_6-x_1)}{2\sqrt{(x_1-x_3)(x_3-x_6)(x_6-x_1)(x_2-x_4)(x_4-x_5)(x_5-x_2)}}$$
$$\left.+\frac{(x_2-x_3)(x_2-2x_5+x_3)(x_1-x_4)(x_4-x_6)(x_6-x_1)}{2\sqrt{(x_1-x_4)(x_4-x_6)(x_6-x_1)(x_2-x_3)(x_3-x_5)(x_5-x_2)}}\right\}.$$

The other three are obtained from these by permuting x_1 and x_2, x_3 and x_4, x_5 and x_6 respectively.

If, now, we expand our determinants, we shall get a fractional function of which the denominator will be composed of products of differences of roots and some power of the product $\xi, \eta, \zeta, \vartheta$. Since $2(\xi\zeta+\eta\vartheta)=\sqrt{(124)(356)}$ and $2(\xi\zeta-\eta\vartheta)=\sqrt{(123)(456)}$, we have

$$8\xi\zeta\eta\vartheta = 124.356-123.456 = 12.56.(24.41.35.63-23.31.45.64)$$
$$= 12.34.56\big((x_3x_4-x_1x_2)(x_5+x_6)+(x_5x_6-x_3x_4)(x_1+x_2)+(x_1x_2-x_5x_6)(x_3+x_4)\big).$$

Since, now, the denominator is either symmetrical or alternating, it is evident that every difference of a pair of roots will occur as often as any other difference. These differences may therefore be grouped together and give some power of the $\sqrt{\Delta}$. There remains over a power of the parenthesis above. This is neither symmetrical nor alternating, but is 30-valued. *The numerator, then, must be of such a nature that this 30-valued function cancels out,* and this shows at once that the denominator is simply some power of the square root of the discriminant. That this is an odd power is evident from the above.

For the rational functions proposed by Klein, nothing but differences of the roots would occur in the denominator from the start, and for them the denominator would also be merely a power of the square root of the discriminant.

The knowledge of the nature of our determinants obtained by this process of reasoning suffices to remove all theoretical difficulties in their computation, which is now perfectly straightforward, but extremely long and tedious. It is necessary to determine to what power the square root of the discriminant enters in the denominator. From this the degree of the numerator is easily ascertained, and the numerator itself is got at once by multiplying our final infinite series by the denominator.

In the following computations, I have aimed at complete accuracy above everything else. Every step has been thoroughly tested by parallel methods, so that there is scarcely any possibility of error in the determination of the infinite series for $\xi, \eta, \zeta, \vartheta$. Errors may have crept in in the final determination of the coefficients in their rational form. These will readily be detected when the coefficients have been finally determined in full. Although the value which these determinations may have depends entirely on their accuracy, I have not thought it best to give them in full, on account of their great length. I have deposited a copy of them in the library of Harvard University. They may be divided into four parts: 1st. The calculation of the roots in terms of the coefficients. This I have carried out, for the case where the coefficients x^5, x^4, x^3 and x^2 are 0, to eleven terms, as follows:

$$x = \varepsilon - \frac{\varepsilon^2}{6}\alpha - \frac{\varepsilon^3}{24}\alpha^2 - \frac{\varepsilon^4}{81}\alpha^3 - \frac{91\varepsilon^5}{3^5.2^7}\alpha^4 + \frac{4301\varepsilon}{3^8.2^{10}}\alpha^6 + \frac{11\varepsilon^2}{3^9}\alpha^7 + \frac{11\varepsilon^3}{2^{15}}\alpha^8 + \frac{247\varepsilon^4}{3^{13}}\alpha^9 + \frac{60911435\varepsilon^5}{3^{14}.2^{18}}\alpha^{10} + 0\alpha^{11}.$$

In the accompanying table I have given only the test of the accuracy of the determination, by actually substituting the value of x in the equation, but I obtained the coefficient in another way. It will be noted that every sixth coefficient, beginning with the fifth, vanishes, which fact is of considerable assistance in the computation, and is a good test of correctness. In getting the expansion, I have supposed the equation to be divided through by $-\delta$, and have put for x, $\frac{x}{(-\delta)^{\frac{1}{6}}}$, and for α $\frac{\alpha}{(-\delta)^{\frac{5}{6}}}$, so that the Borchardt moduli must be multiplied finally by $\sqrt[6]{-\delta}$ and for α we must put $\frac{\alpha}{(-\delta)^{\frac{5}{6}}}$.

2d. The determination of the products (135)(246), etc. This is much facilitated by the following considerations: The first terms of the six series for the roots are 1, ε, ε^2, -1, $-\varepsilon$, $-\varepsilon^2$, in order. Consider now the equation of which the roots are ε^2 times the present ones. This equation will be of the form $x^6 - \varepsilon a x - 1 = 0$. The roots of this equation will be got by putting $-\varepsilon\alpha$ for α in the six series above. If now we calculate for this new system of roots the products (135)(246), etc., these will evidently be equal to the former products, but in a different order; for each root has been moved forward two places in the order 123456, the first being multiplied by ε^2 and becoming the third so far as the first term is concerned. Having obtained (145)(236), we get therefore (146)(235) and (136)(245) by putting for x, $-\varepsilon\alpha$ and $\varepsilon^2\alpha$ respectively. It is therefore only necessary to calculate (135)(246) and (145)(236), and in case of the former, (246) is at once obtainable from (123) by putting $-\varepsilon^2\alpha$ for α. We have

$$(135) = -3(\varepsilon + \varepsilon^2)\left(1 - \frac{7}{216}\alpha^3 + \frac{1957}{3^6 . 2^{10}}\alpha^6 + \frac{240691}{3^9 . 2^{15}}\alpha^9 \ldots\right),$$

$$(246) = -3(\varepsilon + \varepsilon^2)\left(-1 - \frac{7}{216}\alpha^3 - \frac{1957}{3^6 . 2^{10}}\alpha^6 + \frac{240691}{3^9 . 2^{15}}\alpha^9 \ldots\right),$$

$$\begin{aligned}(145) = 2 - 2\varepsilon^2 &+ \frac{2(1+\varepsilon)}{3}\alpha + \frac{\varepsilon - 6}{12}\alpha^2 - \frac{7 - 14\varepsilon}{2^2 . 3^4}\alpha^3 - \frac{350 - 91\varepsilon}{3^5 . 2^6}\alpha^4 \\ &- \frac{196 - 98\varepsilon}{3^6 . 2^7}\alpha^5 + \frac{35814 - 17613\varepsilon}{3^8 . 2^9}\alpha^6 - \frac{34130 + 34130\varepsilon}{3^9 . 2^{10}}\alpha^7 \\ &+ \frac{3210518 - 358723\varepsilon}{3^{10} . 2^{14}}\alpha^8 + \frac{14055494 - 28110988\varepsilon}{3^{13} . 2^{15}}\alpha^9 \\ &+ \frac{214802770 - 57182027\varepsilon}{3^{14} . 2^{17}}\alpha^{10} + \frac{73539718 - 36769859\varepsilon}{3^{15} . 2^{17}}\alpha^{11},\end{aligned}$$

$$\begin{aligned}(236) = 2(\varepsilon + 1) &+ \frac{2 - 4\varepsilon}{3}\alpha + \frac{6\varepsilon - 1}{12}\alpha^2 - \frac{7 - 14\varepsilon}{9 . 36}\alpha^3 + \frac{350 - 259\varepsilon}{3^5 . 2^6}\alpha^4 \\ &+ \frac{98 - 196\varepsilon}{3^6 . 2^7}\alpha^5 + \frac{18201 + 17613\varepsilon}{3^8 . 2^9}\alpha^6 - \frac{34130 - 68260\varepsilon}{3^9 . 2^{10}}\alpha^7 \\ &+ \frac{358723 - 3210518\varepsilon}{3^{10} . 2^{14}}\alpha^8 + \frac{14055494 - 28110988\varepsilon}{3^{13} . 2^{15}}\alpha^9 \\ &- \frac{214802770 - 157620743\varepsilon}{3^{14} . 2^{17}}\alpha^{10} - \frac{36769859 - 73539718\varepsilon}{3^{15} . 2^{17}}\alpha^{11},\end{aligned}$$

$$135 . 246 = -27\left(-1 - \frac{1565}{3^6 . 2^9}\alpha^6 \ldots\right) = 27 + \frac{1565}{3^8 . 2^9}\alpha^6 + \ldots,$$

$$\begin{aligned}145 . 236 = 12 &+ \frac{5(1-\varepsilon)}{6}\alpha^2 + \frac{20(\varepsilon + \varepsilon^2)}{27}\alpha^3 - \frac{2352}{3^3 . 2^5}\alpha^4 + \frac{30095}{3^7 . 2^8}\alpha^6 - \frac{112915(1-\varepsilon)}{3^7 . 2^{13}}\alpha^8 \\ &- \frac{118285(\varepsilon^2 + \varepsilon)}{3^{13} . 2^5}\alpha^9 + \frac{1512905\varepsilon}{3^8 . 2^{16}}\alpha^{10}.\end{aligned}$$

The disappearance of terms in regular order again furnishes a valuable control of the accuracy up to this point.

3d. The extraction of the square roots of the products above:

$$\sqrt{(x_1-x_4)(x_4-x_5)(x_5-x_1)(x_2-x_3)(x_3-x_6)(x_6-x_2)}$$
$$=2\sqrt{3}+\frac{5(1-\varepsilon)}{24\sqrt{3}}\alpha^2+\frac{20(\varepsilon+\varepsilon^2)}{108\sqrt{3}}\alpha^3-\frac{445\varepsilon}{3^9.2^8\sqrt{3}}\alpha^4-\frac{25(\varepsilon+1)}{3^5.2^4\sqrt{3}}\alpha^5+\frac{217205}{3^7.2^{13}\sqrt{3}}\alpha^6$$
$$+\frac{1975(\varepsilon^2-1)}{3^7.2^9\sqrt{3}}\alpha^7-\frac{13763755(1-\varepsilon)}{3^9.2^{18}\sqrt{3}}\alpha^8-\frac{11395315(\varepsilon+\varepsilon^2)}{3^{12}.2^{18}\sqrt{3}}\alpha^9$$
$$+\frac{891732635\varepsilon}{3^{11}.2^{23}\sqrt{3}}\alpha^{10}+\frac{452064875(1+\varepsilon)}{3^{14}.2^{19}\sqrt{3}}\alpha^{11},$$

$$\sqrt{(x_1-x_3)(x_3-x_6)(x_6-x_1)(x_2-x_4)(x_4-x_5)(x_5-x_2)}$$
$$=2\sqrt{3}-\frac{5}{24\sqrt{3}}\alpha^2+\frac{20(\varepsilon+\varepsilon^2)}{108\sqrt{3}}\alpha^3+\frac{445}{3^8.2^8\sqrt{3}}\alpha^4+\frac{25(\varepsilon+\varepsilon^2)}{3^5.2^4\sqrt{3}}\alpha^5+\frac{217205}{3^7.2^{13}\sqrt{3}}\alpha^6$$
$$-\frac{1975(\varepsilon+\varepsilon^2)}{3^7.2^9\sqrt{3}}\alpha^7+\frac{13763755}{3^9.2^{18}\sqrt{3}}\alpha^8-\frac{11395315(\varepsilon+\varepsilon^2)}{3^{12}.2^{13}\sqrt{3}}\alpha^9-\frac{891732635}{3^{11}.2^{23}\sqrt{3}}\alpha^{10}$$
$$-\frac{452064875(\varepsilon+\varepsilon^2)}{3^{14}.2^{19}\sqrt{3}}\alpha^{11},$$

$$\sqrt{(x_1-x_4)(x_4-x_6)(x_6-x_1)(x_2-x_3)(x_3-x_5)(x_5-x_2)}$$
$$=2\sqrt{3}+\frac{5(1+\varepsilon^2)}{24\sqrt{3}}\alpha^2+\frac{20(\varepsilon+\varepsilon^2)}{108\sqrt{3}}\alpha^3+\frac{445\varepsilon^2}{3^8.2^8\sqrt{3}}\alpha^4+\frac{25(1+\varepsilon^2)}{3^5.2^4\sqrt{3}}\alpha^5+\frac{217205}{3^7.2^{13}\sqrt{3}}\alpha^6$$
$$+\frac{1975(1+\varepsilon)}{3^7.2^9\sqrt{3}}\alpha^7-\frac{13763755\varepsilon}{3^9.2^{18}\sqrt{3}}\alpha^8-\frac{11395315(\varepsilon+\varepsilon^2)}{3^{12}.2^{13}\sqrt{3}}\alpha^9-\frac{891732635\varepsilon^2}{3^{11}.2^{23}\sqrt{3}}\alpha^{10}$$
$$+\frac{452064875(\varepsilon^2-1)}{3^{14}.2^{19}\sqrt{3}}\alpha^{11},$$

$$\sqrt{(x_1-x_3)(x_3-x_5)(x_5-x_1)(x_2-x_4)(x_4-x_6)(x_6-x_2)}=3\sqrt{3}\ldots+\frac{1565}{3^4.2^{10}\sqrt{3}}\alpha^6,$$

the combining of the results so as to give ξ^2, η^2, ζ^2, ϑ^2:

$$4\xi^2=9\sqrt{3}+\frac{5(\varepsilon+\varepsilon^2)}{9\sqrt{3}}\alpha^3+\frac{273545}{3^6.2^{12}\sqrt{3}}\alpha^6-\frac{11395315(\varepsilon+\varepsilon^2)}{3^{11}.2^{18}\sqrt{3}}\alpha^9,$$

$$4\zeta^2=\sqrt{3}-\frac{5\varepsilon^2}{12\sqrt{3}}\alpha^2-\frac{5(\varepsilon+\varepsilon^2)}{27\sqrt{3}}\alpha^3-\frac{445\varepsilon}{3^3.2^7\sqrt{3}}\alpha^4-\frac{25(\varepsilon+1)}{3^5.2^3\sqrt{3}}\alpha^5-\frac{48185}{3^7.2^{13}\sqrt{3}}\alpha^6$$
$$+\frac{1975(\varepsilon^2-1)}{3^7.2^8\sqrt{3}}\alpha^7+\frac{13763755(\varepsilon-1)}{3^9.2^{17}\sqrt{3}}\alpha^8+\frac{11395315(\varepsilon+\varepsilon^2)}{3^{12}2^{13}\sqrt{3}}\alpha^9$$
$$+\frac{891732635\varepsilon}{3^{11}.2^{21}\sqrt{3}}\alpha^{10}+\frac{452064875(1+\varepsilon)}{3^{14}.2^{18}\sqrt{3}}\alpha^{11},$$

$$4\vartheta^2=\sqrt{3}-\frac{5}{12\sqrt{3}}\alpha^2-\frac{5(\varepsilon+\varepsilon^2)}{27\sqrt{3}}\alpha^3+\frac{445}{3^3.2^7\sqrt{3}}\alpha^4+\frac{25(\varepsilon+\varepsilon^2)}{3^5.2^3\sqrt{3}}\alpha^5-\frac{48185}{3^7.2^{13}\sqrt{3}}\alpha^6$$
$$-\frac{1975(\varepsilon^2+\varepsilon)}{3^7.2^8\sqrt{3}}\alpha^7+\frac{13763755}{3^9.2^{17}\sqrt{3}}\alpha^8+\frac{11395315(\varepsilon+\varepsilon^2)}{3^{12}.2^{18}\sqrt{3}}\alpha^9$$
$$-\frac{891732635}{3^{11}.2^{21}\sqrt{3}}\alpha^{10}-\frac{452064875(\varepsilon+\varepsilon^2)}{3^{14}.2^{18}\sqrt{3}}\alpha^{11},$$

$$4\eta^2 = \sqrt{3} + \frac{5\varepsilon}{12\sqrt{3}}\alpha^2 - \frac{5(\varepsilon+\varepsilon^2)}{27\sqrt{3}}\alpha^3 + \frac{445\varepsilon^2}{3^3.2^7\sqrt{3}}\alpha^4 + \frac{25(1-\varepsilon^2)}{3^5.2^8\sqrt{3}}\alpha^5 - \frac{48185}{3^7.2^{13}\sqrt{3}}\alpha^6$$
$$+ \frac{1975(\varepsilon+1)}{3^7.2^8\sqrt{3}}\alpha^7 - \frac{13763755\varepsilon}{3^9.2^{17}}\alpha^8 + \frac{11395315(\varepsilon+\varepsilon^2)}{3^{12}.2^{13}\sqrt{3}}\alpha^9$$
$$- \frac{891732635\varepsilon^2}{3^{11}.2^{21}\sqrt{3}}\alpha^{10} + \frac{452064875(\varepsilon^2-1)}{3^{14}2^{18}\sqrt{3}}\alpha^{11},$$

and the extraction of the square roots in each case:

$$\xi = \frac{3}{2}\sqrt[4]{3} + \frac{5(\varepsilon+\varepsilon^2)}{108\sqrt[4]{27}}\alpha^3 + \frac{99715}{3^6.2^{14}\sqrt[4]{27}}\alpha^6 - \frac{402845(\varepsilon+\varepsilon^2)}{3^{12}.2^{10}\sqrt[4]{27}}\alpha^9,$$

$$\zeta = \frac{1}{2}\sqrt[4]{3} - \frac{5\varepsilon^2}{48\sqrt[4]{27}}\alpha^2 - \frac{5(\varepsilon+\varepsilon^2)}{108\sqrt[4]{27}}\alpha^3 - \frac{395\varepsilon}{3^3.2^9\sqrt[4]{27}}\alpha^4 + \frac{99715}{3^7.2^{14}\sqrt[4]{27}}\alpha^6 + \frac{479615\varepsilon^2}{3^6.2^{19}\sqrt[4]{27}}\alpha^8$$
$$+ \frac{402845(\varepsilon+\varepsilon^2)}{3^{12}.2^{10}\sqrt[4]{27}}\alpha^9 + \frac{93445715\varepsilon}{3^9.2^{23}\sqrt[4]{27}}\alpha^{10},$$

$$\vartheta = \frac{1}{2}\sqrt[4]{3} - \frac{5\alpha^2}{48\sqrt[4]{27}} - \frac{5(\varepsilon+\varepsilon^2)}{108\sqrt[4]{27}}\alpha^3 + \frac{395}{3^3.2^9\sqrt[4]{27}}\alpha^4 + \frac{99715}{3^7.2^{14}\sqrt[4]{27}}\alpha^6 + \frac{479615}{3^6.2^{19}\sqrt[4]{27}}\alpha^8$$
$$+ \frac{402845(\varepsilon+\varepsilon^2)}{3^{12}.2^{10}\sqrt[4]{27}}\alpha^9 - \frac{93445715}{3^9.2^{23}\sqrt[4]{27}}\alpha^{10},$$

$$\eta = \frac{1}{2}\sqrt[4]{3} + \frac{5\varepsilon\alpha^2}{48\sqrt[4]{27}} - \frac{5(\varepsilon+\varepsilon^2)}{108\sqrt[4]{27}}\alpha^3 + \frac{395\varepsilon^2}{3^3.2^9\sqrt[4]{27}}\alpha^4 + \frac{99715}{3^7.2^{14}\sqrt[4]{27}}\alpha^6 - \frac{479615\varepsilon}{3^6.2^{19}\sqrt[4]{27}}\alpha^8$$
$$+ \frac{402845(\varepsilon+\varepsilon^2)}{3^{12}.2^{10}\sqrt[4]{27}}\alpha^9 - \frac{93445715\varepsilon^2}{3^9.2^{23}\sqrt[4]{27}}\alpha^{10}.$$

Here again terms vanish in regular order; namely, for ξ, all terms except those whose order is divisible by 3, and for ζ, η, ϑ, the 5th, 7th, 11th, 13th, etc. Further, the relation between the coefficients of ξ and those of the corresponding terms of ζ, η, ϑ, is a valuable check. These I have tested by means of $2(\xi\zeta - \eta\vartheta) = \sqrt{123.456}$, which ensures the correctness of the values of the $\xi\eta\zeta\vartheta$.

4th. The determination of the coefficients of the differential equation. First of all, the degree of the denominator must be obtained. The values of α for which two roots are equal are got by equating the discriminant to $0.6^6.\alpha^6 - 5^5 = 0$.* This gives 6 finite values of α, each of which is, in our case, a point of the first order; *i. e.* only two roots are equal for any one of these values of α. We may therefore expand the roots in ascending powers of $\alpha - \alpha_0$, if α_0 be one of these values. For those roots which do not coincide we shall have only integral powers, while for the two coincident roots the $\frac{1}{2}$ powers of $\alpha - \alpha_0$ will occur. If these series be introduced into the Borchardt moduli, as may

* See Salmon's Higher Algebra, p. 263.

legitimately be done, we shall have the latter expanded in a system of convergent series, in which the $\frac{1}{4}$ and $\frac{1}{8}$ powers may occur, *but no negative powers.* If there be written $a_0 + a_1(\alpha - \alpha_0)^{\frac{1}{4}} + \ldots . \; b_0 + b_1(\alpha - \alpha_0)^{\frac{1}{4}} + \ldots .$, etc., and if these series be differentiated and substituted in the determinants, we may determinate the highest power to which $\alpha - \alpha_0$ appears in the denominator, and this will be the upper limit of the power of any factor $x_1 - x_2$ in the denominator, and therefore of the square root of the discriminant to be determined. This matter is considerably simplified by the following considerations: If two roots are equal, say x_1 and x_2, since $x_1 - x_2$ is not a factor of any of the square roots $\sqrt{(135)(246)}$, etc., occurring in the Borchardt moduli, these square roots will involve only $\frac{1}{2}$ powers and integral powers of $(\alpha - \alpha_0)$. Again, if two roots are equal, two of the moduli vanish, while two do not. The two which do not vanish will contain only $\frac{1}{2}$ powers and integral powers of $\alpha - \alpha_0$, while the other two may also involve $\frac{1}{4}$ powers. Consider now the first coefficient in the differential equation for α. The numbers in the determinant denote the highest negative power of $\alpha - \alpha_0$ which occurs in the corresponding infinite series:

$$\begin{vmatrix} 0 & 0 & \frac{1}{4} & \frac{1}{4} \\ -\frac{1}{2} & -\frac{1}{2} & -\frac{3}{4} & -\frac{3}{4} \\ -\frac{3}{2} & -\frac{3}{2} & -\frac{7}{4} & -\frac{7}{4} \\ -\frac{5}{2} & -\frac{5}{2} & -\frac{11}{4} & -\frac{11}{4} \end{vmatrix} .$$

The highest negative power of $\alpha - \alpha_0$ which will occur in the development of this determinant will be the power whose index is $\frac{11}{4} + \frac{7}{4} + \frac{1}{2} = 5$. But since the square root of the discriminant must occur to an odd power, we see that $\Delta^{\frac{3}{2}}$ will contain the denominator as a factor.

But ξ, η, ζ, ϑ are homogeneous functions of the x's of degree $\frac{3}{2}$, and α is homogeneous of degree 5. Hence the determinant just considered will be of degree $4 . \frac{3}{2} - 5 - 10 - 15 = -24$. Since the square root of the discriminant is of degree 15 in the roots, it must enter at least to the third power in the denominator, which is therefore either $\Delta^{\frac{3}{2}}$, $\Delta^{\frac{5}{2}}$, $\Delta^{\frac{7}{2}}$ or $\Delta^{\frac{9}{2}}$. Having now expanded the determinant in an infinite series, we may try these denominators successively. First, if the denominator is $\Delta^{\frac{3}{2}}$, the numerator will be of degree $3 . 15 - 24 = 21$.

The only combination of α and δ which is of this degree is $\alpha^3\delta$; the numerator could therefore contain only this one term, and all following terms must cancel. This is not the case. As I have obtained only two terms of the infinite series in $\frac{\alpha}{\delta}$, it is impossible to proceed further in this way. I have therefore begun at the other end of the coefficient, and have obtained a development of the moduli in terms of $\frac{\delta}{\alpha}$ instead of $\frac{\alpha}{\delta}$. This will serve as a very perfect test of the work ultimately, and the consideration of the series in this form is instructive as showing that the term which would contain the lowest power of δ must vanish in every coefficient of the differential equation with respect to α. For consider the four series in $\frac{\delta}{\alpha}$, $\frac{a_0}{\alpha^\mu} + a_1 \frac{\delta}{\alpha^\nu} +$ etc., $\frac{b_0}{\alpha^\mu} + b_1 \frac{\delta}{\alpha^\nu}$, etc. Differentiate these, and form the determinant and expand in powers of $\frac{\delta}{\alpha}$. Evidently the constant term will be

$$a_0b_0c_0d_0 \begin{vmatrix} 1 & 1 & \\ \mu - 1 & \mu - 1 & \\ (\mu-2)(\mu-1) & (\mu-2)(\mu-1), & \text{etc.,} \\ (\mu-3)(\mu-2)(\mu-1) & (\mu-3)(\mu-2)(\mu-1) & \end{vmatrix},$$

and is therefore 0. The same will be true for every coefficient which is not obtained from four sets of coefficients of different powers of δ. Evidently the first power of δ for which this is possible is the sixth power, which can be obtained by taking the constant term from one set, the term of 1st degree from another, and the terms of 2d and 3d degrees from the other two. *The lowest power of δ which appears in any coefficient of the differential equation for α will be the sixth power.*

We can now test the hypothesis $\Delta^{\frac{5}{2}}$, since, in this case, the numerator contains only the term $\alpha^3\delta^6$, and this also fails. Finally, I have actually calculated the coefficient of δ^6 in this coefficient by the process indicated above, and which evidently requires only three terms of the series in $\frac{\delta}{\alpha}$. This coefficient having been determined, $\Delta^{\frac{7}{2}}$ can be tested, and it too fails. It follows, therefore, that the denominator of the first coefficient is $\Delta^{\frac{9}{2}}$. This being known, it appears at once that the following denominators are $\Delta^{\frac{11}{2}}$, $\Delta^{\frac{13}{2}}$, $\Delta^{\frac{15}{2}}$, $\Delta^{\frac{17}{2}}$. The numerators can then be written at once with arbitrary coefficients. Thus that of the first coefficient is $p_1\alpha^3\delta^{16} + p_2\alpha^9\delta^{11} + p_3\alpha^{15}\delta^6$. The last coefficient is determined in each case from the $\frac{\delta}{\alpha}$ series, the first two from the $\frac{\alpha}{\delta}$ series.

(In determining the coefficient I have used both the method of undetermined coefficients and that of the expansion of the determinant.)

This gives us at once all the terms of the first coefficient. We notice now that the second coefficient is the derivative of the first, so that all its terms are known. In the other coefficients, two, three and four terms respectively are still wanting. To obtain them both, the series must be extended further. These further extensions will also check each other as they meet, ensuring absolute accuracy in the result. The differential equation for a is, so far as I have determined it,

$$A\frac{d^4\xi}{da^4} - B\frac{d^3\xi}{da^3} + C\frac{d^2\xi}{da^2} - D\frac{d\xi}{da} + E = 0, \text{ where}$$

$$A = \frac{-3^{21}.2^{17}.5^3.79a^3\delta^{16} + 3^{15}.2^8.5^8.4649a^9\delta^{11} \pm 19.3^{11}.5^{13}.2^2.a^{15}\delta^6}{\Delta^{\frac{9}{2}}}\,i,$$

$$B = \frac{dA}{da} = \frac{-3^{28}.2^{23}.5^3.79a^2\delta^{21} + 3^{22}.2^{14}.5^8.8891a^8\delta^{16} + 3^{17}.2^8.5^{13}(9298 \pm 985)\,a^{14}\delta^{11} \pm 19.5^{18}.2^4.3^{12}a^{20}\delta^6}{\Delta^{\frac{11}{2}}}\,i,$$

$$C = \frac{-5^3.79.3^{34}.2^{30}.a\delta^{26} - 5^7.3^{27}.2^{21}.6628a^7\delta^{21} + \ldots \pm 19.3^{11}.2^2.5^{23}.a^{25}.\delta^6}{\Delta^{\frac{13}{2}}}\,i,$$

$$D = \frac{125.79.3^{40}.2^{36}a\delta^{31} + 5^7.3^{33}.2^{26}.393418a^7\delta^{26} + \ldots \pm 19.3^{11}.2^4.5^{27}\,a^{35}.\delta^6}{\Delta^{\frac{15}{2}}},$$

$$E = \frac{3^{39}.2^{32}.125.79.99715a^5\delta^{36} + \ldots \pm 19.3^{16}.2^8.5^{29}a^{41}\delta^6}{\Delta^{\frac{17}{2}}}.$$

In the final results, the coefficients are by no means so irregular as at first appears. The powers of 2 and 3 may readily be combined with the variable, since their variation is so regular. This I have not cared to do until I could obtain the complete coefficients.

The further theory of these equations, which I intend to treat in future papers, will consist in the complete determination of all the coefficients, and the transformation into similar equations, referred to the four absolute invariants of the binary sextic as independent variables. This transformation can probably be effected by means of the "Typische Darstellung" of Clebsch.* The relation of the equations to the theory of the bisection of the hyperelliptic integrals of deficiency 2 constitutes a theory by itself. Finally, the theory of these equations should be studied in the manner which Schwarz has made so well known in the case of the linear differential equations of the second order with rational coefficients, and ultimately I hope to study them in connection with that latest of Sylvester's great methods—The Theory of Reciprocants.

CAMBRIDGE, June, 1886.

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* Clebsch: Binöm. Formen, p. 455.

A Proof of the Elliptic-Function Addition-Theorem.

By J. C. Fields.

The integration of the differential equation

$$\frac{d\phi}{\Delta\phi} + \frac{d\psi}{\Delta\psi} = 0 \quad (\Delta\phi = \sqrt{1 - k^2 \sin^2\phi}, \text{ etc.})$$

can be effected by aid of the factor

$$\frac{\Delta\phi\Delta\psi - k^2 \sin\phi \cos\phi \sin\psi \cos\psi}{1 - k^2 \sin^2\phi \sin^2\psi},$$

which, as will shortly be seen, is an integrating factor; thus,

$$0 = \frac{\Delta\phi\Delta\psi - k^2 \sin\phi \cos\phi \sin\psi \cos\psi}{1 - k^2 \sin^2\phi \sin^2\psi}\left(\frac{d\phi}{\Delta\phi} + \frac{d\psi}{\Delta\psi}\right)$$

$$= \frac{1}{1 - k^2 \sin^2\phi \sin^2\psi}\left(\Delta\psi d\phi - k^2 \sin\phi \cos\phi \sin\psi \cos\psi \frac{d\psi}{\Delta\psi}\right) + (\text{——});\text{*}$$

and since $1 - k^2 \sin^2\phi \sin^2\psi = \cos^2\phi + \sin^2\phi - k^2 \sin^2\phi \sin^2\psi = \cos^2\phi + \sin^2\phi\Delta^2\psi$,

$$\text{this} = \frac{1}{\cos^2\phi + \sin^2\phi\Delta^2\psi}\left(\Delta\psi d\phi - k^2 \sin\phi \cos\phi \sin\psi \cos\psi \frac{d\psi}{\Delta\psi}\right) + (\text{——})$$

$$= \frac{1}{1 + \tan^2\phi\Delta^2\psi}\left(\Delta\psi \sec^2\phi d\phi - k^2 \tan\phi \sin\psi \cos\psi \frac{d\psi}{\Delta\psi}\right) + (\text{——})$$

$$= \frac{1}{1 + \tan^2\phi\Delta^2\psi} d(\tan\phi\Delta\psi) + (\text{——})$$

$$= d(\tan^{-1}\tan\phi\Delta\psi) + d(\tan^{-1}\tan\psi\Delta\phi)$$

$$\therefore \tan^{-1}(\tan\phi\Delta\psi) + \tan^{-1}(\tan\psi\Delta\phi) = \mu;$$

i. e.
$$\frac{\tan\phi\Delta\psi + \tan\psi\Delta\phi}{1 - \tan\phi \tan\psi\Delta\phi\Delta\psi} = \tan\mu.$$

And since evidently $\mu = \phi$ when $\psi = 0$, μ is the amplitude of $(u + v)$, where u, v are the elliptic functions whose amplitudes are ϕ, ψ respectively.

---

* (——) being the same function of (ψ, ϕ) that the preceding term is of (ϕ, ψ).

The formula for sn $(u+v)$ can be very readily derived from above; thus,

$$\begin{aligned}\operatorname{sn}(u+v) = \sin\mu &= \frac{\tan\mu}{\sqrt{1+\tan^2\mu}}\\ &= \frac{\tan\phi\Delta\psi + \tan\psi\Delta\phi}{\sqrt{(\tan\phi\Delta\psi + \tan\psi\Delta\phi)^2 + (1-\tan\phi\,\tan\psi\Delta\phi\Delta\psi)^2}}\\ &= \frac{\tan\phi\Delta\psi + \tan\psi\Delta\phi}{\sqrt{1+\tan^2\phi\Delta^2\psi + \tan^2\psi\Delta^2\phi + \tan^2\phi\,\tan^2\psi\Delta^2\phi\Delta^2\psi}}\\ &= \frac{\tan\phi\Delta\psi + \tan\psi\Delta\phi}{\sqrt{(1+\tan^2\phi\Delta^2\psi)(1+\tan^2\psi\Delta^2\phi)}}\\ &= \frac{\sin\phi\,\cos\psi\Delta\psi + \cos\phi\,\sin\psi\Delta\phi}{\sqrt{(\cos^2\phi + \sin^2\phi\Delta^2\psi)(\cos^2\psi + \sin^2\psi\Delta^2\phi)}}\\ &= \frac{\sin\phi\,\cos\psi\Delta\psi + \cos\phi\,\sin\psi\Delta\phi}{1-k^2\sin^2\phi\,\sin^2\psi},\end{aligned}$$

since $\cos^2\phi + \sin^2\phi\Delta^2\psi = 1 - k^2\sin^2\phi\,\sin^2\psi = \cos^2\psi + \sin^2\psi\Delta^2\phi$;

i. e. $\operatorname{sn}(u+v) = \operatorname{sn} u \operatorname{cn} v \operatorname{dn} v + \operatorname{sn} v \operatorname{cn} u \operatorname{dn} u \div 1 - k^2 \operatorname{sn}^2 u \operatorname{sn}^2 v$.

The cn $(u+v)$ and dn $(u+v)$ can of course be just as readily obtained.

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AUGUST 1886.

Sur les Fonctions Abéliennes.

PAR H. POINCARÉ.

§ 1. *Réduction des Intégrales.*

J'ai donné dans le Bulletin de la Société Mathématique de France (tome 12, page 124) une démonstration et une généralisation de deux théorèmes de M. Weierstrass. Je veux d'abord rappeler ici succinctement, en y ajoutant quelques compléments, ce qu'il y a d'essentiel dans cette démonstration.

Soient $J_1, J_2, \ldots, J_\rho$, ρ intégrales abéliennes de rang ρ.

Soit $$x_1, x_2, \ldots, x_\rho;\ x'_1, x'_2, \ldots, x'_\rho$$
un système de périodes normales de J_1;
$$y_1, y_2, \ldots, y_\rho;\ y'_1, y'_2, \ldots, y'_\rho$$
les périodes correspondantes de J_2;

. . . .

$$t_1, t_2, \ldots, t_\rho;\ t'_1, t'_2, \ldots, t'_\rho$$
les périodes correspondantes de J_μ;

. . . .

et enfin $$u_1, u_2, \ldots, u_\rho;\ u'_1, u'_2, \ldots, u'_\rho$$
celles de J_ρ.

De telle façon que l'on ait:
$$x_1y'_1 - x'_1y_1 + x_2y'_2 - x'_2y_2 + \ldots + x_\rho y'_\rho - x'_\rho y_\rho = 0$$
et $\frac{\rho(\rho-1)}{2} - 1$ autres relations de même forme.

Imaginons maintenant que l'on puisse trouver $2\mu^2$ nombres:
$$\xi_1, \xi_2, \ldots, \xi_{2\mu};$$
$$\eta_1, \eta_2, \ldots, \eta_{2\mu},$$
$$\cdot\ \cdot\ \cdot\ \cdot\ \cdot\ \cdot\ \cdot\ ;$$
$$\tau_1, \tau_2, \ldots, \tau_{2\mu}$$

tels que les périodes de l'intégrale J_1 puissent se mettre sous la forme suivante:

$$x_i = \Sigma_k \alpha_{ik} \xi_k ; \; x'_i = \Sigma_k \alpha'_{ik} \xi_k$$

les $4\mu\rho$ nombres α_{ik} et α'_{ik} étant tous entiers.

Supposons que l'on ait de même pour les périodes de l'intégrale J_2:

$$y_1 = \Sigma \alpha_{ik} \eta_k ; \; y'_i = \Sigma \alpha'_{ik} \eta_k$$

(les nombres α_{ik} et α'_{ik} conservant les mêmes valeurs que plus haut) qu'il en soit de même pour les périodes des intégrales suivantes $J_3 \ldots .$ et qu'enfin pour les périodes de l'intégrale J_μ on ait:

$$t_i = \Sigma \alpha_{ik} \tau_k ; \; t'_i = \Sigma \alpha'_{ik} \tau_k .$$

Nous dirons alors que les μ intégrales $J_1, J_2, \ldots, J_\mu$ sont réductibles au genre μ.

Nous formerons le tableau des $4\mu\rho$ nombres entiers:

$$(1) \qquad \begin{vmatrix} \alpha_{1.1} & \alpha'_{1.1} & \alpha_{2.1} & \alpha'_{2.1} & \ldots & \alpha_{\rho.1} & \alpha'_{\rho.1} \\ \alpha_{1.2} & \alpha'_{1.2} & \alpha_{2.2} & \alpha'_{2.2} & \ldots & \alpha_{\rho.2} & \alpha'_{\rho.2} \\ \cdot & \cdot & \cdot & \cdot & \cdot & \cdot & \cdot \\ \alpha_{1.2\mu} & \alpha'_{1.2\mu} & \alpha_{2.2\mu} & \alpha'_{2.2\mu} & \ldots & \alpha_{\rho.2\mu} & \alpha'_{\rho.2\mu} \end{vmatrix} .$$

Le problème que nous nous proposons est de réduire ce tableau à sa plus simple expression.

Voici comment cette réduction peut se faire:

1°. Au lieu d'envisager un système de périodes normales

$$x_1, x_2, \ldots, x_\rho ; \; x'_1, x'_2, \ldots, x'_\rho$$

de l'intégrale J_1 et les périodes correspondantes des autres intégrales, on aurait pu envisager un *autre* système de périodes normales de J_1.

Par exemple:

$$x_1 + x'_1, x_2, \ldots, x_\rho ; \qquad x'_1, x'_2, \ldots, x'_\rho$$

ou bien $\qquad -x_1, x_2, \ldots, x_\rho ; \quad -x'_1, x'_2, \ldots, x'_\rho$

ou bien $\qquad -x'_1, x_2, \ldots, x_\rho ; \quad x_1, x'_2, \ldots, x'_\rho$

formeront encore trois systèmes de périodes normales de J_1.

Il sera donc permis:

Ou bien d'ajouter dans le tableau (1) aux termes d'une colonne de rang impair, les termes correspondants de la colonne de rang pair qui la suit (nous dirons pour abréger que ces deux colonnes appartiennent à la même paire).

Ou bien de changer de signe tous les termes des deux colonnes d'une même paire.

Ou bien de permuter deux colonnes d'une même paire en changeant tous les signes de l'une d'elles.

Plus généralement on peut faire l'opération suivante que j'appellerai l'opération A_i et qui n'est qu'une combinaison de celles dont nous venons de parler:

Remplacer les nombres α_{ik} et α'_{ik} par les nombres entiers β_{ik} et β'_{ik} définis comme il suit:

$$\beta_{ik} = a_i \alpha_{ik} + b_i \alpha'_{ik}\,; \quad \beta'_{ik} = c_i \alpha_{ik} + d_i \alpha'_{ik}$$

où $(a_1, b_1, c_1, d_1), (a_2, b_2, c_2, d_2), \ldots, (a_\rho, b_\rho, c_\rho, d_\rho)$

sont 4ρ nombres entiers satisfaisant aux ρ conditions

$$a_i d_i - b_i c_i = 1.$$

De même les périodes

$$-x_2, x_1, x_3, x_4, \ldots, x_\rho\,; \quad -x'_2, x'_1, x'_3, x'_4, \ldots, x'_\rho$$

ou bien $x_1 + x_2, x_2, x_3, \ldots, x_\rho\,; \quad x'_1, x'_2 - x'_1, x'_3, \ldots, x'_\rho$

sont encore des périodes normales.

Il est donc permis:

Ou bien de permuter deux paires de colonnes en changeant tous les signes de l'une d'entre elles.

Ou bien d'ajouter la première colonne d'une paire π à la première colonne d'une autre paire π' et de retrancher en même temps la deuxième colonne de la paire π' de la deuxième colonne de la paire π.

Plus généralement on peut faire l'opération suivante que j'appellerai l'opération B:

Remplacer les nombres α_{ik} et α'_{ik} par les nombres β_{ik} et β'_{ik} définis comme il suit:

$$\beta_{ik} = \Sigma_j a_{ij} \alpha_{jk}\,; \quad \beta'_{ik} = \Sigma_j b_{ij} \alpha'_{jk}$$

les a_{ij} et les b_{ij} étant des nombres entiers satisfaisant aux conditions suivantes: le déterminant des a_{ij} est égal à 1 de même que celui des b_{ij}; de plus on a:

$$\Sigma_i a_{ij} b_{ik} = 0 \text{ si } j \gtrless k$$

et

$$\Sigma_i a_{ij} b_{ij} = 1$$

de telle sorte que les deux substitutions linéaires définies, la première par les ρ^2 nombres a_{ij}, la seconde par les ρ^2 nombres b_{ij} soient deux substitutions corrélatives.

2°. Posons maintenant: $\xi'_k = \Sigma \alpha_{ik} \xi_i$

les a_{ik} étant des coëfficients entiers dont le déterminant est égal à 1. Supposons de plus que les quantités $\eta'_k, \ldots, \tau'_k$

soient formées à l'aide des $\eta, \ldots$ et des τ comme les ξ' sont formés avec les ξ.

Par hypothèse les périodes de l'intégrale J_1 sont des combinaisons linéaires à coëfficients entiers des quantités ξ et les périodes des intégrales $J_2, \ldots, J_\mu$ seront formées avec les quantités $\eta, \ldots, \tau$ comme celles de J_1 avec les quantités ξ.

De même (et cela se voit sans peine), les périodes de l'intégrale J_1 seront des combinaisons lineáires à coëfficients entiers des quantités ξ' et les périodes des intégrales $J_2, \ldots, J_\mu$ seront formées avec les quantités $\eta', \ldots, \tau'$ comme celles de J_1 avec les quantités ξ'.

Il est donc permis de faire l'opération suivante que j'appellerai l'opération C:

Remplacer les nombres α_{ik} et α'_{ik} par les nombres β_{ik} et β'_{ik} définis comme il suit:
$$\beta_{ik} = \Sigma_j a_{jk} \alpha_{ij} \, ; \, \beta'_{ik} = \Sigma_j a_{jk} \alpha'_{ij}$$
les coëfficients a_{jk} étant $4\mu^2$ nombres entiers dont le déterminant est égal à 1.

On peut en particulier permuter deux lignes du tableau (1) en changeant tous les signes de l'une d'elles, ou bien ajouter une ligne à une autre.

En d'autres termes, on conserve le même système de périodes normales, mais on remplace le système des quantités ξ auxquelles ces périodes peuvent se réduire par un système équivalent.

Le problème que je me propose est de réduire le tableau (1) à sa plus simple expression par le moyen des opérations A, B et C.

Envisageons d'abord le cas où $\mu = 1$, c'est à dire où l'intégrale J_1 est réductible aux intégrales elliptiques. Nous supposerons de plus, mais seulement pour fixer les idées, $\rho = 3$.

Le tableau (1) s'écrira alors:
$$\begin{vmatrix} \alpha_{1.1} & \alpha'_{1.1} & \alpha_{2.1} & \alpha'_{2.1} & \alpha_{3.1} & \alpha'_{3.1} \\ \alpha_{1.2} & \alpha'_{1.2} & \alpha_{2.2} & \alpha'_{2.2} & \alpha_{3.2} & \alpha'_{3.2} \end{vmatrix}.$$

Il est clair que les opérations A, B, C, appliquées au tableau précédent ne changeront: ni la quantité
$$(2) \qquad \alpha_{1.1}\alpha'_{1.2} - \alpha_{1.2}\alpha'_{1.1} + \alpha_{2.1}\alpha'_{2.2} - \alpha_{2.2}\alpha'_{2.1} + \alpha_{3.1}\alpha'_{3.2} - \alpha_{3.2}\alpha'_{3.1}$$
ni le plus grand commun diviseur des termes de la première ligne, ni celui des termes de la deuxième ligne, ni celui des déterminants formés avec deux des colonnes du tableau.

Cela posé:

1°. On peut, par l'opération A, annuler $\alpha'_{1.1}, \alpha'_{2.1}, \alpha'_{3.1}$. Il est donc toujours permis de supposer que le tableau (1) s'écrit:
$$\begin{vmatrix} \alpha_{1.1} & 0 & \alpha_{2.1} & 0 & \alpha_{3.1} & 0 \\ \alpha_{1.2} & \alpha'_{1.2} & \alpha_{2.2} & \alpha'_{2.2} & \alpha_{3.2} & \alpha'_{3.2} \end{vmatrix}.$$

2°. Appliquons maintenant l'opération B. Les nombres $\alpha'_{1.1}$, $\alpha'_{2.1}$, $\alpha'_{3.1}$ ne cesseront pas d'être nuls ; mais nous pourrons nous servir de cette opération de façon à annuler $\alpha_{2.1}$ et $\alpha_{3.1}$. Il arrivera alors que $\alpha_{1.1}$ sera le seul terme de la première ligne qui ne sera pas nul ; je puis toujours le supposer égal à 1 ; car s'il ne l'était pas on pourrait remplacer la période ξ_1 par son multiple $\alpha_{1.1}\xi_1$ et $\alpha_{1.1}$ deviendrait ainsi égal à 1. Il est donc toujours permis de supposer que le tableau (1) s'écrit :

$$\begin{vmatrix} 1 & 0 & 0 & 0 & 0 & 0 \\ \alpha_{1.2} & \alpha'_{1.2} & \alpha_{2.2} & \alpha'_{2.2} & \alpha_{3.2} & \alpha'_{3.2} \end{vmatrix}.$$

3°. Appliquons de nouveau l'opération A, mais sans toucher à la première paire de colonnes ; les termes déjà annulés resteront nuls. Mais nous pourrons diriger l'opération de façon à annuler $\alpha'_{2.2}$ et $\alpha'_{3.2}$. Le tableau deviendra :

$$\begin{vmatrix} 1 & 0 & 0 & 0 & 0 & 0 \\ \alpha_{1.2} & \alpha'_{1.2} & \alpha_{2.2} & 0 & \alpha_{3.2} & 0 \end{vmatrix}.$$

4°. Appliquons l'opération B sans toucher à la première paire de colonnes ; les termes déjà annulés resteront nuls, et nous pourrons diriger l'opération de façon à annuler $\alpha_{3.2}$.

Le tableau (1) s'écrira alors, en remplaçant les lettres α pourvues d'indices par de simples lettres a, b, c, etc. :

$$\begin{vmatrix} 1 & 0 & 0 & 0 & 0 & 0 \\ a & b & c & 0 & 0 & 0 \end{vmatrix}.$$

5°. Appliquons maintenant l'opération C en retranchant a fois la 1ère ligne de la seconde ; le tableau se simplifie encore et s'écrit :

$$\begin{vmatrix} 1 & 0 & 0 & 0 & 0 & 0 \\ 0 & b & c & 0 & 0 & 0 \end{vmatrix}.$$

On est ainsi conduit au théorème de M. Weierstrass que j'ai démontré et généralisé dans la note citée plus haut.

Mais la simplification peut être encore poussée plus loin ainsi que M. Picard l'a montré dans le cas de $\rho = 2$.

Le nombre b (que je puis toujours regarder comme premier avec c) est une constante absolue à laquelle je ne puis toucher, car ce n'est autre chose que l'invariant (2). Mais je vais montrer que je puis en dirigeant convenablement les opérations rendre c égal à tel nombre entier (premier avec b) que je voudrai et en particulier à 1.

Je puis en combinant les opérations A et B, ajouter α fois la 2ème colonne à la troisième en ajoutant α fois la 4ème colonne à la première. Le tableau devient:

$$\begin{vmatrix} 1 & 0 & 0 & 0 & 0 & 0 \\ 0 & b & c+ab & 0 & 0 & 0 \end{vmatrix}.$$

Je puis de même ajouter β fois la 2ème colonne à la quatrième en retranchant β fois la 3ème colonne de la première; il vient alors:

$$\begin{vmatrix} 1 & 0 & 0 & 0 & 0 & 0 \\ -\beta c - \alpha\beta b & b & c+ab & \beta b & 0 & 0 \end{vmatrix}.$$

Je puis toujours choisir les nombres entiers α et β de telle sorte que le plus grand commun diviseur de:

$$c+ab \text{ et de } \beta b$$

soit tel nombre d (premier avec b) que je voudrai.

Appliquons ensuite l'opération A sans toucher à la première paire de colonnes, mais de façon à annuler le 4ème terme de la 2de ligne; il vient:

$$\begin{vmatrix} 1 & 0 & 0 & 0 & 0 & 0 \\ -\beta e - \alpha\beta b & b & d & 0 & 0 & 0 \end{vmatrix}$$

on en ajoutant $\beta c + \alpha\beta b$ fois la 1ère ligne à la 2ème:

$$\begin{vmatrix} 1 & 0 & 0 & 0 & 0 & 0 \\ 0 & b & d & 0 & 0 & 0 \end{vmatrix}$$

d'où il suit que la forme la plus simple du tableau (1) est la suivante:

$$\begin{vmatrix} 1 & 0 & 0 & 0 & 0 & 0 \\ 0 & b & 1 & 0 & 0 & 0 \end{vmatrix}.$$

Nous allons maintenant passer au cas général.

Voyons d'abord quels sont les invariants que les opérations A, B et C laisseront inaltérés.

Il y a en premier lieu le plus grand commun diviseur des déterminants obtenus en prenant 2μ colonnes dans le tableau (1). Je pourrai toujours supposer que ce commun diviseur est égal à 1.

En effet l'opération C permet de remplacer le système des périodes ξ par un système équivalent. Mais il peut arriver dans certains cas qu'on puisse remplacer ce système par un autre, non pas équivalent mais plus simple.

Si par exemple on avait (en faisant pour plus de simplicité $\rho = 2$, $\mu = 1$):

$$x_1 = 2\xi_1,\ x_1' = b\xi_2,\ x_2 = \xi_2,\ x_3 = 0,$$

il serait plus simple de poser $2\xi_1 = \xi_1'$ et de considérer le système des périodes ξ_1' et ξ_2; c'est d'ailleurs ce que nous avons déjà fait une fois.

Si donc les déterminants définis plus haut n'étaient pas premiers entre eux, il serait possible de remplacer les périodes ξ par un autre système plus simple et on réduirait ainsi à l'unité le plus grand commun diviseur en question.

Cela posé, les $\mu\ (2\mu - 1)$ quantités:

$$\Phi_{kq} = \Sigma_i\ (\alpha_{ik}\alpha'_{iq} - \alpha_{iq}\alpha'_{ik})$$

ne sont pas altérées par les opérations A et B.

La forme bilinéaire

$$F = \Sigma_{kq}\Phi_{kq}\xi_k\eta_q$$

(où l'on donne à k et à q les valeurs $1, 2, \ldots\ldots, 2\mu$ en observant que

$$\Phi_{kq} = -\Phi_{qk},\ \Phi_{qq} = 0)$$

ne sera donc pas altérée par les opérations A et B.

On voit sans peine que l'opération C change cette forme en une autre équivalente, au sens arithmétique du mot.

Soit Δ^2 le déterminant des nombres Φ_{kq} (qui est évidemment un carré parfait); ce sera un premier invariant de la forme F.

Nous allons chercher à réduire cette forme bilinéaire F à sa plus simple expression par une transformation linéaire convenablement choisie.

Si Δ n'est pas nul, la forme F pourra être réduite ainsi qu'il suit:

$$F = \Sigma_k A_k (\xi_{2k-1})\,\eta_{2k} - (\xi_{2k}\eta_{2k-1})$$

où $A_1, A_2, \ldots\ldots, A_\mu$ sont μ nombres entiers tel que:

$$A_1A_2A_3\ldots\ldots A_\mu = \Delta.$$

Il suffit pour s'en assurer de répéter presque sans y rien changer le raisonnement de M. M. Clebsch et Jordan (Theorie der Abelschen Functionen, p. 103).

Il en sera encore de même si Δ est nul; seulement un ou plusieurs des nombres A_k seraient nuls.

Mais on peut pousser plus loin encore la réduction. Supposons pour fixer les idées $\mu = 3$, et imaginons (en supposant Δ différent de 0) que la forme F soit réduite ainsi qu'il suit:

$$A_1(\xi_1\eta_2 - \xi_2\eta_1) + A_2(\xi_3\eta_4 - \xi_4\eta_3) + A_3(\xi_5\eta_6 - \xi_6\eta_5).$$

Soit A le plus grand commun diviseur des trois nombres A_1, A_2, A_3; soit A_2B celui des trois produits $A_1A_2,\ A_1A_3,\ A_2A_3$; soit enfin:

$$A_1A_2A_3 = A^3B^2C = \Delta$$

la forme F peut être réduite encore ainsi qu'il suit:

$$A(\xi_1\eta_2 - \xi_2\eta_1) + AB(\xi_3\eta_4 - \xi_4\eta_3) + ABC(\xi_5\eta_6 - \xi_6\eta_5).$$

Ceci nous fait voir quels sont, outre Δ les invariants de la forme F, Soit Δ_i le plus grand commun diviseur des mineurs d'ordre i du déterminant Δ^2; ce sera un invariant, et la forme F n'en aura pas d'autres; il suffira d'ailleurs d'envisager les mineurs d'ordre impair.

Observons maintenant que les $2\mu^2$ quantités $\xi, \eta, \ldots., \tau$ ne peuvent être choisies arbitrairement. Envisageons en effet la forme bilinéaire:

$$\Sigma_i (x_i y_i' - x_i' y_i).$$

Si on y substitue de la place de x_i ou de x_i':

$$a_1 x_i + a_2 y_i + a_3 z_i + \ldots. + a_\mu t_i,$$
$$a_1 x_i' + a_2 y_i' + a_3 z_i' + \ldots. + a_\mu t_i'$$

et en même temps à la place de y_i ou de y_i':

$$b_1 x_i + b_2 y_i + b_3 z_i + \ldots. + b_\mu t_i,$$
$$b_1 x_i' + b_2 y_i' + b_3 z_i' + \ldots. + b_\mu t_i'$$

les a et les b étant 2μ nombres tout à fait quelconques, le résultat de la substitution devra être nul.

Si on substitue à la place de x_i ou de x_i' les parties réelles de

$$a_1 x_i + a_2 y_i + a_3 z_i + \ldots. + a_\mu t_i,$$
$$a_1 x_i' + a_2 y_i' + a_3 z_i' + \ldots. + a_\mu t_i'$$

et à la place de y_i ou de y_i' les parties imaginaires des mêmes quantités, le résultat de la substitution devra être positif.

Reprenons donc la forme

$$F(\xi_k, \eta_k) = \Sigma \Phi_{kq} \xi_k \eta_q.$$

Nous devrons avoir:

$$F(a_1\xi_k + a_2\eta_k + \ldots. + a_\mu\tau_k,\ b_1\xi_k + b_2\eta_k + \ldots. + b_\mu\tau_k) = 0,$$
$$F[R(a_1\xi_k + a_2\eta_k + \ldots. + a_\mu\tau_k),\ I(a_1\xi_k + a_2\eta_k + \ldots. + a_\mu\tau_k)] > 0$$

quelles que soient les quantités a et b.

Nous désignons pour abréger $R(u)$ et $I(u)$ les parties réelle et imaginaire de u.

Nous pouvons toujours supposer que la forme F est réduite. Nous l'écrirons donc en supposant $\mu = 3$ pour fixer les idées:

$$F(\xi_k, \eta_k) = A_1(\xi_1\eta_2 - \xi_2\eta_1) + A_2(\xi_3\eta_4 - \xi_4\eta_3) + A_3(\xi_5\eta_6 - \xi_6\eta_5).$$

Je dis que les trois nombres A_1, A_2, A_3 sont différents de 0. Supposons en effet que A_3 par exemple soit nul. On pourrait choisir alors les nombres a_1, a_2 et a_3 de telle sorte que:

$$\arg(a_1\xi_1 + a_2\eta_1 + a_3\tau_1) = \arg(a_1\xi_2 + a_2\eta_2 + a_3\tau_2)$$
$$\arg(a_1\xi_3 + a_2\eta_3 + a_3\tau_3) \neq \arg(a_1\xi_4 + a_2\eta_4 + a_3\tau_4).$$

Ces deux conditions jointes à $A_3 = 0$ entraineraient

$$F[R(a_1\xi_k + a_2\eta_k + a_3\tau_k),\ R^I(a_1\xi_k + a_2\eta_k + a_3\tau_k)] = 0$$

ce qui serait contraire à l'inégalité démontrée plus haut. *Donc* Δ *ne peut jamais être nul.*

Cela posé, on voit sans peine que l'on a entre les ξ, les η et les τ la relation :

$$A_1(\xi_1\eta_2 - \xi_2\eta_1) + A_2(\xi_3\eta_4 - \xi_4\eta_3) + A_3(\xi_5\eta_6 - \xi_6\eta_5) = 0$$

et les deux relations analogues.

Cette démonstration se trouve dans une note que nous avons publiée, M. Picard et moi, dans le tome 97 des Comptes Rendus. Dans cette note nous avons établi que toute fonction analytique de n variables et à $2n$ périodes peut s'exprimer à l'aide des fonctions Θ. Mais je ne reproduis ici que la portion de la démonstration qui est utile pour mon objet actuel.

Revenons maintenant au problème de la réduction du tableau (1) et supposons pour fixer les idées, $\mu = 2$, $\rho = 4$.

Nous commencerons au moyen de l'opération C par réduire la forme F à sa plus simple expression. Nous écrirons donc :

$$F = a(\xi_1\eta_4 - \xi_4\eta_1) + ab(\xi_2\eta_3 - \xi_3\eta_2)$$

a et b étant deux entiers différents de 0 ; cela est toujours possible d'après ce qui précède.

Nous avons vu en traitant le cas particulier de $\mu = 1$, qu'on peut par les opérations A et B faire disparaître tous les termes de la 1ère ligne du tableau (1), sauf le premier ; qu'on peut ensuite par les opérations A et B, et en ayant soin de ne pas toucher à la première paire, faire disparaître tous les termes de la 2de ligne sauf les trois premiers. Nous opérerons de même ici et nous ferons disparaître tous termes, sauf le 1er de la 1ère ligne, les trois premiers de la 2de, les cinq premiers de la 3ème et les sept premiers de la 4ème.

Le tableau (1) s'écrira alors :

$$\left|\begin{matrix} 1 & 0 & 0 & 0 & 0 & 0 & 0 & 0 \\ \alpha_{1.2} & \alpha'_{1.2} & \alpha_{2.2} & 0 & 0 & 0 & 0 & 0 \\ \alpha_{1.3} & \alpha'_{1.3} & \alpha_{2.3} & \alpha'_{2.3} & \alpha_{3.3} & 0 & 0 & 0 \\ \alpha_{1.4} & \alpha'_{1.4} & \alpha_{2.4} & \alpha'_{2.4} & \alpha_{3.4} & \alpha'_{3.4} & \alpha_{4.4} & 0 \end{matrix}\right|.$$

Nous n'avons employé que les opérations A et B ; la forme F n'a donc pas changé et l'on a encore :

$$F = a(\xi_1\eta_4 - \xi_4\eta_1) + ab(\xi_2\eta_3 - \xi_3\eta_2).$$

On a donc :

$$\Phi_{1.2} = \alpha'_{1.2} = 0, \qquad \Phi_{1.3} = \alpha'_{1.3} = 0,$$
$$\Phi_{1.4} = \alpha'_{1.4} = a, \qquad \Phi_{2.3} = \alpha_{2.2}\alpha'_{2.3} = ab.$$

De plus $\alpha_{2.2}$ doit être égal à 1 sans quoi le plus grand commun diviseur des déterminants formés avec 4 colonnes du tableau (1) ne serait pas égal à 1.

Le tableau (1) peut donc s'écrire :

$$\begin{vmatrix} 1 & 0 & 0 & 0 & 0 & 0 & 0 & 0 \\ \alpha_{1.2} & 0 & 1 & 0 & 0 & 0 & 0 & 0 \\ \alpha_{1.3} & 0 & \alpha_{2.3} & ab & \alpha_{3.3} & 0 & 0 & 0 \\ \alpha_{1.4} & a & \alpha_{2.4} & \alpha'_{2.4} & \alpha_{3.4} & \alpha'_{3.4} & \alpha_{4.4} & 0 \end{vmatrix}.$$

Retranchons maintenant la $1^{\text{ère}}$ ligne, $\alpha_{1.2}$ fois de la 2^{de}, $\alpha_{1.3}$ fois de la $3^{\text{ème}}$, $\alpha_{1.4}$ fois de la $4^{\text{ème}}$; puis la 2^{de} ligne $\alpha_{2.3}$ fois de la $3^{\text{ème}}$ et $\alpha_{2.4}$ fois de la $4^{\text{ème}}$, le tableau (1) deviendra :

$$1^{\text{bis}} \qquad \begin{vmatrix} 1 & 0 & 0 & 0 & 0 & 0 & 0 & 0 \\ 0 & 0 & 1 & 0 & 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & ab & \alpha_{3.3} & 0 & 0 & 0 \\ 0 & a & 0 & \alpha'_{2.4} & \alpha_{3.4} & \alpha'_{3.4} & \alpha_{4.4} & 0 \end{vmatrix}.$$

Ici deux cas sont à distinguer suivant que a est égal à 1 ou n'est pas égal à 1. Supposons d'abord $a = 1$.

Retranchons la deuxième colonne $\alpha'_{2.4}$ fois de la $4^{\text{ème}}$ et ajoutons en même temps la troisième colonne $\alpha'_{3.4}$ fois à la $1^{\text{ère}}$ (opération B).

Retranchons $\alpha_{3.4}$ fois la 2^{de} colonne de la $5^{\text{ème}}$ et la $6^{\text{ème}}$ de la $1^{\text{ère}}$ (opérations A et B).

Retranchons enfin $\alpha_{4.4}$ fois la 2^{de} colonne de la $7^{\text{ème}}$ et la $8^{\text{ème}}$ de la $1^{\text{ère}}$.

Le tableau 1 devient alors :

$$\begin{vmatrix} 1 & 0 & 0 & 0 & 0 & 0 & 0 & 0 \\ \alpha'_{2.4} & 0 & 1 & 0 & 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & b & \alpha_{3.3} & 0 & 0 & 0 \\ -\alpha'_{3.4}\alpha_{3.4} & 1 & 0 & 0 & 0 & 0 & 0 & 0 \end{vmatrix}.$$

On peut ensuite en retranchant la $1^{\text{ère}}$ ligne un nombre convenable de fois de la 2^{de} et de la $4^{\text{ème}}$, amener le tableau (1) à la forme :

$$\begin{vmatrix} 1 & 0 & 0 & 0 & 0 & 0 & 0 & 0 \\ 0 & 0 & 1 & 0 & 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & b & \alpha_{3.3} & 0 & 0 & 0 \\ 0 & 1 & 0 & 0 & 0 & 0 & 0 & 0 \end{vmatrix}.$$

Enfin une dernière simplification est encore possible. Les nombres b et $\alpha_{3.3}$ sont premiers entre eux, et on peut en opérant comme nous l'avons fait dans le cas particulier de $\mu = 1$, réduire $\alpha_{3.3}$ à l'unité.

Le tableau (1) est alors amené à sa forme définitive:

$$\begin{vmatrix} 1 & 0 & 0 & 0 & 0 & 0 & 0 & 0 \\ 0 & 0 & 1 & 0 & 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & b & 1 & 0 & 0 & 0 \\ 0 & 1 & 0 & 0 & 0 & 0 & 0 & 0 \end{vmatrix}.$$

Supposons maintenant que a ne soit pas égal à 1 et reprenons le tableau (1) sous sa forme (1^{bis}).

Il est clair que ab et $\alpha_{3.3}$ sont premiers entre eux; on pourra donc, en opérant comme nous l'avons fait dans le cas particulier de $\mu = 1$, réduire $\alpha_{3.3}$ à l'unité.

Comme nous avons appliqué plusieurs fois l'opération C, la forme F a été changée en une autre forme équivalente, mais *elle a dû rester divisible par* a. Donc $\Phi_{2.4} = \alpha'_{2.4}$ et $\Phi_{3.4} = \alpha'_{3.4}$ (puisque $\alpha_{3.3} = 1$) doivent être divisibles par a. Soit donc:

$$\alpha'_{2.4} = ac, \qquad \alpha'_{3.4} = ad.$$

Nous retrancherons ensuite la $3^{\text{ème}}$ ligne $\alpha_{3.4}$ fois de la $4^{\text{ème}}$ et le tableau (1^{bis}) deviendra:

$$\begin{vmatrix} 1 & 0 & 0 & 0 & 0 & 0 & 0 & 0 \\ 0 & 0 & 1 & 0 & 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & ab & 1 & 0 & 0 & 0 \\ 0 & a & 0 & ac' & 0 & ad & \alpha_{4.4} & 0 \end{vmatrix},$$

où $c' = c - b\alpha_{3.4}$.

Retranchons maintenant c' fois la 2^{de} colonne de la $4^{\text{ème}}$ en ajoutant c' fois la $3^{\text{ème}}$ à la $1^{\text{ère}}$.

Retranchons d fois la 2^{de} de la $6^{\text{ème}}$ en ajoutant d fois la $5^{\text{ème}}$ à la $1^{\text{ère}}$.

Nous avons introduit ainsi dans la $1^{\text{ère}}$ colonne le terme c' à la seconde ligne et le terme d à la $3^{\text{ème}}$; nous les ferons disparaître en retranchant c' fois la $1^{\text{ère}}$ ligne de la 2^{de} et d fois de la $3^{\text{ème}}$. Le tableau (1) deviendra alors:

$$\begin{vmatrix} 1 & 0 & 0 & 0 & 0 & 0 & 0 & 0 \\ 0 & 0 & 1 & 0 & 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & ab & 1 & 0 & 0 & 0 \\ 0 & a & 0 & 0 & 0 & 0 & \alpha_{4.4} & 0 \end{vmatrix}.$$

Enfin une dernière simplification est encore possible; il est clair que a et $\alpha_{4.4}$ sont premiers entre eux; nous pouvons donc opérer comme nous l'avons fait dans le cas de $\mu = 1$ et comme nous l'avons fait deux fois dans le cas actuel et réduire $\alpha_{4.4}$ à l'unité. Le tableau (1) prendra alors sa forme définitive:

$$\begin{vmatrix} 1 & 0 & 0 & 0 & 0 & 0 & 0 & 0 \\ 0 & 0 & 1 & 0 & 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & ab & 1 & 0 & 0 & 0 \\ 0 & a & 0 & 0 & 0 & 0 & 1 & 0 \end{vmatrix}.$$

On peut généraliser; j'écrirai le tableau (1) réduit à sa plus simple expression en supposant $\rho = 6$, $\mu = 3$:

$$\begin{vmatrix} 1 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 \\ 0 & 0 & 1 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 & 1 & 0 & 0 & 0 & 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 & 0 & abc & 1 & 0 & 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & ab & 0 & 0 & 0 & 0 & 1 & 0 & 0 & 0 \\ 0 & a & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 1 & 0 \end{vmatrix}.$$

Il reste maintenant à écrire le tableau des périodes sous la forme habituelle; nous le ferons dans les quatre cas que nous avons choisis plus haut pour exemples, c'est à dire quand:

$$\mu = 1,\ \rho = 3,$$
$$\mu = 2,\ \rho = 4,\ a = 1,$$
$$\mu = 2,\ \rho = 4,\ a > 1,$$
$$\mu = 3,\ \rho = 6,\ a > 1.$$

Nous supposerons que le tableau (1) a été réduit à sa plus simple expression comme il a été dit plus haut, et nous ferons:

Dans le premier cas:

$$\xi_3 = \frac{1}{b}.$$

Dans le second cas:

$$\xi_4 = 1,\ \xi_3 = 0;\ \eta_4 = 0,\ \eta_3 = \frac{1}{b}.$$

Dans le troisième cas:

$$\xi_4 = \frac{1}{a},\ \xi_3 = 0;\ \eta_4 = 0,\ \eta_3 = \frac{1}{ab}.$$

Enfin dans le quatrième:

$$\xi_6 = \frac{1}{a},\ \xi_5 = 0,\ \xi_4 = 0,$$
$$\eta_6 = 0,\ \eta_5 = \frac{1}{ab},\ \eta_4 = 0,$$
$$\tau_6 = 0,\ \tau_5 = 0,\ \tau_4 = \frac{1}{abc}.$$

Cela est toujours possible en choisissant convenablement les intégrales J_1, $J_2, \ldots, J_\mu$.

Il résulte de là et de la forme du tableau (1) que les périodes $x'_1, x'_2, \ldots, x'_\rho$ de J_1 sont toutes égales à 0, excepté une qui est égale à 1. De même des périodes correspondantes de $J_2, \ldots, J_\mu$. En d'autres termes, $J_1, J_2, \ldots, J_\mu$ sont des *intégrales normales*. On y adjoindra $\rho - \mu$ autres intégrales normales et on écrira le tableau des périodes sous la forme habituelle, c'est à dire dans l'ordre suivant:

$$\begin{array}{l} x'_1, x'_2, \ldots, x'_\rho, x_1, x_2, \ldots, x_\rho \\ y'_1, y'_2, \ldots, y'_\rho, y_1, y_2, \ldots, y_\rho \\ \ldots\ldots\ldots\ldots \\ u'_1, u'_2, \ldots, u'_\rho, u_1, u_2, \ldots, u_\rho. \end{array}$$

On obtiendra ainsi les tableaux suivants:

1^er^ Cas. $\mu = 1, \rho = 3$,

$$\begin{array}{cccccc} 1 & 0 & 0 & \xi_1 & \frac{1}{b} & 0 \\ 0 & 1 & 0 & \frac{1}{b} & G & H \\ 0 & 0 & 1 & 0 & H & G'. \end{array}$$

2^ème^ Cas. $\mu = 2, \rho = 4, a = 1$,

$$\begin{array}{cccccccc} 1 & 0 & 0 & 0 & \xi_1 & \xi_2 & 0 & 0 \\ 0 & 1 & 0 & 0 & \eta_1 & \eta_2 & \frac{1}{b} & 0 \\ 0 & 0 & 1 & 0 & 0 & \frac{1}{b} & G & H \\ 0 & 0 & 0 & 1 & 0 & 0 & H & G'. \end{array}$$

3^ème^ Cas. $\mu = 2, \rho = 4, a > 1$,

$$\begin{array}{cccccccc} 1 & 0 & 0 & 0 & \xi_1 & \xi_2 & 0 & \frac{1}{a} \\ 0 & 1 & 0 & 0 & \eta_1 & \eta_2 & \frac{1}{ab} & 0 \\ 0 & 0 & 1 & 0 & 0 & \frac{1}{ab} & G & H \\ 0 & 0 & 0 & 1 & \frac{1}{a} & 0 & H & G'. \end{array}$$

4[ème] Cas. $\mu = 3, \rho = 6, a > 1,$

$$\begin{matrix} 1 & 0 & 0 & 0 & 0 & 0 & \xi_1 & \xi_2 & \xi_3 & 0 & 0 & \frac{1}{a} \\ 0 & 1 & 0 & 0 & 0 & 0 & \eta_1 & \eta_2 & \eta_3 & 0 & \frac{1}{ab} & 0 \\ 0 & 0 & 1 & 0 & 0 & 0 & \tau_1 & \tau_2 & \tau_3 & \frac{1}{abc} & 0 & 0 \\ 0 & 0 & 0 & 1 & 0 & 0 & 0 & 0 & \frac{1}{abc} & G & H'' & H' \\ 0 & 0 & 0 & 0 & 1 & 0 & 0 & \frac{1}{ab} & 0 & H'' & G' & H \\ 0 & 0 & 0 & 0 & 0 & 1 & \frac{1}{a} & 0 & 0 & H' & H & G'', \end{matrix}$$

a, b, et c sont des entiers; il est clair d'ailleurs qu'on doit avoir:

$$\xi_2 = \eta_1, \; \xi_3 = \tau_1, \; \eta_3 = \tau_2.$$

L'inspection de ces tableaux montre que s'il y a μ intégrales réductibles au rang μ, il y en aura $\rho - \mu$ réductibles au rang $\rho - \mu$.

§ 2. *Cas singuliers de réduction.*

Nous allons désormais nous restreindre au cas où une ou plusieurs des intégrales abéliennes considérées sont réductibles aux intégrales elliptiques.

Soient J_1 et J_2 deux intégrales abéliennes réductibles aux intégrales elliptiques. Soient:

$$x_1 = \alpha_1\xi_1 + \beta_1\xi_2, \; x_2 = \alpha_2\xi_1 + \beta_2\xi_2, \; \ldots\ldots, \; x_\rho = \alpha_\rho\xi_1 + \beta_\rho\xi_2,$$
$$x'_1 = \alpha'_1\xi_1 + \beta'_1\xi_2, \; x'_2 = \alpha'_2\xi_1 + \beta'_2\xi_2, \; \ldots\ldots, \; x'_\rho = \alpha'_\rho\xi_1 + \beta'_\rho\xi_2,$$

les périodes normales de J_1 et

$$y_1 = \gamma_1\eta_1 + \delta_2\eta_2, \; \ldots\ldots, \; y_\rho = \gamma_\rho\eta_1 + \delta_\rho\eta_2,$$
$$y'_1 = \gamma'_1\eta_1 + \delta'_1\eta_2, \; \ldots\ldots, \; y'_\rho = \gamma'_\rho\eta_1 + \delta'_\rho\eta_2,$$

les périodes normales de J_2. Les α, β, γ et δ seront des entiers.

On devra avoir

$$\Sigma\,(x_i y'_i - y_i x'_i) = 0 \quad (i = 1, 2, \ldots\ldots, \rho)$$

et par conséquent

(1) $$A\xi_1\eta_1 + B\xi_1\eta_2 + C\xi_2\eta_1 + D\xi_2\eta_2 = 0,$$

où

$$A = \Sigma\,(\alpha_i\gamma'_i - \alpha'_i\gamma_i),$$
$$B = \Sigma\,(\alpha_i\delta'_i - \alpha'_i\delta_i),$$
$$C = \Sigma\,(\beta_i\gamma'_i - \beta'_i\gamma_i),$$
$$D = \Sigma\,(\beta_i\delta'_i - \beta'_i\delta_i),$$

A, B, C et D sont des entiers.

Cela posé de deux choses l'une:

Ou bien l'égalité (1) n'est pas une identité, ou bien elle est une identité, de sorte que $$A = B = C = D = 0.$$

1°. Supposons d'abord qu'elle ne soit pas une identité.

Je dis alors qu'il y aura une infinité d'intégrales de la forme
$$J_1 + \lambda J_2$$
qui seront réductibles aux intégrales elliptiques. En effet pour que $J_1 + \lambda J_2$ soit réductible, il suffit qu'il y ait entre les quatre quantités:
$$\xi_1,\ \xi_2,\ \lambda\eta_1,\ \lambda\eta_2$$
deux relations linéaires et homogènes à coëfficients commensurables que je pourrai écrire:
$$\xi_1 = \lambda\,(b\eta_1 + c\eta_2)\,,$$
$$\xi_2 = \lambda\,(b'\eta_1 + c'\eta_2),$$
b, c, b', c' étant commensurables.

L'élimination de λ entre ces deux équations donnera la condition:
$$b'\xi_1\eta_1 + c'\xi_1\eta_2 - b\xi_2\eta_1 - c\xi_2\eta_2 = 0.$$
Cette condition sera satisfaite si l'on pose:
$$b' = \mu A,\ c' = \mu B,\ b = -\mu C,\ c = -\mu D,$$
μ étant un nombre commensurable quelconque.

On aura alors: $$\lambda = -\frac{\xi_1}{\mu\,(C\eta_1 + D\eta_2)}.$$
Si donc nous posons $$\frac{\xi_1}{C\eta_1 + D\eta_2} = h$$
l'intégrale: $$J_1 + \mu h J_2$$
sera réductible toutes les fois que μ sera commensurable.

On peut toujours supposer que $h = 1$; cas si cela n'était pas, on remplacerait l'intégrale J_2 par l'intégrale hJ_2 qui n'en diffère que par le facteur constant h. On a alors: $$\xi_1 = C\eta_1 + D\eta_2,$$
$$\xi_2 = -(A\eta_1 + B\eta_2).$$

2°. Supposons maintenant que la relation (1) soit une identité. Imaginons que le tableau des périodes ait été réduit comme il a été dit au paragraphe précédent et qu'il s'écrire (en supposant $\rho = 3$ pour fixer les idées).

	1	0	0	ξ_2	$\frac{1}{a}$	0	périodes de l'intégrale J_1,
(2)	0	1	0	$\frac{1}{a}$	G	H	périodes d'une intégrale que j'appelle J_2',
	0	0	1	0	H	G'	périodes d'une intégrale que j'appelle J_3',

a étant un entier.

Si donc on pose: $$\xi_1 = \frac{1}{a}$$
on aura:

(3) $$\begin{aligned} &\alpha_1 = 0,\ \alpha_2 = 1,\ \alpha_3 = 0;\ \alpha'_1 = a,\ \alpha'_2 = 0,\ \alpha'_3 = 0, \\ &\beta_1 = 1,\ \beta_2 = 0,\ \beta_3 = 0;\ \beta'_1 = 0,\ \beta'_2 = 0,\ \beta'_3 = 0. \end{aligned}$$

Il viendra alors:
$$A = -a\gamma_1 + \gamma'_2;\ B = -a\delta_1 + \delta'_2;\ C = \gamma'_1;\ D = \delta'_1.$$
Pour que la relation (1) soit une identité, on doit donc avoir:
$$\gamma'_1 = \delta'_1 = 0,\ \ \gamma'_2 = a\gamma_1,\ \ \delta'_2 = a\delta_1$$
ce qui entraîne évidemment
$$y'_1 = 0,\quad y'_2 = ay_1.$$

Or si l'intégrale J_2 qui doit être une combinaison linéaire de J_1, J'_2 et J'_3 s'écrit par exemple:
$$J_2 = hJ_1 + kJ'_2 + lJ'_3$$
on aura: $$y'_1 = h,\ \ y_1 = h\xi_2 + \frac{k}{a};\ \ y'_2 = k.$$

Les conditions précédentes équivalent donc à la suivante
$$h = 0.$$
On a donc $$J_2 = kJ'_2 + lJ'_3.$$

Revenons maintenant au cas où la relation (1) n'est pas une identité, mais en supposant que le tableau des périodes ait été ramené à la forme (2) de façon que les α et les β prennent les valeurs (3).

On aura alors une infinité d'intégrales réductibles
$$J_1 + \mu J_2$$
où μ est un nombre commensurable quelconque.

On aura d'ailleurs:
$$\xi_1 = \frac{1}{a} = C\eta_1 + D\eta_2 = \gamma'_1\eta_1 + \delta'_1\eta_2 = y'_1.$$
La première période de l'intégrale $J_1 + \mu J_2$ (dans l'ordre du tableau (2)) est alors:
$$x'_1 + \mu y'_1 = 1 + \frac{\mu}{a}.$$

Nous pouvons donc choisir le nombre commensurable μ, de telle sorte que cette première période soit nulle; il suffit pour cela de poser:
$$\mu = -a.$$
On a alors: $$J_1 - aJ_2 = kJ'_2 + lJ'_3$$
k et l étant des coëfficients numériques convenablement choisis.

Si donc parmi les intégrales abéliennes:

$$hJ_1 + kJ'_2 + lJ'_3$$

il y en a une (autre que J_1) qui soit réductible aux intégrales elliptiques, ou bien h sera nul, ou bien il y en aura une infinité d'autres comprises dans la formule générale suivante:

$$\mu hJ_1 + kJ'_2 + lJ'_3$$

(où μ est un nombre commensurable quelconque) qui seront également réductibles. En particulier:

$$kJ'_2 + lJ'_3$$

est une intégrale réductible.

Si donc on désigne par

$$y'_1,\, y'_2,\, y'_3;\, y\ ,\, y_2,\, y_3$$

les périodes de cette intégrale

$$kJ'_2 + lJ'_3$$

on devra avoir $y'_1 = 0,\ y'_2 = ay_1$

ce qui prouve que pour cette intégrale la relation (1) est une identité.

D'où la conséquence suivante:

S'il existe deux intégrales réductibles J_2 et J_1 pour lesquelles la relation (1) ne soit pas une identité, il y en aura une infinité d'autres et parmi celles-là, il y en aura encore une pour laquelle la relation (1) sera une identité.

Je vais maintenant démontrer le théorème suivant:

Si dans un système d'intégrales abéliennes de rang ρ, il y en a $\rho - 1$ linéairement indépendantes qui sont réductibles aux intégrales elliptiques, il y en aura une $\rho^{\text{ème}}$ qui sera également réductible.

En effet ces $\rho - 1$ intégrales abéliennes linéairement indépendantes et réductibles aux intégrales elliptiques, peuvent être regardées comme formant un système de $\rho - 1$ intégrales réductibles au genre $\rho - 1$. Donc d'après le théorème énoncé à la fin du paragraphe précédent, il devra y avoir une $\rho^{\text{ème}}$ intégrale, linéairement indépendante des $\rho - 1$ premières, et qui sera réductible au genre 1.

C. Q. F. D.

Je dis maintenant que si l'on a $\mu + 1$ intégrales réductibles aux intégrales elliptiques (et que ces $\mu + 1$ intégrales ne soient pas linéairement indépendantes) il y en a une infinité.

Soient en effet:

$$y_1,\, y_2,\, \ldots\, ,\, y_\mu$$

μ intégrales réductibles linéairement indépendantes et:

$$J = \alpha_1 y_1 + \alpha_2 y_2 + \ldots + \alpha_\mu y_\mu$$

une $\mu + 1^{\text{ème}}$ intégrale qui est une combinaison linéaire des μ premières et que je suppose également réductible.

Les périodes de l'une quelconque des μ intégrales $y_1, y_2, \ldots, y_\mu$, celles de y_i par exemple, devront être des combinaisons linéaires à coëfficients entiers de deux quantités que j'appelle ξ_i et η_i et qui sont les périodes de l'intégrale elliptique à laquelle peut se réduire y_i.

Il résulte de là que les périodes de

$$J = \alpha_1 y_1 + \alpha_2 y_2 + \ldots + \alpha_\mu y_\mu$$

seront des combinaisons linéaires à coëfficients entiers des 2μ quantités

$$\alpha_1\xi_1,\ \alpha_2\xi_2,\ \ldots,\ \alpha_\mu\xi_\mu$$
$$\alpha_1\eta_1,\ \alpha_2\eta_2,\ \ldots,\ \alpha_\mu\eta_\mu.$$

Pour que cette intégrale J soit réductible aux intégrales elliptiques, il faut et il suffit qu'il y ait entre ces 2μ quantités, $2\mu - 2$ relations linéaires homogènes à coëfficients entiers.

Supposons qu'il en soit ainsi, et soient:

$$\beta_1,\ \beta_2,\ \ldots,\ \beta_\mu$$

μ nombres commensurables *quelconques* différents de 0; il y aura alors aussi $2\mu - 2$ relations linéaires à coëfficients entiers entre les 2μ quantités:

$$\alpha_1\beta_1\xi_1,\ \alpha_2\beta_2\xi_2,\ \ldots,\ \alpha_\mu\beta_\mu\xi_\mu$$
$$\alpha_1\beta_1\eta_1,\ \alpha_2\beta_2\eta_2,\ \ldots,\ \alpha_\mu\beta_\mu\eta_\mu$$

et par conséquent l'intégrale:

$$\alpha_1\beta_1 y_1 + \alpha_2\beta_2 y_2 + \ldots + \alpha_\mu\beta_\mu y_\mu$$

sera réductible quels que soient les nombres commensurables $\beta_1, \beta_2, \ldots, \beta_\mu$.

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Si en particulier il y a plus de ρ intégrales réductibles, il y en aura une infinité.

Et en effet, entre $\rho + 1$ intégrales il y a toujours une relation linéaire, puisqu'un système d'intégrales abéliennes de rang ρ ne contient que ρ intégrales linéairement indépendantes.

Dans un mémoire inséré aux Acta Mathematica, Mme. Kowalevski étudie le cas de réduction des intégrales de rang 3 au rang 1, en supposant que le nombre caractéristique de la réduction que nous avons ici appelé b soit égal à 2.

Dans un cas remarquable, elle trouve 4 intégrales réductibles et n'en trouve que 4; il n'y en a que 4 en effet où b soit égal à 2, mais il y en a une infinité pour lesquelles b est supérieur à 2.

Je vais enfin pour terminer ce paragraphe démontrer que tout système d'intégrales abéliennes diffère infiniment peu d'un système réductible.

Voici ce que j'entends par là :

Soit (en supposant $\rho = 4$ pour fixer les idées),

$$\begin{matrix} 1 & 0 & 0 & 0 & \tau_{1.1} & \tau_{2.1} & \tau_{3.1} & \tau_{4.1} \\ 0 & 1 & 0 & 0 & \tau_{1.2} & \tau_{2.2} & \tau_{3.2} & \tau_{4.2} \\ 0 & 0 & 1 & 0 & \tau_{1.3} & \tau_{2.3} & \tau_{3.3} & \tau_{4.3} \\ 0 & 0 & 0 & 1 & \tau_{1.4} & \tau_{2.4} & \tau_{3.4} & \tau_{4.4} \end{matrix} \qquad (\tau_{ik} = \tau_{ki})$$

le tableau des périodes normales d'un système d'intégrales abéliennes. On n'obtiendra un système d'intégrales abéliennes proprement dites, c'est à dire un système engendré par une courbe algébrique, que s'il y a une certaine relation entre les ρ; car le nombre des modules d'une courbe de rang (4) étant 9, est inférieur à 10, nombre des τ.

Il convient toutefois de s'affranchir de cette difficulté, et pour cela il suffit d'étendre un peu le sens du mot *intégrales abéliennes.*

Soient x_1, x_2, x_3, x_4, x_5, cinq fonctions abéliennes (8 fois périodiques) de quatre variables u_1, u_2, u_3 et u_4.

Il y aura entre ces cinq fonctions une relation algébrique :

$$\phi(x_1, x_2, x_3, x_4, x_5) = 0,$$

et il est aisé de voir que u_1, u_2, u_3 et u_4 sont des intégrales de différentielles totales dépendant de x_1, x_2, x_3, x_4 et x_5. On aura :

$$u_1 = \int \psi_1 dx_1 + \psi_2 dx_2 + \psi_3 dx_3 + \psi_4 dx_4,$$

où ψ_1, ψ_2, ψ_3 et ψ_4 sont des fonctions rationnelles de x_1, x_2, x_3, x_4 et x_5.

Nous conviendrons d'appeler encore intégrales abéliennes les intégrales de différentielles totales ainsi engendrées.

Alors quels que soient les τ (pourvu qu'ils satisfassent à certaines inégalités et sans qu'ils soient assujettis à aucune égalité) ils pourront être regardés comme les périodes d'un système d'intégrales abéliennes et la difficulté signalée plus haut disparaîtra.

Considérons donc le système suivant de périodes :

$$\begin{matrix} 1 & 0 & 0 & 0, & \tau_{1.1} + r\epsilon_{1.1}, & \tau_{2.1} + r\epsilon_{2.1}, & \tau_{3.1} + r\epsilon_{3.1}, & \tau_{4.1} + r\epsilon_{4.1} \\ 0 & 1 & 0 & 0, & \tau_{1.2} + r\epsilon_{1.2}, & \tau_{2.2} + r\epsilon_{2.2}, & \tau_{3.2} + r\epsilon_{3.2}, & \tau_{4.2} + r\epsilon_{4.2} \\ 0 & 0 & 1 & 0, & \tau_{1.3} + r\epsilon_{1.3}, & \tau_{2.3} + r\epsilon_{2.3}, & \tau_{3.3} + r\epsilon_{3.3}, & \tau_{4.3} + r\epsilon_{4.3} \\ 0 & 0 & 0 & 1, & \tau_{1.4} + r\epsilon_{1.4}, & \tau_{2.4} + r\epsilon_{2.4}, & \tau_{3.4} + r\epsilon_{3.4}, & \tau_{4.4} + r\epsilon_{4.4} \end{matrix}$$

Ici r est une quantité positive donnée. Quant aux ε ce sont des quantités satisfaisant bien entendu à la condition:

$$\varepsilon_{ik} = \varepsilon_{ki}$$

et que j'assujettis de plus aux inégalités

$$|\varepsilon| < 1.$$

Laissons r constant et faisons varier les ε en leur donnant toutes les valeurs compatibles avec les inégalités précédentes. Nous obtiendrons ainsi une infinité de systèmes d'intégrales abéliennes.

Parmi ces systèmes, il y en aura, *quelque petit que soit* r, une infinité qui contiendront quatre intégrales réductibles au rang 1.

C'est ce que j'entendais en disant que tout système d'intégrales abéliennes est infiniment voisin d'une infinité de systèmes réductibles.

Le résultat que je viens d'énoncer est presque évident. En effet une intégrale sera évidemment réductible au genre 1 si toutes ses périodes sont de la forme

$$\alpha + \beta\sqrt{-1},$$

α et β étant commensurables; si en d'autres termes, toutes les périodes sont des nombres complexes commensurables.

Mais parmi les nombres

$$\tau_{ik} + r\varepsilon_{ik}$$

qui satisfont à la condition

$$|\varepsilon_{ik}| < 1$$

il y aura, quelque petit que soit r, une infinité de nombres complexes commensurables.

On peut donc choisir les ε_{ik} d'une infinité de manières et de telle façon que les quatre intégrales normales soient réductibles aux intégrales elliptiques.

Je pourrais ajouter que l'on peut choisir les ε_{ik} de telle sorte qu'il y ait une infinité d'intégrales réductibles, mais je n'ai pas besoin pour mon objet de cette extension du résultat précédent.

§ 3. *Généralisation du Théorème d'Abel.*

Je suis obligé ici de faire une digression et de donner avant d'aller plus loin, une généralisation du théorème d'Abel qui me sera utile dans la suite. Soit:

$$f(x, y) = 0 \tag{1}$$

une courbe plane quelconque. Soit $u(x, y)$ une intégrale abélienne de 1[ère] espèce attachée à cette courbe. Soit c une courbe variable de degré donné m qui coupe la courbe (1) en q points variables:

$$x_1, y_1; x_2, y_2; \ldots\ldots; x_q, y_q.$$

La somme

$$u(x_1, y_1) + u(x_2, y_2) + \ldots\ldots + u(x_q, y_q)$$

sera une constante (quelle que soit la courbe c, pourvu toutefois que son degré m ne change pas).

Tel est le théorème d'Abel que je me propose d'étendre aux surfaces.

Je vais d'abord l'étendre aux courbes gauches.

Soit c une courbe gauche quelconque et x, y, z un point mobile sur cette courbe. Nous pourrons mettre l'équation de cette courbe c sous la forme suivante:

$$f(x, y) = 0, \quad z = \frac{\phi(x, y)}{\psi(x, y)},$$

f, ϕ et ψ étant des polynomes entiers en x et en y. L'intersection complète des deux surfaces:

$$f = 0, \quad \psi z - \phi = 0,$$

se compose alors de la courbe c et d'un certain nombre de droites parallèles à l'axe des z et qui sont les droites communes aux trois cylindres:

$$f(x, y) = 0, \quad \psi(x, y) = 0, \quad \phi(x, y) = 0.$$

D'ailleurs toutes les droites communes aux deux premiers de ces cylindres, doivent également appartenir au troisième. Je renverrai pour plus de détails au Mémoire de M. Halphen sur les Courbes gauches algébriques, couronné par l'Académie de Berlin.

Il existera un certain nombre d'intégrales:

$$u(x, y, z) = \int R(x, y, z)\, dx$$

(où R est une fonction rationnelle de x, de y et de z; y et z étant définis en fonctions de x par les équations de la courbe c) qui resteront finies en tous les points de c.

Ce seront les intégrales de 1[ère] espèce attachées à la courbe c.

Le théorème d'Abel s'applique à ces intégrales. Considérons une surface algébrique d'ordre m qui coupe c en q points:

$$(x_1, y_1, z_1), \quad (x_2, y_2, z_2), \ldots\ldots, (x_q, y_q, z_q)$$

la somme

$$u(x_1, y_1, z_1) + u(x_2, y_2, z_2) + \ldots\ldots + u(x_q, y_q, z_q)$$

restera constante quand on fera varier cette surface, pourvu que m reste constant.

Cela est presque évident et pourrait se démontrer directement par le même procédé que le théorème d'Abel relatif aux courbes planes. On peut aussi le déduire aisément de ce théorème:

L'intégrale u peut se mettre sous la forme:

$$\int R\left[x, y, \frac{\phi(x, y)}{\psi(x, y)}\right] dx = \int R_1(x, y)\, dx.$$

R_1 étant rationnel en x et en y. C'est alors une intégrale abélienne attachée à la courbe plane $f = 0$.

Soit maintenant

$$\theta(x, y, z) = z^m + \theta_1(x, y) z^{m-1} + \theta_2(x, y) z^{m-2} + \ldots + \theta_m(x, y) = 0$$

l'équation d'une surface quelconque S d'ordre m.

Si on y remplace z par sa valeur $\frac{\phi}{\psi}$, il vient:

(2) $$\phi^m + \theta_1 \psi \phi^{m-1} + \theta_2 \psi^2 \phi^{m-2} + \ldots + \theta_m \psi^m = 0.$$

Si l'on appelle n et $n+1$ les degrés des deux polynômes ψ et ϕ, l'équation (2) sera l'équation d'une courbe plane de degré $m(n+1)$. Cette courbe plane coupera la courbe $f = 0$ en un certain nombre de points. Parmi les points d'intersection, il y en aura q

$$(x_1, y_1), (x_2, y_2), \ldots, (x_q, y_q)$$

qui correspondront aux q points communs à la courbe C et à la surface S. Les autres correspondront aux droites communes aux trois cylindres:

$$f = 0, \ \phi = 0, \ \psi = 0$$

la trace de chacune de ces droites sur le plan des xy comptant pour m points d'intersection.

Les points d'intersection de la première sorte sont mobiles, les autres sont fixes. Mais dans l'application du théorème d'Abel aux courbes planes les points d'intersection fixes ne doivent pas intervenir. Nous pouvons donc écrire:

$$u(x_1, y_1) + u(x_2, y_2) + \ldots + u(x_q, y_q) = \text{const.}$$

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Supposons en particulier que la courbe C soit l'intersection complète de deux surfaces algébriques:

$$f(x, y, z) = 0, \quad \phi(x, y, z) = 0$$

de degrés m et n. Il est aisé de former alors les intégrales de 1ère espèce.

Supposons que la courbe C n'ait pas de point singulier, ce qui veut dire que les trois déterminants fonctionnels:

$$\frac{df}{dy}\frac{d\phi}{dz}-\frac{df}{dz}\frac{d\phi}{dy},\quad \frac{df}{dz}\frac{d\phi}{dx}-\frac{df}{dx}\frac{d\phi}{dz},\quad \frac{df}{dx}\frac{d\phi}{dy}-\frac{df}{dy}\frac{d\phi}{dx}$$

ne peuvent pas s'annuler à la fois en un point de la courbe.

Je dis que l'intégrale:

$$u=\int\frac{P(x,y,z)\,dx}{\frac{df}{dy}\frac{d\phi}{dz}-\frac{df}{dz}\frac{d\phi}{dy}},$$

où P est un polynôme quelconque de degré $m+n-4$, sera de première espèce.

En effet elle reste finie quand x, y et z deviennent infinis; elle ne pourrait donc devenir infinie que si le dénominateur

$$\frac{df}{dy}\frac{d\phi}{dz}-\frac{df}{dz}\frac{d\phi}{dy}$$

s'annulait. Mais on a:

$$u=\int\frac{Pdx}{\frac{df}{dy}\frac{d\phi}{dz}-\frac{df}{dz}\frac{d\phi}{dy}}=\int\frac{Pdy}{\frac{df}{dz}\frac{d\phi}{dx}-\frac{df}{dx}\frac{d\phi}{dz}}=\int\frac{Pdz}{\frac{df}{dx}\frac{d\phi}{dy}-\frac{df}{dy}\frac{d\phi}{dx}}.$$

L'intégrale u ne pourrait donc devenir infinie que si les trois déterminants fonctionnels s'annulaient à la fois, ce qui n'a pas lieu, par hypothèse.

Donc elle restera toujours finie.

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Passons maintenant aux surfaces. Soit:

$$f(x,y,z)=0$$

une surface algébrique S.

M. Picard a démontré qu'il n'existe pas pour toutes les surfaces d'intégrales de différentielle exacte de 1ère espèce, c'est à dire d'intégrale de la forme:

$$u=\int(Rdx+R_1dy)$$

qui reste toujours finie, où la quantité sous le signe $\int$ est une différentielle exacte, où enfin R et R_1 sont rationnels en x, y et z.

Je dis que dans ce cas le théorème d'Abel est encore applicable.

Voici ce que j'entends par là.

On sait que pour définir une famille de courbes gauches, il ne suffit pas de s'en donner le degré, mais qu'il faut connaître également plusieurs autres nombres caractéristiques. Je renverrai d'ailleurs pour plus de détails au Mémoire cité de

M. Halphen. Quoi qu'il en soit, nous dirons que deux courbes gauches, sans point singulier, appartiennent à la même famille quand tous les nombres définis par M. Halphen seront les mêmes pour les deux courbes. On peut alors passer de l'une à l'autre par variation continue.

Soit alors une courbe gauche C qui varie, mais de façon à appartenir toujours à la même famille. Soient:

$$(x_1, y_1, z_1), \ldots\ldots, (x_q, y_q, z_q)$$

ses q points d'intersection avec la surface S. Soit $u_1, u_2, \ldots\ldots, u_q$ les valeurs de l'intégrale u en ces q points. La somme

$$u_1 + u_2 + \ldots\ldots + u_q$$

sera une constante.

Commençons par envisager le cas particulier où la courbe C est l'intersection complète de deux surfaces algébriques d'ordre m et n, et que j'appellerai S_1 et S_2. Alors les q points d'intersection de C et de S seront les q points communs aux trois surfaces:

$$S, S_1, S_2.$$

L'intersection de S et de S_1 est une courbe gauche et l'intégrale u d'après sa définition même, sera une intégrale de 1[ère] espèce attachée à cette courbe gauche. Nous pouvons donc faire varier la surface S_2 sans que la somme:

$$u_1 + u_2 + \ldots\ldots + u_q$$

varie. Pour la même raison, cette même somme ne variera pas quand on fera varier S_1. Donc quelles que soient les surfaces S_1 et S_2 on aura:

$$u_1 + u_2 + \ldots\ldots + u_q = k;$$

k étant une constante.

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En particulier, si $m = n = 1$, la courbe C se réduit à une droite, et on a pour toutes les droites de l'espace

$$u_1 + u_2 + \ldots\ldots + u_q = k;$$

k étant une constante.

Je dis maintenant que $k = mnk$.

En effet faisons dégénérer la surface S_1 en m plans et la surface S_2 en n plans; la courbe C dégénérera en mn droites. La somme Σu étant égale à k lorsqu'on envisage une droite isolée, devra être égale à mnk quand on envisagera un système de mn droites.

Envisageons maintenant une courbe gauche de degré d qui ne soit pas une intersection complète. Son équation pourra toujours s'écrire :

$$F(x, y) = 0, \quad \psi(x, y) z - \phi(x, y) = 0.$$

F, ψ et ϕ étant des polynômes dont le degré est respectivement d, n et $n + 1$.

Les deux surfaces algébriques

$$F = 0, \quad \psi z - \phi = 0$$

sont de degré d et $n + 1$. Leur intersection complète se compose de la courbe C et de nd droites parallèles à l'axe des z. Soient:

$$(x_1, y_1, z_1), \ldots, (x_q, y_q, z_q)$$

les q points d'intersection de C et de S et

$$(x_{q+1}, y_{q+1}, z_{q+1}), \ldots, (x_p, y_p, z_p)$$

les $p - q$ points d'intersection des nd droites dont je viens de parler et de S.

On aura alors d'après ce que nous venons de voir:

$$u(x_1, y_1, z_1) + u(x_2, y_2, z_2) + \ldots + u(x_p, y_p, z_p) = (n + 1) dk.$$

D'autre part on aura pour les nd droites

$$u(x_{q+1}, y_{q+1}, z_{q+1}) + u(x_{q+2}, y_{q+2}, z_{q+2}) + \ldots + u(x_p, y_p, z_p) = ndk.$$

On a donc:

$$u(x_1, y_1, z_1) + u(x_2, y_2, z_2) + \ldots + u(x_q, y_q, z_q) = dk.$$

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Cela montre en même temps que la somme Σu est la même pour deux courbes de même degré, quand même ces deux courbes n'appartiennent pas à la même famille.

On comprendra, sans que j'insiste d'avantage, que le théorème d'Abel s'applique encore aux intégrales de 1ère espèce de différentielles totales de la forme :

$$\int R_1 dx_1 + R_2 dx_2 + \ldots + R_n dx_n,$$

où les R sont des fonctions rationnelles de $x_1, x_2, \ldots, x_n$ et z, et où z est défini par une équation algébrique :

$$F(x_1, x_2, \ldots, x_n, z) = 0.$$

Je vais maintenant, quoique cela ne soit pas nécessaire pour mon objet principal, montrer comment et dans quelle mesure le théorème d'Abel peut s'étendre aux surfaces qui n'admettent pas d'intégrale de 1ère espèce.

En ce qui concerne les courbes planes, sans point singulier, ce théorème peut s'énoncer de la façon suivante:

Soit $f=0$ une courbe algébrique de degré m; soient $\phi=0$, $\phi+\varepsilon\psi=0$ deux autres courbes algébriques de même degré et infiniment peu différentes l'une de l'autre.

Soient (x_1, y_1), (x_2, y_2), ..., (x_q, y_q) les q points d'intersection de $f=0$, $\phi=0$; soient (x_1+dx_1, y_1+dy_1), ..., (x_q+dx_q, y_q+dy_q) les q points d'intersection de $f=0$, $\phi=\varepsilon\psi=0$. On aura:

$$\sum_{\nu=1}^{q}\frac{P(x_\nu, y_\nu)\,dx_\nu}{\dfrac{df}{dy_\nu}}.$$

P est un polynôme quelconque d'ordre $m-3$.

De même, en ce qui concerne les courbes gauches, sans point singulier, le théorème d'Abel peut s'énoncer comme il suit:

Nous ne considérerons qu'une courbe gauche, intersection complète de deux surfaces algébriques:

$$f=0,\quad f_1=0,$$

de degrés m et n.

Soient encore $\phi=0$, $\phi+\varepsilon\psi=0$, deux surfaces algébriques infiniment voisines l'une de l'autre. La courbe gauche coupe la surface $\phi=0$ en q points (x_ν, y_ν, z_ν) et la surface $\phi+\varepsilon\psi=0$ en q points $(x_\nu+dx_\nu,\ y_\nu+dy_\nu,\ z_\nu+dz_\nu)$ et l'on a:

$$\sum\frac{P(x_\nu, y_\nu, z_\nu)\,dx_\nu}{\dfrac{df}{dy_\nu}\dfrac{df_1}{dz_\nu}-\dfrac{df_1}{dy_\nu}\dfrac{df}{dz_\nu}}=0.$$

P étant un polynôme quelconque de degré $m+n-4$.

Passons maintenant aux surfaces; soit $f=0$ une surface algébrique d'ordre m. Considérons son intersection avec une courbe gauche variable, intersection complète de deux surfaces $\phi=0$, $\phi_1=0$ d'ordres n et p. La surface $f=0$ coupera une de ces courbes gauches C en q points (x_ν, y_ν, z_ν) et une courbe gauche C', infiniment voisine de C en q points $(x_\nu+dx_\nu, y_\nu+dy_\nu, z_\nu+dz_\nu)$.

Soient $\phi=0$, $\phi_1=0$ les équations de la courbe C et $\phi+\varepsilon\psi=0$, $\phi_1+\varepsilon\psi_1=0$ les équations de C', ε étant infiniment petit.

Nous envisagerons la courbe C'' qui a pour équations

$$\phi+\varepsilon\psi=0,\quad \phi_1=0$$

et nous appellerons

$$(x_\nu + \delta x_\nu,\ y_\nu + \delta y_\nu,\ z_\nu + \delta z_\nu)$$

ses q points d'intersection avec la surface $f = 0$. Nous poserons ensuite :

$$dx_\nu = \delta x_\nu + \partial x_\nu,\quad dy_\nu = \delta y_\nu + \partial y_\nu,\quad dz_\nu = \delta z_\nu + \partial z_\nu.$$

Les deux courbes C et C'' étant sur la même surface $\phi_1 = 0$, on aura :

$$\frac{d\phi_1}{dx_\nu}\delta x_\nu + \frac{d\phi_1}{dy_\nu}\delta y_\nu + \frac{d\phi_1}{dz_\nu}\delta z_\nu = 0.$$

De même les deux courbes C'' et C' étant sur la même surface $\phi + \varepsilon\psi = 0$, on peut, en négligeant ε, écrire :

$$\frac{d\phi}{dx_\nu}\partial x_\nu + \frac{d\phi}{dy_\nu}\partial y_\nu + \frac{d\phi}{dz_\nu}\partial z_\nu = 0. \tag{2}$$

Appliquons le théorème d'Abel à l'intersection de la courbe gauche $f = 0$, $\phi_1 = 0$, avec les deux surfaces infiniment voisines $\phi = 0$, $\phi + \varepsilon\psi = 0$. Il viendra :

$$\sum \frac{P_\nu \delta x_\nu}{\dfrac{\partial(f, \phi_1)}{\partial(y_\nu, z_\nu)}} = 0.$$

Dans cette équation P_ν désigne un polynôme de degré $m + p - 4$, où x, y, z ont été remplacés par x_ν, y_ν, z_ν et $\dfrac{\partial(f, \phi_1)}{\partial(y_\nu, z_\nu)}$ représente suivant l'usage le déterminant fonctionnel de f et de ϕ_1 par rapport à y_ν et à z_ν.

Mais on a identiquement :

$$\frac{\delta x_\nu}{\dfrac{\partial(f,\phi_1)}{\partial(y_\nu, z_\nu)}} = \frac{\delta y_\nu}{\dfrac{\partial(f,\phi_1)}{\partial(z_\nu, x_\nu)}} = \frac{\delta z_\nu}{\dfrac{\partial(f,\phi_1)}{\partial(x_\nu, y_\nu)}} = \frac{\dfrac{d\phi}{dx_\nu}\delta x_\nu + \dfrac{d\phi}{dy_\nu}\delta y_\nu + \dfrac{d\phi}{dz_\nu}\delta z_\nu}{\dfrac{\partial(f,\phi,\phi_1)}{\partial(x_\nu, y_\nu, z_\nu)}}.$$

Je désignerai pour abréger par Δ_ν le dénominateur de la dernière de ces fractions.

On aura alors :

$$\sum_{\nu=1}^{q} \frac{P_\nu}{\Delta_\nu}\left(\frac{d\phi}{dx_\nu}\delta x_\nu + \frac{d\phi}{dy_\nu}\delta y_\nu + \frac{d\phi}{dz_\nu}\delta z_\nu\right) = 0.$$

Mais on a de même, à cause de (2) :

$$\sum \frac{P_\nu}{\Delta_\nu}\left(\frac{d\phi}{dx_\nu}\partial x_\nu + \frac{d\phi}{dy_\nu}\partial y_\nu + \frac{d\phi}{dz_\nu}\partial z_\nu\right) = 0.$$

Il vient donc:

$$\sum \frac{P_\nu}{\Delta_\nu}\left(\frac{d\phi}{dx_\nu}dx_\nu + \frac{d\phi}{dy_\nu}dy_\nu + \frac{d\phi}{dz_\nu}dz_\nu\right) = 0;$$

c'est la généralisation du théorème d'Abel.

On trouve de même:

$$\sum \frac{Q_\nu}{\Delta_\nu}\left(\frac{d\phi_1}{dx_\nu}dx_\nu + \frac{d\phi_1}{dy_\nu}dy_\nu + \frac{d\phi_1}{dz_\nu}dz_\nu\right) = 0.$$

Q_ν étant un polynôme de degré $m + n - 4$.

Un cas particulier intéressant est celui où la surface $f = 0$ se réduit à un plan; si $\phi = 0$, $\phi_1 = 0$ sont deux courbes planes de degré m se coupant en m^2 points (x_ν, y_ν) et si deux courbes de degré m infiniment voisines se coupent en m^2 points $(x_\nu + dx_\nu, y_\nu + dy_\nu)$, on aura:

$$\sum \frac{P(x_\nu, y_\nu)\left[\frac{d(\phi + \lambda\phi_1)}{dx_\nu}dx_\nu + \frac{d(\phi + \lambda\phi_1)}{dy_\nu}dy_\nu\right]}{\frac{d\phi}{dx_\nu}\frac{d\phi_1}{dy_\nu} - \frac{d\phi_1}{dx_\nu}\frac{d\phi}{dy_\nu}} = 0,$$

P étant un polynôme quelconque de degré $m - 3$ et λ étant une constante quelconque.

§ 4. *Fonctions intermédiaires.*

Envisageons un système de fonctions abéliennes à n variables $x_1, x_2, \ldots, x_n$ et à $2n$ périodes. A l'exemple de MM. Briot et Bouquet, j'appellerai fonction intermédiaire toute fonction entière des n variables qui se reproduit multipliée par une exponentielle quand les n variables augmentent d'une période.

Soit par exemple: $a_{1k}, a_{2k}, \ldots, a_{nk}$ une période. Une fonction entière Φ sera une fonction intermédiaire si l'on a:

$$\Phi(x_1 + a_{1k}, x_2 + a_{2k}, \ldots, x_n + a_{nk}) = \Phi(x_1, x_2, \ldots, x_n)\, e^{a_{1k}x_1 + a_{2k}x_2 + \cdots + a_{nk}x_n + \gamma_k}$$

(les α et les γ étant des constantes) et cela pour toutes les périodes.

Je vais supposer pour fixer les idées qu'il n'y a que deux variables x et y; j'appellerai les périodes fondamentales

$$\begin{matrix} a_1, & a_2, & a_3, & a_4 \\ b_1, & b_2, & b_3, & b_4 \end{matrix}$$

et les multiplicateurs correspondants seront:

$$e^{a_1x + \beta_1y + \gamma_1},\quad e^{a_2x + \beta_2y + \gamma_2},\quad e^{a_3x + \beta_3y + \gamma_3},\quad e^{a_4x + \beta_4y + \gamma_4}.$$

Si l'on augmente x et y d'abord de la période (a_1, b_1) puis de la période (a_2, b_2), les deux multiplicateurs successifs ont pour exposants d'abord

$$(\alpha_2 x + \beta_2 y + \gamma_2) \text{ puis } (\alpha_1 x + \beta_1 y + \gamma_1 + \alpha_1 a_2 + \beta_1 b_2).$$

Le résultat devant être le même dans les deux cas, le nombre :

$$\alpha_2 a_1 + \beta_2 b_1 - \alpha_1 a_2 - \beta_1 b_2 = M_{1.2}$$

devra être égal à un entier multiplié par $2i\pi$. Il en sera de même des expressions analogues M_{ik} où les indices i et k ou des valeurs quelconques.

Considérons en particulier une exponentielle de la forme suivante :

$$e^P \text{ où } P = Ax^2 + 2Bxy + Cy^2 + 2Dx + 2Ey + F.$$

Si l'on augmente x et y de a et de b, l'exponentielle se trouvera multipliée par :

$$e^{\alpha x + \beta y + \gamma}$$

où

$$\alpha = 2Aa + 2Bb,$$
$$\beta = 2Ba + 2Cb,$$
$$\gamma = Aa^2 + 2Bab + Cb^2 + 2Da + 2Eb.$$

Il vient donc :

$$\frac{1}{2} M_{ik} = (Aa_i + Bb_i)\, a_k + (Ba_i + Cb_i)\, b_k - (Aa_k + Bb_k)\, a_i - (Ba_k + Cb_k)\, b_i = 0.$$

Ainsi pour une exponentielle e^P, tous les M_{ik} sont nuls. Il est aisé de voir que ces exponentielles sont les seules fonctions intermédiaires qui jouissent de cette propriété. Car si une autre fonction intermédiaire Φ en jouissait, on pourrait trouver une exponentielle e^{-P} telle que la fonction Φe^{-P}, qui est également intermédiaire, eût tous ses multiplicateurs égaux à 1. Il en résulterait que cette fonction Φe^{-P} qui est *entière* serait quadruplement périodique ; elle se réduirait donc à une constante, de sorte qu'on devrait avoir :

$$\Phi = Ce^P.$$

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Supposons maintenant que les périodes :

$$\begin{matrix} a_1, & a_2, & a_3, & a_4 \\ b_1, & b_2, & b_3, & b_4 \end{matrix}$$

que nous considérons soient des périodes normales, de sorte que l'on ait :

(1) $$a_1 b_3 - a_3 b_1 + a_2 b_4 - a_4 b_2 = 0.$$

Posons :

$$\begin{matrix} a_1' = a_3 & a_3' = -a_1 & a_2' = a_4 & a_4' = -a_2 \\ b_1' = b_3 & b_3' = -b_1 & b_2' = b_4 & b_4' = -b_2. \end{matrix}$$

On aura alors : $$\Sigma a_i a_i' = \Sigma a_i b_i' = 0.$$

Je dis maintenant qu'on a:

(2) $$\Sigma_{ik} M_{ik} (a'_i b'_k - a'_k b'_i) = 0,$$

la somme étant étendue aux 6 combinaisons possibles des nombres i et k.

En effet cela peut s'écrire:

$$\Sigma_{ik} M_{ik} a'_i b'_k = 0,$$

où cette fois on donnera à i et à k, indépendamment l'un de l'autre les valeurs 1, 2, 3, 4; cela fera donc en tout 16 termes dont 4 seront nuls parce que $M_{ii} = 0$.

La relation précédente devient alors:

$$\Sigma_{ik} (\alpha_i a_k + \beta_i b_k - \alpha_k a_i - \beta_k b_i) a'_i b'_k = 0$$

ou bien

$$\Sigma_i (\alpha_i a'_i \Sigma_k a_k b'_k) + \Sigma_i (\beta_i b_i \Sigma_k b_k b'_k) - \Sigma_k (\alpha_k b'_k \Sigma_i a_i a'_i) - \Sigma_k (\beta_k \beta'_k \Sigma_i b_i a'_i) = 0.$$

La relation est donc vérifiée, puisqu'on a:

$$\Sigma_k a_k b'_k = \Sigma_k b_k b'_k = \Sigma_i a_i a'_i = \Sigma_i b_i a'_i = 0.$$

Cela posé, de deux choses l'une, ou bien les relations (1) et (2) sont distinctes, où elles ne le sont pas.

Je ne veux pas démontrer ici que ces relations ne seront jamais distinctes à moins que les intégrales abéliennes considérées ne soient réductibles aux intégrales elliptiques. La démonstration serait sans doute fort longue.

Laissons de côté ce cas exceptionnel et supposons que les deux relations ne sont pas distinctes, et par conséquent que les M_{ik} sont nuls, à l'exception de $M_{1.3}$ et de $M_{2.4}$ qui sont égaux entre eux.

$$M_{1.3} = M_{2.4} = 2mi\pi.$$

Le nombre m devra toujours être de même signe; il sera positif si comme on le suppose d'ordinaire, on a:

$$a_1^0 a_3^1 - a_1^1 a_3^0 + a_2^0 a_4^1 - a_2^1 a_4^0 > 0$$

en désignant par a_i^0 et a_i^1 les parties réelle et imaginaire de a_i.

Nous dirons alors que la fonction intermédiaire envisagée est d'ordre m.

La fonction intermédiaire sera une fonction Θ d'ordre m si les quatre multiplicateurs sont:

$$1, \ 1, \ e^{mx+\gamma_3}, \ e^{my+\gamma_4},$$

les huit périodes étant:

pour x:	$2i\pi$	0	G	H,
pour y:	0	$2i\pi$	H	G'.

Il est aisé de voir:

1°. Que toute fonction intermédiaire peut être regardée comme le produit d'une fonction Θ et d'une exponentielle de la forme e^P.

2°. Que toutes les fonctions Θ d'ordre m qui ont mêmes multiplicateurs sont des fonctions linéaires de m^2 d'entre elles.

Il en résulte que toutes les fonctions linéaires qui ont mêmes multiplicateurs sont des fonctions linéaires de m^2 d'entre elles.

Cela posé, on peut définir le multiplicateur correspondant à une période quelconque (a, b) en se donnant les trois nombres (α, β, γ).

Si par exemple on a:

$$\Phi(x+a, y+b) = \Phi(x, y)\, e^{\alpha x + \beta y + \gamma}$$

le multiplicateur serait défini par les trois nombres (α, β, γ). Mais il est préférable d'envisager trois autres nombres (α, β, δ) dont le dernier δ est défini comme il suit:

$$\delta = \gamma - \frac{1}{2}(a\alpha + b\beta)$$

ou plutôt

$$\delta \equiv \gamma - \frac{1}{2}(a\alpha + b\beta) \pmod{2i\pi}.$$

Car on peut évidemment augmenter δ d'un multiple de $2i\pi$ sans changer le multiplicateur.

Soient donc (a, b), (a', b') deux périodes quelconques. Les multiplicateurs seront définis par les deux systèmes de nombres (α, β, γ), $(\alpha', \beta', \gamma')$ ou bien encore par les deux systèmes de nombres (α, β, δ), $(\alpha', \beta', \delta')$ en posant:

$$\delta = \gamma - \frac{1}{2}(a\alpha + b\beta), \quad \delta' = \gamma' - \frac{1}{2}(a'\alpha' + b'\beta').$$

Considérons maintenant la période $(a + a', b + b')$; le multiplicateur correspondant sera défini par les trois nombres $(\alpha'', \beta'', \gamma'')$ ou bien encore par les nombres $(\alpha'', \beta'', \delta'')$, on trouve aisément:

$$\alpha'' = \alpha + \alpha', \quad \beta'' = \beta + \beta',$$

$$\gamma'' \equiv \gamma + \gamma' + a'\alpha + b'\beta \equiv \gamma + \gamma' + a\alpha' + b\beta' \pmod{2i\pi},$$

$$\delta'' \equiv \gamma'' - \frac{1}{2}[\alpha'''(a + a') + \beta''(b + b')] \equiv \delta + \delta' + \frac{1}{2}(a\alpha' + b\beta' - a\alpha' - b\beta').$$

Or d'après ce que nous avons vu:

$$a'\alpha + b'\beta - a\alpha' - b\beta' = 2ki\pi,$$

k étant un entier. On aura donc, selon que l'entier k sera pair ou impair:

$$\delta''' \equiv \delta + \delta'$$

ou

$$\delta''' \equiv \delta + \delta' + i\pi \pmod{2i\pi}.$$

Cherchons donc à déterminer le nombre k.

Supposons que l'on ait :

$$\left.\begin{array}{l} a = \xi_1 a_1 + \xi_2 a_2 + \xi_3 a_3 + \xi_4 a_4 \\ a' = \eta_1 a_1 + \eta_2 a_2 + \eta_3 a_3 + \eta_4 a_4 \end{array}\right\} \text{(les } \xi \text{ et les } \eta \text{ étant entiers)},$$

et de même pour b et b'. Il viendra :

$$k = m(\xi_1\eta_3 - \xi_3\eta_1 + \xi_2\eta_4 - \xi_4\eta_2).$$

Si donc m est pair, on aura toujours :

$$\delta'' \equiv \delta + \delta'.$$

Si au contraire, m est impair, tout dépend de la parité de la parenthèse.

Considérons donc une période quelconque

$$a = \xi_1 a_1 + \xi_2 a_2 + \xi_3 a_3 + \xi_4 a_4,$$
$$b = \xi_1 b_1 + \xi_2 b_2 + \xi_3 b_3 + \xi_4 b_4,$$

et proposons nous de trouver le multiplicateur correspondant (α, β, δ). On trouvera aisément :

$$\alpha = \xi_1\alpha_1 + \xi_2\alpha_2 + \xi_3\alpha_3 + \xi_4\alpha_4,$$
$$\beta = \xi_1\beta_1 + \xi_2\beta_2 + \xi_3\beta_3 + \xi_4\beta_4.$$

On aura de plus :

$$\delta \equiv \xi_1\delta_1 + \xi_2\delta_2 + \xi_3\delta_3 + \xi_4\delta_4 \quad (\text{mod. } 2i\pi),$$

si m est pair ou si

$$\xi_1\xi_3 + \xi_2\xi_4$$

est pair.

On aura au contraire :

$$\delta \equiv \xi_1\delta_1 + \xi_2\delta_2 + \xi_3\delta_3 + \xi_4\delta_4 + i\pi,$$

si m et $\xi_1\xi_3 + \xi_2\xi_4$ sont impairs.

Nous allons maintenant introduire six nombres (3):

$$(a, \alpha), \ (a, \beta), \ (a, \delta), \ (b, \alpha), \ (b, \beta), \ (b, \delta)$$

définis comme il suit ; considérons la forme bilinéaire

$$x_1y_3 - x_3y_1 + x_2y_4 - x_4y_2$$

et substituons a_1, a_2, a_3, a_4 à la place de x_1, x_2, x_3, x_4 et $\alpha_1, \alpha_2, \alpha_3, \alpha_4$ à la place de y_1, y_2, y_3, y_4 ; nous aurons (a, α) ; de même pour les autres.

Nous poserons de plus

$$(a, \delta) = 2Ri\pi,$$
$$(b, \delta) = 2Si\pi.$$

Nous conviendrons d'écrire :

$$(x', y') \equiv (x, y),$$

lorsque la différence $x' - x, \ y' - y$ sera une période.

Si donc on augmente chacun des δ d'un multiple de $2i\pi$, ce qui ne change pas les multiplicateurs, les nombres R et S deviendront R' et S'; mais on aura :

$$(R', S') \equiv (R, S).$$

Voyons maintenant ce que deviennent nos 6 nombres (3).

1°. Quand on change de variables.

2°. Quand on change de périodes.

3°. Quand on multiplie la fonction intermédiaire par une exponentielle de la forme e^P.

1°. *Changement d'origine.*

Imaginons qu'on pose :

$$x = x' + h, \quad y = y' + k,$$

quels seront les 6 nombres relatifs aux nouvelles variables x', y' ?

Il est évident que les 4 nombres

$$(a, \alpha), \quad (a, \beta), \quad (b, \alpha), \quad (b, \beta)$$

ne changeront pas. Quant à R et à S, ils augmenteront respectivement par

$$\frac{1}{2i\pi}[h(a, \alpha) + k(a, \beta)] \text{ et } \frac{1}{2i\pi}[h(b, \alpha) + k(b, \beta)].$$

2°. *Changement linéaire.*

Posons:

$$\begin{aligned} x' &= \lambda x + \mu y, & \Delta x &= \mu_1 x' - \mu y', \\ y' &= \lambda_1 x + \mu_1 y, & \Delta y &= -\lambda_1 x' + \lambda y', \\ & & \Delta &= \lambda\mu_1 - \lambda_1\mu. \end{aligned}$$

On aura alors, en appelant a', b', α', β' les nouvelles valeurs de a, b, α, β:

$$\begin{aligned} a' &= \lambda a + \mu b, & \Delta\alpha' &= \mu_1\alpha - \lambda_1\beta, \\ b' &= \lambda_1 a + \mu_1 b, & \Delta\beta' &= -\mu\alpha + \lambda\beta. \end{aligned}$$

Si nous formons alors les nouveaux nombres :

$$(a', \alpha'), (a', \beta'), \text{ etc.,}$$

il viendra : $$(a', \alpha') = \frac{\lambda\mu_1}{\Delta}(a, \alpha) + \frac{\mu\mu_1}{\Delta}(b, \alpha) - \frac{\lambda\lambda_1}{\Delta} - \frac{\mu\lambda_1}{\Delta}(b, \beta)$$

et de même pour les trois autres.

Les quatre nouveaux nombres tels que (a', α') s'exprimeront donc linéairement en fonctions des anciens.

Voici le tableau des coëfficients:

	$\frac{(a,\,\alpha)}{\Delta}$	$\frac{(b,\,\alpha)}{\Delta}$	$\frac{(a,\,\beta)}{\Delta}$	$\frac{(b,\,\beta)}{\Delta}$
$(a',\,\alpha')$	$\lambda\mu_1$	$\mu\mu_1$	$-\lambda\lambda_1$	$-\mu\lambda_1$
$(b',\,\alpha')$	$\lambda_1\mu_1$	μ_1^2	$-\lambda_1^2$	$-\lambda_1\mu_1$
$(a',\,\beta')$	$-\lambda\mu$	$-\mu^2$	$+\lambda^2$	$\lambda\mu$
$(b',\,\beta')$	$-\lambda_1\mu$	$-\mu_1\mu$	$\lambda\lambda_1$	$\lambda\mu_1$.

Ce tableau montre immédiatement que:

$$(a,\,\alpha)+(b,\,\beta)$$

est un invariant et il est aisé de voir en effet que cette expression est égale à:

$$M_{1.3}+M_{1.4}=4mi\pi.$$

Mais il y a plus. Supposons que l'on ait

$$(a,\,\alpha)=(b,\,\beta)=2mi\pi$$

et

$$(a,\,\beta)=(b,\,\alpha)=0,$$

le tableau précédent montre qu'on aura encore:

$$(a',\,\alpha')=(b',\,\beta')=2mi\pi$$

et

$$(a',\,\beta')=(b',\,\alpha')=0.$$

Nous allons voir maintenant que les quatre nombres $(a,\,\alpha)$, etc., ont effectivement les valeurs que j'indique plus haut.

Pour une exponentielle de la forme e^P, on voit aisément que ces quatre nombres sont nuls.

Pour une fonction Θ les quatre périodes ont pour valeurs respectivement:

$$\begin{matrix} 2i\pi & 0 & G & H \\ 0 & 2i\pi & H & G' \end{matrix}$$

et les nombres α et β ont pour valeurs:

$$\begin{matrix} 0 & 0 & m & 0 \\ 0 & 0 & 0 & m \end{matrix}$$

Il en résulte que l'on a

$$(a,\,\alpha)=(b,\,\beta)=2mi\pi,$$
$$(a,\,\beta)=(b,\,\alpha)=0.$$

Ces valeurs resteront encore les mêmes quand on multipliera une fonction Θ par une exponentielle e^P; elles ne changeront pas non plus (ainsi que nous venons de le voir) quand on changera de variables. Elles sont donc les mêmes pour une fonction intermédiaire quelconque.

Qu'arrive-t-il maintenant des nombres R et S?

Il est aisé de voir que par suite du changement linéaire que nous envisageons, les γ et les δ ne changent pas. On aura donc, en appelant R' et S' les nouvelles valeurs de R et de S:

$$R' = \lambda R + \mu S,$$
$$S' = \lambda_1 R + \mu_1 S.$$

Revenons au changement de variables que nous avions envisagé d'abord, c'est à dire au changement d'origîne. Nous avons vu que R et S augmentaient respectivement de

$$\frac{1}{2i\pi}[h(a, \alpha) + k(a, \beta)],$$
$$\frac{1}{2i\pi}[h(b, \alpha) + k(b, \beta)].$$

Ces deux quantités, en tenant compte des valeurs des nombres (a, α), etc., se réduisent à: mh et mk.

Si donc on change de variables en posant:

$$x' = \lambda x + \mu y + h,$$
$$y' = \lambda_1 x + \mu_1 y + k,$$

il viendra:

$$R' = \lambda R + \mu S - mh,$$
$$S' = \lambda_1 R + \mu_1 S - mk,$$

ce qui peut s'énoncer ainsi:

Les nombres — $\frac{R}{m}$ et $\frac{S}{m}$ subissent le même changement que les variables ellesmêmes.

3°. Changement de périodes. Supposons que l'on remplace le système des périodes

$$a_1, \quad a_2, \quad a_3, \quad a_4,$$
$$b_1, \quad b_2, \quad b_3, \quad b_4,$$

que nous supposons être un système de périodes normales, par un autre système équivalent et également formé de périodes normales.

Les nouvelles valeurs des nombres α et β seront formées avec les anciennes, comme les nouvelles périodes avec les anciennes et il en sera encore de même des nouvelles valeurs des nombres δ à un multiple près de $i\pi$.

Il en résulte que les nouvelles valeurs des quatre nombres (a, α), (a, β), (b, α), (b, β) sont les mêmes que les anciennes; et que les nouvelles valeurs de R et de S sont aussi les mêmes que les anciennes à une demi-période près.

C'est cette demi-période qu'il faut maintenant chercher à déterminer.

Pour cela il faut envisager les changements simples de périodes qui sont les suivants :

1°. Permutation de a_1 et de a_3 en changeant le signe d'une des deux périodes.

2°. Changement simultané des signes a_1 et a_3.

3°. Permutation des deux paires de périodes (a_1, a_3), (a_2, a_4).

4°. Changement de a_1 en $a_1 + a_3$.

5°. Changement de a_1 en $a_1 + a_3$ et de a_4 en $a_4 - a_3$.

Les trois premières opérations ne peuvent altérer les nombres R et S. Il n'en est pas de même des deux dernières.

Si l'on change a_1 en $a_1 + a_3$, δ_1 va se changer en $\delta_1 + \delta_3 + \frac{1}{2} M_{1.3}$ ou en $\delta_1 + \delta_3 + mi\pi$. De sorte que le nouveau δ_1 sera congru à $\delta_1 + \delta_3$ ou à $\delta_1 + \delta_3 + i\pi$ selon que m sera pair ou impair. Si on appelle R' et S' les nouvelles valeurs de R et de S, on aura:

$$(R', S') \equiv (R, S),$$

si m est pair, et

$$\left(R', S'\right) \equiv \left(R + \frac{a_3}{2}, S + \frac{b_3}{2}\right),$$

si m est impair.

Dans la 5ème opération, le nouveau δ_1 sera $\delta_1 + \delta_4 + \frac{1}{2} M_{1.4}$ et le nouveau δ_4 sera $\delta_4 - \delta_3 - \frac{1}{2} M_{4.3}$.

Mais comme on a $M_{1.4} = M_{4.3} = 0$, on voit que cette cinquième opération n'altère pas R et S.

Ainsi donc dans le cas où m est pair, on a toujours

$$(R', S') \equiv (R, S)$$

et l'on a dans tous les cas possibles :

$$(2R', 2S') \equiv (2R, 2S).$$

4°. Multiplication par une exponentielle.

Supposons que l'on multiplie la fonction intermédiaire envisagée par l'exponentielle e^P, où

$$P = Ax^2 + 2Bxy + Cy^2 + 2Dx + 2Ey + F.$$

Nous avons trouvé pour l'exponentielle :

$$\alpha = 2Aa + 2Bb,$$
$$\beta = 2Ba + 2Cb,$$
$$\gamma = Aa^2 + 2Bab + Cb^2 + 2Da + 2Eb,$$

et par conséquent $$\delta = 2Da + 2Eb,$$

ce qui montre que les 6 nombres (a, α), (a, β), (a, δ), etc., sont tous nuls.

On ne change donc pas ces six nombres en multipliant une fonction intermédiaire par une exponentielle.

On voit donc dans quelle mesure nos six nombres peuvent être regardés comme des invariants.

Ces résultats se généralisent immédiatement et s'étendent au cas des fonctions abéliennes de plus de deux variables, mais il est tout à fait inutile que j'insiste sur ce point.

§ 5. *Transformation.*

Soit:

$$(1) \qquad \begin{matrix} a_1, & a_2, & a_3, & a_4, \\ b_1, & b_2, & b_3, & b_4, \end{matrix}$$

un système de périodes normales, tel que:

$$a_1,\ b_3 - a_3,\ b_1 + a_2,\ b_4 - a_4,\ b_2 = 0\,.$$

Envisageons maintenant un second système de périodes:

$$(2) \qquad \begin{matrix} c_1, & c_2, & c_3, & c_4, \\ d_1, & d_2, & d_3, & d_4, \end{matrix}$$

qu'on obtiendra en posant:

$$\begin{matrix} c_i = \Sigma\lambda_{ik}a_k \\ d_i = \Sigma\lambda_{ik}b_k \end{matrix} \qquad (i, k = 1, 2, 3, 4)$$

les λ_{ik} étant des nombres entiers.

Nous supposerons que ces nombres entiers λ aient été choisis de telle sorte que l'on ait identiquement:

$$c_1d_3 - c_3d_1 + c_2d_4 - c_4d_2 = \mu\,(a_1b_3 - a_3b_1 + a_2b_4 - a_4b_2).$$

Le déterminant des λ sera alors égal à μ^2 et l'opération par laquelle on passe du système des périodes (1) au système (2) s'appellera une transformation d'ordre μ.

Il est évident que toute fonction qui admettra le système de périodes (1), admettra également le système (2), d'où il suit que les fonctions abéliennes engendrées par le système (1) ne sont que des cas particuliers de celles qui sont engendrées par le système (2).

Si l'on considère une fonction intermédiaire relative au système (1), ce sera aussi une fonction intermédiaire par rapport au système (2), mais la réciproque n'est pas vraie. Une fonction intermédiaire relative au système (2) n'est pas toujours une fonction intermédiaire par rapport au système (1).

Soit
$$e^{\alpha_i x + \beta_i y + \gamma_i}$$
le multiplicateur d'une fonction intermédiaire relative au système (1) par rapport à la période (a_i, b_i) et
$$e^{\alpha'_i x + \beta'_i y + \gamma'_i}$$
son multiplicateur par rapport à la période (c_i, d_i).

Nous poserons comme dans le paragraphe précédent:
$$\delta_i = \gamma_i - \frac{1}{2}(a_i\alpha_i + b_i\beta_i);\ \delta'_i = \gamma'_i - \frac{1}{2}(c_i\alpha'_i + d_i\beta'_i),$$
et il viendra

(3) $$\alpha'_i = \Sigma\lambda_{ik}\alpha_k,\ \beta'_i = -\Sigma\lambda_{ik}\beta_k,$$

(4) $$\delta \equiv \Sigma\lambda_{ik}\delta_k \quad (\text{mod. } i\pi).$$

Il en résulte que si l'on forme les six nombres:
$$(c, \alpha'), (c, \beta'), (c, \delta'), (d, \alpha'), \text{etc.};$$
on aura: $$(c, \alpha') = \mu(a, \alpha), (c, \beta') = \mu(a, \beta), \text{etc.}$$

On en conclut que si une fonction intermédiaire est d'ordre m par rapport au système (1), elle sera d'ordre $m\mu$ par rapport au système (2).

Posons maintenant:
$$R_1 = \frac{1}{2i\pi}(c, \delta'),\ S_1 = \frac{1}{2i\pi}(d, \delta');$$
on trouvera: $$(2R_1, 2S_1) \equiv (2\mu R, 2\mu S);$$
c'est à dire que les nombres R_1, S_1 ne différeront des nombres μR, μS que d'une demi-période du système (2).

Cela posé, imaginons qu'on se donne les α', les β' et les δ' et qu'on se propose de déterminer les α, les β et les δ. Les α et les β se calculeront sans ambiguité par le moyen des équations (3). Pour déterminer les δ, il faut mettre les congruences (4) sous une forme plus précise.

Nous les écrirons:

(4) $$\delta'_i \equiv \Sigma\lambda_{ik}\delta_k + \varepsilon_i \quad (\text{mod. } 2i\pi)$$

ε_i ne dépendra que des λ_{ik} et sera égal tantôt à 0, tantôt à $i\pi$ (cf. §§ précédent).

Nous n'avons besoin de connaître les δ qu'à un multiple près de $2i\pi$. Il est aisé de conclure que les congruences (4) comportent un nombre de solutions égal au déterminant des λ, c'est à dire à μ^2.

Je vais maintenant résoudre deux problèmes inverses.

1°. Former les fonctions intermédiaires du système (2) à l'aide de celles du système (1).

Nous nous donnons les nombres α', β' et δ' correspondant à un système de fonctions intermédiaires d'ordre μ, relatives au système (2).

Des égalités et congruences (3) et (4) nous déduirons μ^2 systèmes de valeurs des nombres α, β et δ.

A chacun de ces systèmes de valeurs, correspondront m^2 fonctions intermédiaires d'ordre m relatives au système (1).

Par rapport au système (2), ces fonctions intermédiaires seront d'ordre $m\mu$ et correspondront aux nombres donnés α', β', δ'.

On aura donc en tout $m^2\mu^2$ pareilles fonctions d'ordre $m\mu$ qui seront linéairement indépendantes et à l'aide desquelles toutes les autres pourront par conséquent s'exprimer linéairement.

2°. Former les fonctions intermédiaires du système (1) à l'aide de celles du système (2).

Nous nous proposons de trouver une fonction d'ordre m admettant pour chacune des périodes du système (1) un multiplicateur défini par trois nombres donnés (α, β, δ).

Pour les périodes du système (2), le multiplicateur sera défini par les trois nombres $(\alpha', \beta', \delta')$ que l'on peut déduire des relations (3) et (4).

Toute combinaison des périodes (2) est aussi une combinaison des périodes (1); mais la réciproque n'est pas vraie. Parmi les combinaisons des périodes (1), on peut en choisir μ^2 que j'appellerai périodes principales et qui jouiront de la propriété suivante:

Une période quelconque, je veux dire une combinaison quelconque des périodes (1), est toujours égale à une période principale, augmentée d'une combinaison des périodes (2).

Soient alors:

$$(0, 0), (a_1'', b_1''), (a_2'', b_2''), \ldots\ldots, (a_\nu'', b_\nu''),$$

les μ^2 périodes principales ($\nu = \mu^2 - 1$) et

$$(0, 0, 0), (\alpha_1'', \beta_1'', \delta_1''), (\alpha_2'', \beta_2'', \delta_2''), \ldots\ldots, (\alpha_\nu'', \beta_\nu'', \delta_\nu''),$$

les nombres qui définissent les multiplicateurs correspondants.

Quant aux multiplicateurs eux-mêmes, nous les appellerons pour abréger:

$$1, \quad P_1, \quad P_2, \ldots\ldots, P_\nu.$$

Cela posé, soit $\Phi(x, y)$ une fonction intermédiaire d'ordre $m\mu$ par rapport au système (2) et admettant pour les périodes de ce système les mêmes multiplicateurs $(\alpha', \beta', \delta')$ que la fonction qu'il s'agit de construire.

Les fonctions suivantes :

$$P_1\Phi(x-a_1'', y-b_1''),\ P_2\Phi(x-a_2'', y-b_2''),\ \ldots\ldots,\ P_\nu\Phi(x-a_\nu'', y-b_\nu''),$$

admettront les mêmes multiplicateurs que la fonction Φ elle-même. Il est aisé de voir alors que la fonction :

$$\Phi(x, y) + \sum_{i=1}^{\nu} P_i\Phi(x-a_i'', y-b_i''),$$

est une fonction intermédiaire d'ordre m par rapport au système (1) et admettant les multiplicateurs (α, β, δ).

Le problème proposé est donc résolu.

Nous allons maintenant étudier plus particulièrement le cas où les intégrales abéliennes qui ont donné naissance au système de fonctions abéliennes envisagées, sont susceptibles d'être ramenées aux intégrales elliptiques.

Supposons pour fixer les idées qu'il n'y ait que trois variables x, y et z et écrivons le tableau des périodes sous la forme suivante :

pour x :	1	0	0	G	H''	H',
pour y :	0	1	0	H''	G'	H,
pour z :	0	0	1	H'	H	G''.

La variable $\alpha x + \beta y + \gamma z$ aura alors pour périodes :

$$\alpha, \beta, \gamma, \ (\alpha G + \beta H'' + \gamma H'),\ (\alpha H'' + \beta G' + \gamma H),\ (\alpha H' + \beta H + \gamma G'').$$

Si ces six périodes se réduisent à deux, nous dirons que la variable $\alpha x + \beta y + \gamma z$ est réductible. Les variables réductibles correspondent ainsi aux intégrales abéliennes réductibles aux intégrales elliptiques que nous avons envisagées dans les §§1 et 2.

Supposons qu'il y ait une variable réductible ; je puis toujours par les procédés du §1 ramener le tableau des périodes à la forme :

(5)

pour x :	1	0	0	G	$\frac{1}{a}$	0,	
pour y :	0	1	0	$\frac{1}{a}$	G'	H	(a étant un entier),
pour z :	0	0	1	0	H	G'',	

de telle sorte que la variable réductible soit précisement x.

Supposons maintenant qu'il y ait une autre variable réductible ; il y aura certainement une 3[ème] variable réductible, d'après les conclusions du paragraphe 2, et de plus, ou bien ces variables seront de la forme :

$$\beta y + \gamma z, \quad \beta' y + \gamma' z,$$

ou bien elle sera de la forme générale

$$\alpha x + \beta y + \gamma z, \quad \alpha' x + \beta' y + \gamma' z;$$

mais il y aura une infinité d'autres variables réductibles parmi lesquelles les deux suivantes: $\beta y + \gamma z, \; \beta' y + \gamma' z,$
ne dépendront que d'y et de z.

Dans tous les cas, s'il y a trois variables réductibles, ou s'il y en a une infinité, l'une de ces variables sera x, et deux autres seront des combinaisons d'y et de z seulement.

Cela posé, reprenons le tableau des périodes (5) et multiplions les 4ème, 5ème et 6ème périodes par a. Cette opération sera une transformation d'ordre a. Le tableau des périodes deviendra ainsi:

$$\begin{matrix} 1 & 0 & 0 & aG & 1 & 0, \\ 0 & 1 & 0 & 1 & aG' & aH, \\ 0 & 0 & 1 & 0 & aH & aG''. \end{matrix}$$

Retranchons maintenant la 1ère période de la 5ème et la 2ème de la 4ème, le tableau des périodes se simplifiera et deviendra:

pour x:	1	0	0	aG	0	0,
pour y:	0	1	0	0	aG'	aH,
pour z:	0	0	1	0	aH	aG''.

Les périodes de x sont ainsi rendues indépendantes de celles d'y et de z. La 1ère et la 4ème périodes appartiennent à x seulement, les quatre autres périodes à y et à z seulement.

Les deux variables: $\beta y + \gamma z, \quad \beta' y + \gamma' z,$
resteront d'ailleurs réductibles. Il en résulte que si nous envisageons le système de périodes suivant:

pour y:	1	0	aG'	aH,
pour z:	0	1	aH	aG'',

qui n'est plus que de genre (2), le système d'intégrales abéliennes correspondant admettra encore deux intégrales réductibles aux intégrales elliptiques.

On pourra donc d'après les principes du §1 réduire le tableau des périodes, en prenant pour nouvelles variables:

$$y_1 = \beta y + \gamma z, \quad z_1 = \beta' y + \gamma' z.$$

On trouvera ainsi:

pour y_1:	1	0	G_1'	$\frac{1}{b}$,
pour z_1:	0	1	$\frac{1}{b}$	G_1'' (b étant un entier).

Le tableau complet des périodes sera alors:

pour x:	1	0	0	aG	0	0,
pour y_1:	0	1	0	0	G_1'	$\frac{1}{b}$,
pour z_1:	0	0	1	0	$\frac{1}{b}$	G_1''.

Faisons ensuite l'opération suivante qui est une transformation d'ordre b:

Multiplier les trois dernières périodes par b; puis retrancher la 2ème de la 6ème et la 3ème de la 5ème.

Le tableau des périodes devient ainsi:

pour x:	1	0	0	abG	0	0,
pour y_1:	0	1	0	0	bG_1'	0,
pour z_1:	0	0	1	0	0	bG_1''.

On voit que les périodes des trois variables x, y_1 et z_1 sont ainsi rendues indépendantes.

On reconnaît sans peine comment le raisonnement précédent s'étend au cas général et on peut énoncer le résultat suivant:

Si dans un système de variables abéliennes de genre ρ, il y a ρ variables réductibles, on peut par une transformation d'ordre convenable, simplifier le tableau des périodes de manière à rendre indépendantes les périodes des variables définitives.

Nous avons été obligés de faire dans le raisonnement précédent $\rho = 3$ et non $\rho = 2$, parce que la généralisation en partant du cas de $\rho = 2$ aurait pu présenter des difficultés.

Nous allons revenir à l'hypothèse $\rho = 2$ et supposer de nouveau qu'il n'y a que deux variables x et y.

Considérons en particulier le cas où les périodes de x et de y sont indépendantes et où le tableau des périodes s'écrit:

$$\begin{matrix} 2i\pi & 0 & G & 0, \\ 0 & 2i\pi & 0 & G'. \end{matrix}$$

Cherchons à former toutes les fonctions Θ d'ordre m qui admettent pour multiplicateurs relativement à nos quatre périodes:

$$1, \ 1, \ e^{mx+\gamma_3}, \ e^{my+\gamma_4}.$$

On pourra former m fonctions Θ elliptiques, dépendant de la variable x, admettant les deux périodes: $2i\pi$ et G

avec les multiplicateurs 1 et $e^{mx+\gamma_3}$.

Soient: $\Theta_1(x),\ \Theta_2(x), \ldots\ldots, \Theta_m(x),$

ces m fonctions.

On pourra de même former m fonctions Θ elliptiques, dépendant de la variable y, admettant les deux périodes:

$$2i\pi \text{ et } G'$$

avec les multiplicateurs 1 et $e^{my+\gamma_1}$.

Soient: $\Theta'_1(y),\ \Theta'_2(y), \ldots\ldots, \Theta'_m(y):$

ces m fonctions.

Les m^2 produits: $\Theta_i(x)\,\Theta'_k(y)\quad (i,\, k = 1,\, 2, \ldots\ldots, m)$

seront alors des fonctions Θ d'x et d'y, à l'aide desquelles toutes les autres s'exprimeront linéairement.

Une fonction Θ quelconque d'ordre m est alors égale à:

$$\Sigma A_{ik}\Theta_i(x)\,\Theta'_k(y),$$

où les A_{ik} sont des constantes numériques.

Une fonction intermédiaire quelconque sera de la forme:

$$\Sigma A_{ik}e^P\Theta_i(x)\,\Theta'_k(y),$$

où P est un polynôme entier du 2de degré en x et en y.

Supposons maintenant que les périodes de x et de y ne soient plus indépendantes, mais que les deux variables:

$$\alpha x + \beta y,\quad \alpha' x + \beta' y,$$

soient réductibles.

On pourra alors en prenant pour variables nouvelles,

$$x' = \alpha x + \beta y,\quad y' = \alpha' x + \beta' y,$$

et en opérant sur les périodes une transformation d'ordre μ, rendre indépendantes les périodes de x' et de y'.

Les fonctions intermédiaires d'ordre m relatives au premier système de périodes seront des fonctions intermédiaires d'ordre $m\mu$ par rapport au système transformé.

Il en résulte que toute fonction intermédiaire d'ordre m par rapport au premier système pourra se mettre sous la forme:

(6) $$\Sigma A_{ik}e^P\Theta_i(\alpha x + \beta y)\,\Theta'_k(\alpha' x + \beta' y).$$

Les A_{ik} sont des constantes; les indices i et k varient de 1 à $m\mu$; les Θ_i et les Θ'_k sont des fonctions Θ elliptiques d'ordre $m\mu$; P est un polynôme du second degré en x et y.

Une expression de la forme (6) n'est d'ailleurs pas toujours une fonction intermédiaire relative au premier système de périodes.

Il faut pour cela qu'il y ait entre les A_{ik}

$$m^2(\mu^2-1),$$

relations linéaires qu'il est aisé de former.

Ces résultats s'étendent immédiatement au cas de ρ variables.

§6. *Somme des zéros.*

Soient $\Phi_1, \Phi_2, \ldots, \Phi_\rho$, ρ fonctions intermédiaires de ρ variables $x_1, x_2, \ldots, x_\rho$ admettant les mêmes périodes et étant respectivement d'ordre $m_1, m_2, \ldots, m_\rho$. Considérons les solutions communes aux ρ équations simultanées

$$\Phi_1=\Phi_2=\Phi_3, \ldots, \Phi_\rho=0. \tag{1}$$

Soit:

$$x_1=y_1, \quad x_2=y_2, \ldots, x_\rho=y_\rho,$$

un système de solutions des équations (1). Soit:

$$a_1, a_2, \ldots, a_\rho,$$

une des périodes.

Il est clair que

$$x_1=y_1+a_1, \quad x_2=y_2+a_2, \ldots, x_\rho=y_\rho+a_\rho,$$

sera un autre système de solutions. Mais nous ne regarderons pas ces deux solutions comme distinctes.

Dans un travail inséré au tome X du Bulletin de la Société Mathématique de France, j'ai déterminé le nombre des solutions distinctes et démontré qu'il est égal à:

$$N=\rho!\ m_1.m_2. \ldots .m_\rho.$$

Soit maintenant:

$$x_i=y_{ik}, \quad (i=1, 2, \ldots, \rho;\ k=1, 2, \ldots, N),$$

un quelconque de ces N systèmes de solutions.

Je me propose de déterminer la somme de ces solutions, c'est à dire de calculer les ρ quantités:

$$H_1=\sum_{k=1}^{N} y_{1k}, \quad H_2=\sum_{k=1}^{N} y_{2k}, \ldots, H_\rho=\sum_{k=1}^{N} y_{\rho k}.$$

Il est clair que ces quantités H ne pourront être déterminées qu'à un multiple près des périodes. En effet nous ne regardons pas comme distinctes les deux solutions:

$$y_{1k}, y_{2k}, \ldots, y_{\rho k}$$

et

$$y_{1k}+a_1, y_{2k}+a_2, \ldots, y_{\rho k}+a_\rho.$$

Mais si on les remplace l'une par l'autre:

$$H_1, H_2, \ldots, H_\rho$$

se changeront en:

$$H_1 + a_1,\ H_2 + a_2,\ \ldots,\ H_p + a_p.$$

Les H nous seront donc donnés non par une égalité mais par une congruence.

Je vais maintenant montrer que les H ne dépendent que des périodes et des multiplicateurs.

Pour cela nous allons, pour fixer les idées, supposer qu'il n'y a que deux variables x et y.

Nous aurons alors deux fonctions intermédiaires Φ_1 et Φ_2 d'ordre m_1 et m_2 et nous envisagerons les deux equations:

$$\Phi_1 = \Phi_2 = 0,$$

qui ont $2m_1m_2$ solutions distinctes.

Soient alors:

$$\begin{matrix} a_1, & a_2, & a_3, & a_4, \\ b_1, & b_2, & b_3, & b_4, \end{matrix} \tag{2}$$

les périodes et

$$\begin{matrix} (\alpha_1, \beta_1, \delta_1), & (\alpha_2, \beta_2, \delta_2), & (\alpha_3, \beta_3, \delta_3), & (\alpha_4, \beta_4, \delta_4), \\ (\alpha_1', \beta_1', \delta_1'), & (\alpha_2', \beta_2', \delta_2'), & (\alpha_3', \beta_3', \delta_3'), & (\alpha_4', \beta_4', \delta_4'), \end{matrix}$$

les multiplicateurs de Φ_1 et de Φ_2, en conservant aux lettres α, β et δ la même signification que dans les deux paragraphes précédents.

Je dis que H_1 et H_2 ne dépendront que des a, des b, des α, des β et des δ.

Soient en effet Φ_1' et Φ_2' deux fonctions intermédiaires admettant respectivement les mêmes multiplicateurs que Φ_1 et Φ_2. Formons les équations

$$\Phi_1 + \lambda_1\Phi_1' = \Phi_2 + \lambda_2\Phi_2' = 0, \tag{3}$$

qui auront $2m_1m_2$ solutions distinctes; faisons la somme (H_1, H_2) de ces $2m_1m_2$ solutions. Je dis que H_1 et H_2 ne dépendent pas des λ.

Soient Φ_1'' une fonction intermédiaire ayant mêmes multiplicateurs que Φ_1 et Φ_1', et que Φ_1'' une fonction intermédiaire ayant mêmes multiplicateurs que Φ_2 et Φ_2'. Alors les quotients:

$$\frac{\Phi_1 + \lambda_1\Phi_1'}{\Phi_1''} \quad \text{et} \quad \frac{\Phi_2 + \lambda_2\Phi_2'}{\Phi_2''},$$

seront des fonctions abéliennes.

Soient maintenant X, Y et Z trois fonctions abéliennes quelconques admettant les périodes (2); il y aura entre ces trois fonctions une relation algébrique que j'écris:

$$F(X, Y, Z) = 0, \tag{4}$$

et toute fonction abélienne sera une fonction rationnelle de X, Y et Z.

Il en sera ainsi en particulier des six dérivées partielles:

$$\frac{dX}{dx}, \frac{dY}{dx}, \frac{dZ}{dx}, \frac{dX}{dy}, \frac{dY}{dy}, \frac{dZ}{dy},$$

ce qui permet de poser:

$$x = \int (AdX + BdY),$$
$$y = \int (A_1 dX + B_1 dY).$$

A, B, A_1, B_1 étant des fonctions rationnelles de X, Y et Z.

En d'autres termes, si l'on regarde X, Y et Z comme les coordonnées d'un point dans un plan, l'équation (4) représente une surface algébrique et x et y sont des intégrales de différentielles totales de 1ère espèce attachées à cette surface.

De plus je pourrai écrire:

$$\frac{\Phi_1 + \lambda_1 \Phi_1'}{\Phi_1''} = \frac{P_1 + \lambda_1 P_1'}{Q_1}, \quad \frac{\Phi_2 + \lambda_2 \Phi_2'}{\Phi_2''} = \frac{P_2 + \lambda_2 P_2'}{Q_2},$$

P_1, P_1', Q_1, P_2, P_2' et Q_2 étant des polynômes entiers en X, Y et Z.

Il en résulte que les équations (3) équivalent aux suivantes:

(3 bis) $$P_1 + \lambda P_1' = P_2 + \lambda_2 P_2' = 0.$$

Ces dernières représentent une courbe gauche algébrique, variable avec les paramètres λ_1 et λ_2.

Pour avoir H_1, il faut envisager les différents points d'intersection de cette courbe et de la surface (4), et faire la somme des différentes valeurs que prend l'intégrale de 1ère espèce x en ces différents points:

Le théorème d'Abel généralisé, nous apprend que cette somme est une constante. Donc H_1 ne dépend pas de λ_1 et de λ_2.

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Cela posé, proposons nous d'évaluer cette somme en fonction des a, des b, des α, des β et des δ.

Commençons par le cas des fonctions elliptiques et imaginons que l'on ait deux périodes:

$$a_1 \text{ et } a_2$$

avec les multiplicateurs (α_1, γ_1), (α_2, γ_2), ou encore

$$(\alpha_1, \delta_1), (\alpha_2, \delta_2),$$

les nombres α, γ et δ ayant même signification que plus haut.

On trouve aisément que si m est le nombre des zéros de la fonction intermédiaire envisagée et H leur somme, on aura:

$$2mi\pi = \alpha_2 a_1 - \alpha_1 a_2$$

et

$$2Hi\pi = \frac{\alpha_2 a_1^2}{2} + \gamma_2 a_1 - \frac{\alpha_1 a_2^2}{2} - \gamma_1 a_2,$$

ou en tenant compte des relations

$$\gamma = \delta + \frac{1}{2} a\alpha,$$

$$2Hi\pi = \left(\frac{\alpha_2 a_1}{2} - \frac{\alpha_1 a_2}{2}\right)(a_1 + a_2) + \delta_2 a_1 - \delta_1 a_2,$$

ou

$$H = \frac{m}{2}(a_1 + a_2) + \frac{\delta_2 a_1 - \delta_1 a_2}{2i\pi}.$$

Si m est pair, on obtient ainsi:

$$H \equiv \frac{\delta_2 a_1 - \delta_1 a_2}{2i\pi} \quad (\text{mod. } a_1, a_2),$$

et si m est impair:

$$H \equiv \frac{\delta_2 a_1 - \delta_1 a_2}{2i\pi} + \frac{a_1 + a_2}{2} \quad (\text{mod. } a_1, a_2).$$

Revenons maintenant au cas de deux variables, mais en supposant que les périodes de x et de y soient indépendantes. Soient alors:

pour x: $\quad a_1, \; 0, \; a_3, \; 0,$
pour y: $\quad 0, \; b_2, \; 0, \; b_4,$
les 4 périodes et

$$(\alpha_1, \beta_1, \delta_1), (\alpha_2, \beta_2, \delta_2), (\alpha_3, \beta_3, \delta_3), (\alpha_4, \beta_4, \delta_4),$$

les multiplicateurs correspondants. On aura:

$$\alpha_2 a_1 - \beta_1 b_2 = M_{1.2} = 0, \quad \alpha_3 a_1 - \alpha_1 a_3 = M_{1.3} = 2mi\pi,$$
$$\alpha_4 a_1 - \beta_1 b_4 = 0, \qquad \alpha_2 a_3 - \beta_3 b_2 = 0,$$
$$\beta_4 b_2 - \beta_2 b_4 = 2mi\pi, \qquad \beta_3 b_4 - \alpha_4 a_3 = 0,$$

ce qui permet de poser, μ étant une constante convenablement choisie:

$$\alpha_2 = \mu b_2, \; \alpha_4 = \mu b_4, \; \beta_1 = \mu a_1, \; \beta_3 = \mu a_3.$$

Multiplions maintenant notre fonction intermédiaire par:

$$e^{-\mu xy},$$

α_2, α_4, β_1, β_3 s'annuleront et les autres nombres α, β, δ ne changeront pas. Nous ne changerons pas d'ailleurs les zéros de notre fonction intermédiaire.

Nous pouvons donc toujours supposer que l'on a:

$$\alpha_2 = \alpha_4 = \beta_1 = \beta_3 = 0.$$

Soient maintenant Φ et Φ' deux fonctions intermédiaires d'ordres m et m' ayant respectivement pour multiplicateurs:

$$(\alpha_1, 0, \delta_1), \; (0, \beta_2, \delta_2), \; (\alpha_3, 0, \delta_3), \; (0, \beta_4, \delta_4),$$
$$(\alpha'_1, 0, \delta'_1), \; (0, \beta'_2, \delta'_2), \; (\alpha'_3, 0, \delta'_3), \; (0, \beta'_4, \delta'_4),$$

et envisageons les deux équations:

(5) $$\Phi = \Phi' = 0,$$

qui ont $2mm'$ solutions distinctes.

Il existe m^2 fonctions intermédiaires qui ont mêmes multiplicateurs que Φ, et m'^2 fonctions qui ont mêmes multiplicateurs que Φ'. Mais d'après ce que nous venons de voir, H_1 et H_2 ne dépendent que des multiplicateurs et ne changeront pas si on remplace Φ et Φ' par d'autres fonctions intermédiaires ayant mêmes multiplicateurs.

Parmi les fonctions qui ont mêmes multiplicateurs que Φ, il y en a qui sont de la forme: $$\Phi_1(x)\,\Phi_2(y);$$

$\Phi_1(x)$ est une fonction intermédiaire elliptique, admettant les périodes a_1, a_3 et les multiplicateurs (α_1, δ_1), (α_3, δ_3).

De même $\Phi_2(y)$ a pour périodes b_2, b_4 et pour multiplicateurs (β_2, δ_2), (β_4, δ_4).

Je pourrai de même remplacer Φ' par le produit:

$$\Phi_1'(x)\,\Phi_2'(y);$$

$\Phi_1'(x)$ aura pour périodes a_1, a_3 et pour multiplicateurs (α_1', δ_1'), (α_3', δ_3') et $\Phi_2'(y)$ aura pour périodes b_2, b_4 et pour multiplicateurs (β_2', δ_2'), (β_4', δ_4').

Nous remplacerons donc les équations (5) par les suivantes:

$$\Phi_1(x)\,\Phi_2(y) = \Phi_1'(x)\,\Phi_2'(y) = 0,$$

ou ce qui revient au même par les suivantes:

(6) $$\Phi_1(x) = \Phi_2'(y) = 0 \text{ et } \Phi_1'(x) = \Phi_2(y) = 0.$$

Soit h_1, h_2, h_1', h_2' la somme des zéros de Φ_1, Φ_2, Φ_1', Φ_2'.

Je dis qu'on aura $$H_1 = m'h_1 + mh_1',$$ $$H_2 = m'h_2 + mh_2'.$$

En effet énumérons les $2mm'$ solutions des équations (6).

On les obtiendra:

1°. En combinant les m zéros de Φ_1 avec les m' zéros de Φ_2'.

2°. En combinant les m' zéros de Φ_1' avec les m zéros de Φ_2.

Donc chacun des zéros de Φ_1 paraîtra m' fois et chacun des zéros de Φ_1' paraîtra m fois. C'est pourquoi l'on a:

$$H_1 = m'h_1 + mh_1'.$$

Nous aurons d'autre part :

$$h_1 \equiv \frac{\delta_3 a_1 - \delta_1 a_3}{2i\pi} + m\,\frac{a_1 + a_3}{2},$$

$$h'_1 \equiv \frac{\delta'_3 a_1 - \delta'_1 a_3}{2i\pi} + m'\,\frac{a_1 + a_3}{2},$$

d'où
$$H_1 \equiv \frac{m'}{2i\pi}\,(\delta_3 a_1 - \delta_1 a_3) + \frac{m}{2i\pi}\,(\delta'_3 a_1 - \delta'_1 a_3),$$

le terme $2mm'\left(\frac{a_1 + a_3}{2}\right)$ disparaît comme étant un multiple des périodes.

Posons maintenant, en reprenant les notations des paragraphes précédents :

$(a, \delta) = a_1\delta_3 - a_3\delta_1 + a_2\delta_4 - a_4\delta_2 = a_1\delta_3 - a_3\delta_1$ (puisque $a_2 = a_4 = 0$),

$(a, \delta) = 2Ri\pi,$ $\qquad (b, \delta) = 2Si\pi,$

$(a, \delta') = 2R'i\pi,$ $\qquad (b, \delta') = 2S'i\pi.$

Il viendra
$$H_1 \equiv m'R + mR',$$
et de même :
$$H_2 \equiv m'S + mS'.$$

Nous pouvons donc écrire plus succinctement et conformément aux notations adoptées plus haut :

(7)
$$(H_1,\ H_2) \equiv (m'R + mR',\ m'S + mS').$$

Cette formule est démontrée pour deux fonctions intermédiaires quelconques, toutes les fois que les périodes de x et de y sont indépendantes.

Passons maintenant à un cas plus général, et supposons que les périodes de x et de y ne soient plus indépendantes, mais que parmi les variables de la forme $cx + dy$, il y en ait deux qui soient réductibles.

Nous supposerons d'abord que ce soient x et y qui soient réductibles et que les périodes s'écrivent :

(8)
$$\begin{aligned} &\text{pour } x: && a_1, \ a_2, \ a_3, \ a_4, \\ &\text{pour } y: && b_1, \ b_2, \ b_3, \ b_4, \end{aligned}$$

de telle façon qu'il y ait deux relations linéaires à coëfficients entiers entre a_1, a_2, a_3 et a_4 et de même deux autres relations semblables entre b_1, b_2, b_3 et b_4.

On pourra alors, par une transformation d'ordre μ amener les périodes de x et de y à être indépendantes et remplacer le système des périodes (8) par un nouveau système (9) de périodes indépendantes.

Soient alors Φ et Φ' deux fonctions intermédiaires d'ordres m et m'.

Envisageons encore les équations :

$$\Phi = \Phi' = 0.$$

Soient R, S et R', S' les nombres R et S relatifs aux deux fonctions Φ et Φ'.

Par rapport au nouveau système de périodes (9), Φ et Φ' seront des fonctions intermédiaires d'ordres $m\mu$ et $m'\mu$ et par rapport à ce nouveau système, leurs nombres caractéristiques seront devenus respectivement R_1, S_1 et R'_1, S'_1.

On aura d'ailleurs (cf. § précédent):

$$(2R_1,\ 2S_1) \equiv (2\mu R,\ 2\mu S),$$
$$(2R'_1,\ 2S'_1) \equiv (2\mu R',\ 2\mu S'),$$

par rapport aux périodes (9) et *a fortiori* par rapport aux périodes (8).

Les équations $$\Phi = \Phi' = 0$$

auront $2mm'$ solutions distinctes par rapport aux périodes (8); la somme des $2mm'$ valeurs de x sera H_1, celle des $2mm'$ valeurs de y sera H_2.

Ces mêmes équations auront $2\mu^2 mm'$ solutions distinctes par rapport aux périodes (9); la somme des $2\mu^2 mm'$ valeurs de x sera k_1, celle des $2\mu^2 mm'$ valeurs de y sera k_2.

On aura d'ailleurs évidemment:

$$(k_1,\ k_2) \equiv (\mu^2 H_1,\ \mu^2 H_2),$$

la congruence étant prise par rapport aux périodes (8).

Nous pourrons appliquer au calcul de k_1 et de k_2 la formule démontrée précédemment pour le cas où les périodes x et y sont indépendantes, ce qui donne:

$$(k_1,\ k_2) \equiv (\mu m' R_1 + \mu m R'_1,\ \mu m' S_1 + \mu m S'_1),$$

ou

$$(2k_1,\ 2k_2) \equiv (2\mu^2 [m'R + mR'],\ 2\mu^2 [m'S + mS']),$$

ou enfin

$$(2\mu^2 H_1,\ 2\mu^2 H_2) \equiv (2\mu^2 [m'R + mR'],\ 2\mu^2 [m'S + mS']). \tag{10}$$

Envisageons donc le système des deux quantités:

$$H_1 - m'R - mR',\quad H_2 - m'S - mS'.$$

D'après la congruence (10), ces deux quantités devront toujours être égales à une période divisée par $2\mu^2$. Or ces deux quantités sont évidemment des fonctions continues des δ et des δ'. Ce ne peuvent donc être que des constantes, ce qui nous permet d'écrire:

$$(H_1,\ H_2) \equiv (m'R + mR' + \Delta_1,\ m'S + mS' + \Delta_2),$$

Δ_1 et Δ_2 sont des constantes ne dépendant que des périodes a et b, des α et des β, mais indépendantes des δ et des δ'.

Il reste à déterminer ces constantes. Pour cela, j'envisagerai un cas particulier, celui où les δ et les δ' sont tous nuls. Dans ce cas on a:

$$R = R' = S = S' = 0.$$

De plus la fonction $\Phi(x, y)$ aura mêmes multiplicateurs que $\Phi(-x, -y)$ et par conséquent que:

$$\Phi(x, y) + \Phi(-x, -y),$$

qui est une fonction paire.

Cela nous permet de supposer que dans les équations

$$\Phi = \Phi' = 0,$$

Φ et Φ' sont des fonctions paires.

Si alors: $x = x_0, \ y = y_0,$

est une solution de ces équations, il en sera de même de

$$x = -x_0, \ y = -y_0.$$

Si ces deux solutions sont distinctes, leur somme est nulle; elles entrent toutes deux dans l'expression de H_1 et de H_2 et elles s'y détruisent.

Si elles ne sont pas distinctes, une seule d'entre elles devra entrer dans l'expression de H_1 et de H_2 et ce sera évidemment une demi-période.

Ainsi dans le cas qui nous occupe (H_1, H_2) est toujours une demi-période et comme on a dans ce cas:

$$(H_1, H_2) \equiv (\Delta_1, \Delta_2),$$

on voit que (Δ_1, Δ_2) est aussi une demi-période; on a donc dans le cas général

$$(2H_1, 2H_2) \equiv (2m'R + 2mR', \ 2m'S + 2mS').$$

Cette formule subsiste encore quand on fait un changement linéaire de variables, ou un changement de périodes (cf. § 4).

Cette formule est ainsi établie pour tous les cas où il y a deux variables réductibles; mais il est aisé de l'étendre au cas le plus général.

En effet, d'après un théorème démontré au § 2, tout système de périodes diffère infiniment peu d'un système correspondant à un cas de réductibilité. De plus il est clair que H_1 et H_2 sont des fonctions continues des périodes.

La formule précédente doit donc s'appliquer quelles que soient les périodes.

C'est ainsi que si $f(x)$ et $\Phi(x)$ sont deux fonctions continues de x qui sont égales pour toutes les valeurs commensurables de x, elles seront encore égales pour toutes les valeurs incommensurables. La façon de raisonner est toute pareille dans les deux cas.

Ainsi dans tous les cas la différence

$$(H_1 - m'R - mR', \quad H_2 - m'S - mS'),$$

est toujours une demi-période.

Ceci permet d'écrire:

$$(H_1, H_2) \equiv \Big(m'R + mR' + \frac{1}{2}(\nu_1 a_1 + \nu_2 a_2 + \nu_3 a_3 + \nu_4 a_4),$$
$$m'S + mS' + \frac{1}{2}(\nu_1 b_1 + \nu_2 b_2 + \nu_3 b_3 + \nu_4 b_4)\Big),$$

les ν étant des entiers.

Par raison de continuité, les entiers ν devront avoir toujours la même valeur, ou plutôt puisqu'il s'agit, non d'une égalité, mais d'une congruence, ν_1 devra toujours être de même parité; de même pour ν_2, ν_3 et ν_4.

Pour déterminer la parité des nombres ν, il suffit donc d'envisager un cas particulier. Or quand les périodes de x et de y sont indépendantes, nous avons trouvé:

$$(H_1, H_2) \equiv (m'R + mR', \quad m'S + mS').$$

Cette formule subsiste donc dans le cas général.

Ainsi, si l'on envisage les deux équations:

$$\Phi(x, y) = \Phi'(x, y) = 0,$$

où Φ et Φ' sont deux fonctions intermédiaires d'ordres m et m', elles auront $2mm'$ solutions distinctes. La somme des $2mm'$ valeurs de x sera égale à $m'R + mR'$, et celle des $2mm'$ valeurs de y sera égale à $m'S + mS'$ à un multiple près des périodes.

Envisageons en particulier la fonction Θ de Riemann, c'est à dire la fonction Θ du 1[er] ordre.

Soit donc:

$$\Theta(x, y) = \sum e^{mx + ny - \frac{1}{2}(\lambda m^2 + 2\mu mn + \nu n^2)},$$

ce qui nous donne pour les quantités a, b, α, β, γ, δ les valeurs suivantes:

Valeurs de	a	b	α	β	γ	δ,
	$2i\pi$	0	0	0	0	0,
	0	$2i\pi$	0	0	0	0,
	λ	μ	1	0	$\frac{1}{2}\lambda$	0,
	μ	ν	0	1	$\frac{1}{2}\nu$	0.

On a donc aussi:

$$R = S = 0.$$

Considérons maintenant la fonction :

$$\Theta(x-h,\ y-k).$$

Nous avons vu au §4 que si l'on faisait subir aux variables x et y un certain changement linéaire, les nombres $-\frac{R}{m}$, $-\frac{S}{m}$ subiraient précisément le même changement, d'où il suit que R et S qui avaient pour valeurs 0 pour la fonction :

$$\Theta(x,\ y),$$

auront pour valeurs h et k pour la fonction :

$$\Theta(x-h,\ y-k).$$

Considérons alors deux équations simultanées :

$$\Theta(x-h,\ y-k)=\Theta(x-h',\ y-k')=0.$$

Ces équations auront deux solutions distinctes. La somme des deux valeurs de x sera $h+h'$, celle des deux valeurs de y sera $k+k'$ à un multiple près des périodes.

Il serait aisé de voir comment ces résultats peuvent s'étendre au cas où il y a plus de deux variables.

Supposons d'abord qu'il y ait trois variables x, y et z et trois fonctions intermédiaires Φ, Φ' et Φ'' d'ordres m, m' et m''.

On formera pour chacune d'elles trois nombres analogues à R et S que j'appellerai R, S, T pour la première, R', S', T' pour la seconde et R'', S'', T'' pour la troisième.

Si l'on appelle H_1 la somme des $6mm'm''$ valeurs de x, H_2 celles des $6mm'm''$ valeurs de y et H_3 celle des $6mm'm''$ valeurs de z qui satisfont aux trois équations :

$$\Phi=\Phi'=\Phi''=0,$$

on aura :

$$(H_1,\ H_2,\ H_3)\equiv(2m'm''R+2mm''R'+2mm'R'',\ 2m'm''S+2mm''S'+2mm'S'',\ 2m'm''T+2mm''T'+2mm'T'').$$

Soient en particulier trois fonctions Θ du 1[er] ordre. Supposons que l'on ait les trois équations :

$$\Theta(x-h,\ y-k,\ z-l)=\Theta(x-h',\ y-k',\ z-l')=\Theta(x-h'',\ y-k'',\ z-l'')=0.$$

Elles auront 6 solutions distinctes. La somme des 6 valeurs de x sera $2(h+h'+h'')$, celle des 6 valeurs de y sera $2(k+k'+k'')$ et celle des 6 valeurs de z sera $2(l+l'+l'')$ à un multiple près des périodes.

Plus généralement. Soit:

$$\Theta(x_1, x_2, \ldots, x_\rho),$$

une fonction Θ à ρ variables. Formons les ρ équations:

$$\Theta(x_1 - h_{1.k}, x_2 - h_{2.k}, \ldots, x_\rho - h_{\rho.k}) = 0, \tag{11}$$

$$(k = 1, 2, \ldots, \rho).$$

Elles auront $\rho!$ solutions distinctes.

Si on néglige les multiples des périodes, la somme des $\rho!$ valeurs de x_i sera:

$$(\rho - 1)! \; (h_{i.1} + h_{i.2} + \ldots + h_{i.\rho}).$$

Si les ρ fonctions Θ qui entrent dans les équations (11) sont toutes paires ou impaires, tous les h sont des demi-périodes. La somme des valeurs de x_i sera donc une demi-période si $\rho = 2$ et si par conséquent $(\rho - 1)!$ est impair. Si au contraire $\rho > 2$ et si par conséquent $(\rho - 1)!$ est pair, la somme des valeurs de x_i sera une période entière. Or nous n'avons déterminé cette somme qu'à un multiple près des périodes; on peut donc la regarder comme nulle.

Si donc $\rho > 2$ et si l'on forme ρ équations en annulant ρ des 4^ρ fonctions Θ paires ou impaires, la somme des $\rho!$ solutions distinctes de ces ρ équations sera nulle.

PARIS, le 18 Juin, 1886.

A Generalized Theory of the Combination of Observations so as to Obtain the Best Result.

By Simon Newcomb.

§ 1. *Introductory Remarks.*

The accepted practice of combining observations rests upon the hypothesis that the frequency of errors follows a certain well-known law which may be expressed as follows: Let Δ be the amount by which the result of an observation may differ from the value obtained by taking the mean result of an infinity of similar observations. Δ will then be the *error* of the observation. The infinitesimal probability that an error will be contained between the limits Δ and $\Delta + d\Delta$ is supposed to be given by the equation

$$dp = \frac{h}{\sqrt{\pi}} e^{-h^2\Delta^2} d\Delta,$$

in which h is the "modulus of precision" depending upon the accuracy of the observations, and e is the Naperian base.

The modulus h is commonly replaced by a probable error r, which term signifies such a magnitude that the number of errors less than r in absolute value is equal to the number which exceed r. The value of r is given in terms of the modulus h by the equation

$$r = \pm \frac{0.4769}{h}.$$

When the errors really follow the law in question, they diminish with extreme rapidity as Δ increases. For example, only one per cent. of the errors should fall without the limits $\pm 4r$.

As a matter of fact, however, the cases are quite exceptional in which the errors are found to really follow the law. The general rule is that much more than one per cent. of the errors exceed four times the probable error. In other words, it is nearly always found that some of the outstanding errors seem abnormally large. The method of dealing with these abnormal errors has always been one of the most difficult questions in the treatment of observations. The common practice has been to consider the observations affected by them as abnormal, and to reject them in obtaining the final result. But we here meet with the difficulty that no positive criterion for determining whether an observation should or should not be treated as abnormal is possible. Several attempts have indeed been made to formulate such a criterion, the best known of which is that of Peirce.*

Peirce's criterion has always seemed to me subject to two serious objections. One is that it takes no account of any probable error of the observations under consideration which may be known beforehand, but proceeds as if the value of the probable error could be deduced from the comparison of the observations *inter se*. An immediate general consequence of this is that if all the errors of a system are multiplied by the same factor, the same observations are rejected as before, how small or great soever the factor may be.†

The second objection is that it takes no account of the fact that the *a priori* probability that an observer should make an abnormal observation varies with the observer, and places all observers on a level by regarding that probability as determined by a general mathematical principle applicable to all cases.

It is, however, well known that some observers make very few abnormal observations, while others are extremely liable to them. It is evident that if we are dealing with an observation whose error is so large that we doubt whether it should or should not be considered abnormal, our judgment must depend very largely upon any knowledge we may have of the carefulness of the observer.

The fact is, however, that any system of rejecting supposed abnormal observations is subject to the objection of leading to a result which is a discontinuous function of the separate errors of observation, and hence to results

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* Gould's Astronomical Journal, Vol. II, p. 161.

† Certain results of Peirce's criterion in special cases, when applied to sets of three or four observations, do not seem to have been hitherto noticed. The following are cases in point:

Of a set of three observations none are ever rejected by it, no matter how much one may deviate from the mean of the other two.

In a set of four observations, if three agree exactly, the fourth will always be rejected if it differs ever so little from the others. More generally, if no one of the three results which agree best among themselves differs from the mean of the three by more than ε, then a fourth, which differs from that mean by more than 4ε, will be rejected. For example, if the results of four observations with a meridian circle were 0″.3; 0″.4; 0″.5; 0″.8, the last would be rejected.

which are sometimes indeterminate. Suppose, for example, that we are dealing with the mean of three observations, two of which are closely accordant, while the third differs from the mean of the other two by the quantity x. Let us represent the mean of the two accordant ones by the symbol m'; then, if we include the discordant observation, the general expression for the mean result in terms of x will be

$$m = m' + \tfrac{1}{3}x.$$

In ordinary astronomical practice we retain this value of m so long as x does not exceed the limit which we consider that of a normal error. But, as soon as this limit is reached, we drop x entirely and take m' for the value of m. In other words, if we consider x to increase from zero, the adopted value of m will increase one-third as fast until the assigned limit is reached, and will then suddenly spring back from $m' + \frac{1}{3}x$ to m'. If the critical point at which x is to be rejected could be satisfactorily defined, this course would be less objectionable. But, as a matter of fact, it is to be determined by the judgment of the investigator, with the result that between certain wide limits the investigator must himself be doubtful whether he should take m' or $m' + \frac{1}{3}x$ as his result. Of course different investigators would reach different conclusions in special cases, and thus the most probable result is frequently indeterminate.

There are classes of important observations in which the proportion of large errors is so great that no separation into normal and abnormal observations is possible. This is the case in observations of transits of Venus and Mercury over the sun. A noteworthy instance has been given by the writer in his discussion of transits of Mercury.* By a comparison of 684 observations it was found that the errors of one-half of them were contained between the limits $\pm 6''.8$. If the errors followed the commonly assumed law, then only 5 of them should have exceeded $\pm$ 27 seconds. As a matter of fact, however, it was found that 49 exceeded these limits. Yet these 49 observations cannot be considered as wholly worthless, because their results are not scattered entirely at random, and are mostly included between comparatively narrow limits. They differ from the other observations only in having a larger probable error.

The case may be made clearer by reflecting that the law in question presupposes that the observations under consideration are all of the same general quality as regards liability to error; in other words, that they are all liable to

* Astronomical Papers of the American Ephemeris, Vol. I, pp. 379–388.

the same errors, and differ only in the accidental circumstances which give rise to the errors. If, however, this is not true; if, for example, we are furnished with a system of observations of which one portion have a small probable error, another a larger probable error, a third a yet larger one, and so on, then the errors of the whole system will not follow the law in question, but we shall find that large errors are disproportionately frequent. Now, this must be the case in nearly all astronomical and physical work.

From this another conclusion follows. In such a mixed system of observations the most probable result will be, not the arithmetical mean, but a mean obtained by giving less weight to the more discordant observations. This will be evident on reflecting that in such a case the more discordant results will probably belong to the observations having a larger probable error and therefore the less weight.

§ 2. *Modified Curves of Probability.*

The preceding considerations lead us to the further conclusion that the commonly received theory which presupposes that there must always be some one "most probable value" of a quantity determined by observations, lacks generality. The fact is that, in special cases, owing to a possibility of abnormal observations, the curve of probability may have a great variety of forms. As one example, let it be supposed that two mean declinations of a star, determined with a good meridian circle the micrometer-head of which is numbered at intervals of 5″, differ from each other by a quantity approximating to 5″. We then may make three hypotheses: that the observations are both normal, or that one or the other of them is in error by 5″ through a mistake in recording.

According to the probability of the first hypothesis, and of either of the other two, we may have different curves. Assuming the instrument and observer to be so accurate that a difference of 5″ between two normal observations is nearly out of the question, we shall have a curve of the form A. As the probability of the first hypothesis increases, the curve may assume the form B. If the observer is one never known to make mistakes in reading, the curve will approximate to its usual form.

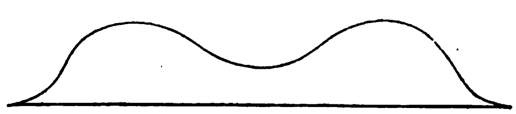

FIG. A.

FIG. B.

Now, it is evident that in such a case as that indicated by curve A, we can assign no one most probable value of the observed quantity. The only complete statement we could make would be embodied in a table showing for each separate possible value of the required quantity the probability that the quantity had a value not differing from it by more than a small arbitrary amount. Assuming the intervals of the table to be taken to equal double this amount, the sum of all these probabilities would be unity.

Looking further into the matter, we see that this is a general method of expressing the conclusion to be derived in all cases from an observation or a series of observations. No matter how definite the primary value given by an observation may be, the actual conclusion to be drawn is always a series of separate probabilities that the quantity observed has some one of an infinite series of values. If the law of probability is that commonly assumed, then the probability of each assignable value is completely determined when the most probable value and the probable error are given. But such is not the case when the law of probability deviates from this form.

§ 3. *Evil and Worth of Erroneous Results.*

The question now arises whether, when we consider the most general case, in which there may be several maxima of probability, and when, therefore, no one most probable value can be assigned, it is possible to formulate any general principle by which a single value shall be preferred for acceptance above all others. Taking as an example such a case as A just given, it is clear that no such principle is possible without some antecedent hypothesis determining a law of choice between errors of different magnitudes. If, to fix the ideas, we suppose that in case A the results of the two separate observations were 0″.0 and 5″.0, then the three hypotheses will give us 0″.0 ; 2″.5 ; 5″.0, as three values between which we are to choose. If we choose either the first or third, we shall have a probability of slightly less than one-half of being very near the truth, and an equal probability of being 5″ in error, together with a very small probability of being about 2″.5 in error. If, on the other hand, we take 2″.5 as our result, we shall be almost sure of being between 2″ and 3″ in error, and no more. Our choice, then, must depend on whether a certainty of being 2″.5 in error, or an even chance of each of the errors 0″ and 5″, is preferable. This again turns upon the question whether an error of 5″ involves more or less than twice the evil of an error of 2″.5.

The ordinary requirements of practical life are in favor of the view that the evil increases in a higher ratio than the simple magnitude of the error. As examples, if it is a case of an error in the position of a ship arising from an error of the Nautical Almanac, we readily see that the probability of the error leading to a shipwreck increases in a higher ratio than that of the simple error itself. Again, in the case in which, by the labor of continually increasing observations of a single quantity, we lessen the probable error, we know that it requires fourfold labor to reduce the probable error to one-half. It would seem, therefore, that the best hypothesis that we can adopt is that the evil of an error is proportional to the square of its magnitude.

A determination has more or less value according as it is less or more liable to errors. The simplest definition of the value of an observation that we can adopt is that it is inversely as the sum total of the evils to which it is subjected, each evil being multiplied by its probability. This also is in strict accord with the ordinary law of probability of a number of observations, since it involves the hypothesis that the value of a result is proportional to the number of observations on which it depends. As, however, the word "value," if used to express this conception, would be ambiguous in consequence of being applied to the simple amount of a quantity, we shall use the term *worth* to express the economic value of an assigned value as just described. We therefore have the definitions:

The *evil* of any value assigned to a quantity is equal to the sum of all the products obtained by multiplying the square of each possible error of that assigned value by the probability of its occurrence.*

The *worth* of any such value is inversely proportioned to its evil.

The value to which we are to give the preference is that whose worth is a maximum, or, which amounts to the same thing, whose evil is a minimum. This value, and the magnitude of the evil with which it is affected, will be two elements corresponding to "most probable value" and "probable error" in the usual theory.

§ 4. *Algebraic Expressions for the Evil.*

The general expression for these elements is readily obtained. Let us represent all possible values which the required quantity can have by the series

$$x_1, x_2, x_3, \ldots . x_n,$$

and let

$$p_1, p_2, p_3, \ldots . p_n$$

* This idea of an evil attached to each error, and proportional to the square of its magnitude, is due to Gauss (Theoria Combinationis Observationum, etc., Pars prior, §6), who applies the term *jactura* to the conception here called evil.

be the several probabilities of these values; we necessarily have in this case

$$\sum_{1}^{n} p_i = 1.$$

Putting x for any value of the quantity in question, the evil of this value will be, by definition,

$$\begin{aligned} e &= p_1(x - x_1)^2 + p_2(x - x_2)^2 + \dots + p_n(x - x_n)^2 \\ &= x^2 - 2Ax + B, \end{aligned}$$

where

$$A = \sum_{1}^{n} p_i x_i,$$

$$B = \sum_{1}^{n} p_i x_i^2.$$

This quantity is a minimum when

$$x = A,$$

which is, therefore, the value we are to prefer. The evil of this value is

$$e_0 = B - A^2.$$

The respective values of A^2 and B may be written, since $\Sigma p_i = 1$,

$$A^2 = \sum_{1}^{n}{}^{2}_{i,j} p_i p_j x_i x_j,$$

$$B = \sum_{1}^{n}{}^{2}_{i,j} p_i p_j x_i^2.$$

The value of the minimum evil then becomes

$$e_0 = \frac{1}{2} \sum_{1}^{n}{}^{2}_{i,j} p_i p_j (x_i - x_j)^2.$$

It therefore appears that the inverse of this expression is the worth of the best value of the required quantity, which best value is given by the equation

$$x_0 = \sum_{1}^{n}{}_{i} p_i x_i. \tag{1}$$

In what precedes, the form of our equations is based upon the supposition that $x_1, x_2 \dots x_n$ are a finite number of discrete values which x may have. In the usual case, however, the unknown quantity may have all values between certain wide limits, and the probability that it is contained between the limits x and $x + dx$ is given by an equation of the form

$$dp = \phi(x)\, dx,$$

dp being an infinitesimal probability. Since this is a pure number, it follows from this equation that, whatever physical quantity may be represented by x, $\phi(x)$ must be of the dimension -1 in this quantity.

Reducing the formula (1) to the present case, we find that the preferable value of x is given by the equation

$$x_0 = \int_{-\infty}^{+\infty} x\phi(x)\, dx. \tag{2}$$

The evil of this value is

$$e_0 = \frac{1}{2}\int_{-\infty}^{+\infty}\int_{-\infty}^{+\infty} \phi(y)\,.\,\phi(z)\,(y - z)^2\, dydz \tag{3}$$

$$= \int_{-\infty}^{+\infty} y^2\phi y dy - \int_{-\infty}^{+\infty}\int_{-\infty}^{+\infty} yz\phi(y)\phi(z)\, dydz.$$

From the definition of evil, it is a quantity of the dimensions $+2$ in those of the physical quantity in question. Its square root is therefore a definite quantity of the magnitude to be determined, which we may regard as an error.

An example of the results of the present theory will be found by applying it to the case of the usually assumed law of error, namely:

$$\phi(x) = \frac{h}{\sqrt{\pi}} e^{-h^2x^2}.$$

We then have $x_0 = 0$ as the preferable value of x, in accordance with the usual theory. In the expression for the evil of this value we have

$$A = x_0 = 0,$$

$$B = \frac{h}{\sqrt{\pi}}\int_{-\infty}^{+\infty} y^2 e^{-h^2y^2} dy = \frac{1}{2h^2}\,. \tag{4}$$

This value of B is identical with the square of what is commonly called the *mean error*, which again is equal to $1.5 \times$ probable error nearly. Hence, in this special case, the evil is identical with the square of the mean error.

If, instead of taking zero as the value of x, we wish to express the evil of any other assumed value, we have the expression

$$e = x^2 + B = \varepsilon^2 + x^2,$$

ε being the mean error. If, instead of ε, we use r, the probable error, the expression will be

$$e = x^2 + 2.198 r^2.$$

We readily find that if, instead of using the most probable value of a quantity, we adopt a value differing from it by its probable error, the evil will be increased by a little less than half its whole amount.

It appears, therefore, that, whatever the law of error, we may always find two quantities corresponding to "most probable value," and "probable error" of that value in the usual theory. One of these quantities will naturally be the best value of the required magnitude, or A itself. The other may be either the evil of the value A, or the change in the value of the magnitude required to increase its evil in a definite ratio, these last quantities being functions of the same quantity, and therefore of each other. If we present the result in the form

$$x = A \pm \sqrt{B - A^2},$$

the last term will be the "mean error" in the usual theory, and the change in x which would double its evil in the generalized theory. If we wish to express the quantity corresponding to the "probable error," we write

$$x = A \pm 0.6745\sqrt{B - A^2}.$$

§ 5. *Modified Law of Probability.*

The whole problem now before us is reduced to finding a curve of probability in the case of a number of observations of the same quantity. This problem naturally involves that of the law of error of the separate observations, and leads us to inquire what modification should be made in the usually assumed law in order that it may be applicable to all cases whatever.

The defect of the commonly assumed law, as represented by the equation

$$\phi(x) = \frac{h}{\sqrt{\pi}} e^{-h^2x^2},$$

is that, in practice, large errors are more frequent than this equation would indicate them to be. This defect might be remedied by substituting some other function of x than h^2x^2 as the exponent of e. The requirements of this function would be :

1. That it should be an even function of x, or of the form $f(x^2)$.
2. That it should become infinite when x did.
3. That it should increase less rapidly than h^2x^2, or, more exactly, that the second derivative $\frac{\partial^2 f}{\partial x^2}$, instead of being a constant, should diminish with an increase

of x. Such a function can be formed by writing, instead of h^2, an expression of the form

$$\frac{h^2(1+h^2x^2)}{1+h'^2x^2},$$

so that we should have, for the exponent $f(x^2)$, an expression of the form

$$-\frac{h^2(x^2+h^2x^4)}{1+h'^2x^2}.$$

The management of such an exponent might, however, prove inconvenient, and I shall adopt a law of error founded on the very probable hypothesis that we are dealing with a mixture of observations having various measures of precision. Let us put

$$h_1,\ h_2\ \ldots\ .\ h_n,$$

the possible separate values of the different measures of precision;

$$p_1,\ p_2\ \ldots\ .\ p_n,$$

the corresponding probabilities that an observation selected at random has any one of these several measures of precision.

Then, for an observation selected at random, the law of error will be

$$\phi(x) = \frac{1}{\sqrt{\pi}}\{p_1h_1e^{-h_1^2x^2} + p_2h_2e^{-h_2^2x^2} + \ldots . + p_nh_ne^{-h_n^2x^2}\}. \tag{5}$$

§6. *Deduction of Best Result.*

If we have m observations of the same quantity giving $x_1, x_2 \ldots . x_m$ as the observed values, then, assuming the law of error expressed in (5), the probability that the quantity is contained between the limits η and $\eta + d\eta$ is given by the equation $dp = a\psi(\eta)\,d\eta$,

in which $\psi(\eta) = \phi(x_1-\eta)\,\phi(x_2-\eta) \ldots . \phi(x_m-\eta)$, (6)

while a is a constant so chosen as to make

$$a\int_{-\infty}^{+\infty}\psi(\eta)\,d\eta = 1.$$

The formula (2) then gives for the best value of x the expression

$$x = \int \eta\,dp = a\int_{-\infty}^{+\infty}\eta\psi(\eta)\,d\eta = \frac{\int_{-\infty}^{+\infty}\eta\,\psi(\eta)\,d\eta}{\int_{-\infty}^{+\infty}\psi(\eta)\,d\eta}. \tag{7}$$

If $\psi(\eta)$ be multiplied by any constant factor, it will disappear from this expression; we may therefore disregard all such factors in forming $\psi(\eta)$. We may therefore take for $\psi(\eta)$ the product of the m quantities

$$\begin{aligned}
& h_1' e^{-h_1^2(x_1-\eta)^2} + h_2' e^{-h_2^2(x_1-\eta)^2} + \ldots\ldots + h_n' e^{-h_n^2(x_1-\eta)^2} \\
\times\ & h_1' e^{-h_1^2(x_2-\eta)^2} + h_2' e^{-h_2^2(x_2-\eta)^2} + \ldots\ldots + h_n' e^{-h_n^2(x_2-\eta)^2} \\
\times\ & \ldots\ldots\ldots\ldots\ldots\ldots\ldots\ldots\ldots\ldots \\
& \ldots\ldots\ldots\ldots\ldots\ldots\ldots\ldots\ldots\ldots \\
\times\ & h_1' e^{-h_1^2(x_m-\eta)^2} + h_2' e^{-h_2^2(x_m-\eta)^2} + \ldots\ldots + h_n' e^{-h_n^2(x_m-\eta)^2},
\end{aligned}$$

in which we have written, for brevity,

$$h' \equiv hp.$$

This product will be formed of n^m terms, each of the form

$$Pe^{-k^2\eta^2+2b\eta-c},$$

where

$$\left.\begin{aligned}
P &\equiv h_i' h_j' h_l' \ldots\ldots h_r', \\
k^2 &\equiv h_i^2 + h_j^2 + h_l^2 + \ldots\ldots + h_r^2, \\
b &= h_i^2 x_1 + h_j^2 x_2 + h_l^2 x_3 + \ldots\ldots + h_r^2 x_m, \\
c &= h_i^2 x_1^2 + h_j^2 x_2^2 + h_l^2 x_3^2 + \ldots\ldots + h_r^2 x_m^2,
\end{aligned}\right\} \qquad (8)$$

$i, j, l \ldots\ldots r$ being any m of the n indices $1, 2, 3 \ldots\ldots n$, with repetition. We therefore write

$$\psi(\eta) = \Sigma P e^{-k^2\eta^2+2b\eta-c}. \qquad (9)$$

We then have, by integration,

$$\begin{aligned}
\int_{-\infty}^{+\infty} \psi(\eta)\, d\eta &= \sum \frac{P\sqrt{\pi}}{k} e^{\frac{b^2}{k^2}-c}, \qquad (10) \\
\int_{-\infty}^{+\infty} \eta\psi(\eta)\, d\eta &= \sum \frac{Pb\sqrt{\pi}}{k^3} e^{\frac{b^2}{k^2}-c}.
\end{aligned}$$

If we distinguish the n^m values of each of the quantities P, k, b and c by the suffixes $1, 2, 3 \ldots\ldots l$, so that $l = n^m$, and if we put, for perspicuity,

$$\left.\begin{aligned}
\eta_t &= \frac{b_t}{k_t^2}, \\
w_t &= \frac{P_t}{k_t} e^{\frac{b_t^2}{k_t^2}-c_t} \quad (t = 1, 2, 3 \ldots\ldots l),
\end{aligned}\right\} \qquad (11)$$

the equation (7) will give

$$x = \frac{w_1\eta_1 + w_2\eta_2 + \ldots\ldots + w_l\eta_l}{w_1 + w_2 + \ldots\ldots + w_l}. \qquad (12)$$

This result admits of a statement which will make the principles of the method quite clear, independently of the analytic processes. The n^m quantities η_1, η_2, etc., are so many *means by weights* of the observed quantities $x_1, x_2, x_3 \ldots\ldots x_m$, each mean being obtained by making an hypothesis respecting the distribution of the measures of precision $h_1, h_2 \ldots . h_n$ among the m separate observations. Since each observation may, independently of all the others, have any one of the n measures of precision, there will be n^m such hypotheses, each leading to a different mean, η. The final value of x is again a mean by weights of the results of the different hypotheses, the weight of each result being proportional to the probability of the hypothesis on which it depends, which is represented by w. This probability is a product of two factors, of which one, $\frac{P}{k}$, is proportional to the probability of the combination, while the other, $e^{\frac{b^2}{B}-c}$, is the probability of the combination of outstanding errors to which the hypothesis leads.

§ 7. *Application to an Example.*

Before showing how the preceding method may be simplified in practice, it may be of interest to give a simple numerical example of its rigorous application. Let it be granted that we have three observations of a class for which there is a probability of $\frac{2}{3}$ that an observation is good, and of $\frac{1}{3}$ that it is poor. Let the measure of precision of a good observation be 4, and of a poor one 1. Let the results of the three observations be

I, 0; II, 0; III, 1.

Since we have, *a priori*, no reason to distinguish between these results, the usual method of treatment would lead either to $\frac{1}{3}$ as the best result, or to the rejection of the third observation, and hence to the result 0.

From the point of view of the present paper, the agreement of the first two observations and the discordance of the third give color to the hypothesis that the first two observations are good and the third poor. On this hypothesis the best result would be $\frac{1}{33}$, the weights of the results being 16, 16 and 1. But, since every other hypothesis we can make would lead to a larger result, the best result must be greater than this.

The rigorous treatment of the problem gives

$$\phi(x) = \frac{1}{3\sqrt{\pi}}\left\{8e^{-16x^2} + e^{-x^2}\right\}.$$

Hence, when $x_1 = x_2 = 0$ and $x_3 = 1$,

$$\begin{aligned}\psi(\eta) = 512e^{-48\eta^2+32\eta-16} &+ 64e^{-33\eta^2+2\eta-1} \\ + 128e^{-33\eta^2+32\eta-16} &+ 16e^{-18\eta^2+2\eta-1} \\ + \quad 8e^{-18\eta^2+32\eta-16} &+ \quad e^{-3\eta^2+2\eta-1}.\end{aligned}$$

These six terms correspond respectively to the six essentially different hypotheses which can be made respecting the distribution of the measures of precision among the different observations, it being observed that among the eight hypotheses there are two pairs such that the members of each pair lead to identical results. The results are tabulated as follows:

Hypothesis.	Results. $\frac{b}{k^2}$	Probability. $\frac{P}{k}e^{\frac{b^2}{k^2}-c}$	Product.
I, II and III all good,	$\frac{1}{3} = 0.3333$	0.003	0.001
I $\begin{cases}\text{good}\\\text{bad}\end{cases}$; II $\begin{cases}\text{bad}\\\text{good}\end{cases}$; III good,	$\frac{16}{33} = 0.4848$	0.006	0.003
I and II bad; III good,	$\frac{8}{9} = 0.8889$	0.318	0.283
I and II good; III bad,	$\frac{1}{33} = 0.0303$	4.224	0.128
I $\begin{cases}\text{good}\\\text{bad}\end{cases}$; II $\begin{cases}\text{bad}\\\text{good}\end{cases}$; III bad,	$\frac{1}{18} = 0.0555$	1.467	0.082
I, II and III all bad,	$\frac{1}{3} = 0.3333$	0.296	0.099
		$\Sigma = 6.314$	0.596

Hence, for the value of maximum worth, we have

$$x = 0.0944.$$

§8. *Modification of the Method when the Observations are Numerous.*

In order to apply the preceding method, it is necessary to know the respective probabilities that the measure of precision of any one observation has the several values $h_1, h_2 \ldots . h_n$. These probabilities are determined from the actual distribution of the residuals with respect to magnitude as found by the study of large masses of observations. If it were found that in any class of observations the magnitudes of the residuals followed the commonly assumed law, we should have but one value of h. If, as will commonly be the case, we find a larger number of large residuals than would be given by the common theory, we assume one, two, three or more additional values of h, and determine how many observations we must assign to each class in order that the distribution may be represented by an equation of the form (5).

To carry out the rigorous process of finding the best mean value of x, we should form, by the equations (8) and (11), n^m different values of P, k, b, c, η and w, and thence, from (12), the required value of x.

To effect this, let us attach, or suppose to be attached, to each of the quantities P, k, etc., a system of m indices, i, j, k, etc., each index taking the values $1, 2 \ldots\ldots n$. The system of indices

$$i, j, k \ldots\ldots q$$

attached to a quantity will then indicate the special value of that quantity which results from assigning

$$\begin{array}{llll} \text{to } x_1 & \text{the precision} & & h_i, \\ \text{to } x_2 & \text{"} & \text{"} & h_j, \\ \text{to } x_3 & \text{"} & \text{"} & h_k, \\ \ldots & \ldots & \ldots & \ldots \\ \ldots & \ldots & \ldots & \ldots \\ \ldots & \ldots & \ldots & \ldots \\ \text{to } x_m & \text{"} & \text{"} & h_q. \end{array}$$

Moreover, we shall, for brevity, represent the combination of indices

$$(i, j, k \ldots\ldots q)$$

by the single symbol t.

Any one value of w in (11) may then be written in the form

$$w_t = \frac{P_t}{k_t} e^{-\frac{\Delta_t}{k_t^2}},$$

where we put, for brevity,

$$\Delta = k^2 c - b^2.$$

If we here substitute for k^2, b and c their values from (8), we find that this expression reduces to

$$\Delta = \sum\nolimits_{i,j}^{2} h_i^2 h_j^2 (x_i - x_j)^2,$$

$$(i = 1, 2, 3 \ldots\ldots m-1; \quad j = i+1, i+2 \ldots\ldots m),$$

where h_i for the moment indicates the special value of h which, in any combination, is assigned to x_i. We may equally represent Δ in the form

$$\frac{1}{2} \sum_{i=1}^{i=m} \sum_{j=1}^{j=m} h_i^2 h_j^2 (x_i - x_j)^2.$$

The first form will consist of $\frac{m(m-1)}{2}$ terms; the second of m^2 terms, if we count those which vanish through $i = j$.

Since Δ depends only on the differences between the x's, it will remain unchanged when we subtract any constant from all of them. Let us then subtract η from all of them, putting

$$\xi_i = x_i - \eta;$$

we shall then have, for each combination of indices,

$$\frac{\Delta}{k^2} = h_1^2\xi_1^2 + h_2^2\xi_2^2 + h_3^2\xi_3^2 + \ldots + h_m^2\xi_m^2.$$

In strictness, each of the values of ξ_i should be affected by the m indices $i, j, k \ldots r$, because the values of η are so affected. But, when m is large, the different values of η arising from different combinations of indices will differ but slightly in the large majority of combinations. We may therefore, to form the m ξ's, take one general mean value of η for all the combinations of the indices.

We have now to substitute the above value of Δ in the exponential expression for w. The result may be expressed in the following form. If we put

$$\begin{aligned} w_i^{(1)} &= h_i' e^{-h_i^2\xi_1^2}, \\ w_i^{(2)} &= h_i' e^{-h_i^2\xi_2^2} \quad (i = 1, 2, 3 \ldots n), \\ w_i^{(m)} &= h_i' e^{-h_i^2\xi_m} \end{aligned} \tag{13}$$

(in which, it will be noted, the suffix i has its original signification), we shall have

$$w_{(i,j,k\ldots r)} = w_i^{(1)} w_j^{(2)} w_k^{(3)} \ldots w_r^{(m)} \div k_{(i,j,k\ldots r)},$$

$$\eta_{(i,j,k\ldots r)} = h_i^2 x_1 + h_j^2 x_2 + \ldots + h_r^2 x_m \div k^2_{(i,j,k\ldots r)}.$$

We have now to introduce another approximate hypothesis, namely, that the various values of k are so nearly equal that they may be regarded as having a common value. We note that k^2 is the sum of the squares of the m h's, and that, in the large majority of cases, the sum will approximate to a certain mean value, found by distributing the m values among the n classes proportionally to their several probabilities.

If we now substitute the preceding values of w and η in (12) we find, on the above hypothesis, that the value of x may be reduced to the form

$$x = \frac{W_1x_1 + W_2x_2 + W_3x_3 + \ldots + W_mx_m}{W_1 + W_2 + W_3 + \ldots + W_m},$$

where

$$\begin{aligned} W_1 &= h_1^2 w_1^{(1)} + h_2^2 w_2^{(1)} + h_3^2 w_3^{(1)} + \ldots + h_n^2 w_n^{(1)} \div \Sigma w_i^{(1)}, \\ W_2 &= h_1^2 w_1^{(2)} + h_2^2 w_2^{(2)} + h_3^2 w_3^{(2)} + \ldots + h_n^2 w_n^{(2)} \div \Sigma w_i^{(2)}, \\ &\ldots\ldots\ldots\ldots\ldots\ldots\ldots\ldots \\ &\ldots\ldots\ldots\ldots\ldots\ldots\ldots\ldots \\ &\ldots\ldots\ldots\ldots\ldots\ldots\ldots\ldots \\ W_m &= h_1^2 w_1^{(m)} + h_2^2 w_2^{(m)} + h_3^2 w_3^{(m)} + \ldots + h_n^2 w_n^{(m)} \div \Sigma w_i^{(m)}. \end{aligned} \tag{14}$$

These values of W are the modified weights of the several results $x_1, x_2 \ldots x_m$ arising from the probable variability of h from one observation to another. Were there no such variability; were there but one value of h, then these weights would all be equal. But, in the case for which the preceding theory is constructed, each result x_i may have any one of the weights h_1^2, h_2^2, etc.; and the equation (14) determines a certain mean among these weights which we are to assign to x_i. The coefficients W are functions of ξ, and admit of being tabulated as such in any special case.

In what precedes we have presupposed no difference of weights among the results $x_1 \ldots . x_m$ to be known *a priori*. But, since each observation may have any one of the weights $h_1 \ldots . h_n$, a certain mean weight w of each is determined *a posteriori*, as a function of its deviation from the general mean, by the equations (14). This mean weight can be tabulated as a function of ξ, and thus taken out from a table with a single argument.

If, however, we have some knowledge of an observation which leads us to assign it one precision rather than another, we may utilize this knowledge so as to modify the values of $w_j^{(i)}$. If h_1, h_2, etc., are taken in the descending order of magnitude, then h_1^2 will be the weight of each observation of the best class. The theoretically best mode of dealing with such cases will, however, depend upon the circumstances of that case. Simplicity is so important an advantage that it will probably be found well to adopt the rule of replacing W by its product by the weight fixed from *a priori* considerations.

§ 9. *Application to Transits of Mercury over the Sun's Disc.*

I now propose to apply the preceding theory to the case of observed contacts of the inferior planets, Mercury and Venus, with the limb of the Sun. A peculiarity of the observations of these phenomena is the great number of them which investigators have had to reject on account of discordance of individual observations from the general mean. By suitable rejection very different final results may be obtained, and it is impossible to draw any line between those observations which should be rejected and those which should be retained.

In my discussion of observations of transits of Mercury,* I have shown that the residuals of 684 observations of the interior contact of the limbs of the Sun and Mercury are distributed as follows, the value of each residual being considered only to the nearest round 5 seconds:

* Astronomical Papers of the American Ephemeris, Vol. I.

Below — 27 seconds were . . . . 20 residuals.

of	-27^{s},	-26^{s},	-25^{s},	-24^{s},	-23^{s}	were	11	"
"	−22,	−21,	−20,	−19,	−18	"	15	"
"	−17,	−16,	−15,	−14,	−13	"	44	"
"	−12,	−11,	−10,	− 9,	− 8	"	77	"
"	− 7,	− 6,	− 5,	− 4,	− 3	"	132	"
"	− 2,	− 1,	− 0,	+ 1,	+ 2	"	147	"
"	+ 3,	+ 4,	+ 5,	+ 6,	+ 7	"	89	"
"	+ 8,	+ 9,	+ 10,	+ 11,	+ 12	"	52	"
"	+ 13,	+ 14,	+ 15,	+ 16,	+ 17	"	33	"
"	+ 18,	+ 19,	+ 20,	+ 21,	+ 22	"	23	"
"	+ 23,	+ 24,	+ 25,	+ 26,	+ 27	"	12	"

Exceeding + 27 . . . . " 29 "

Collecting those of equal absolute value and classifying them according to the middle value of each group, we have the following comparison of actual and probable numbers, the latter being obtained on the usual theory, assuming a probable error of $\pm 6^{s}.67$, or a value of $\frac{1}{h} = 14^{s}.0$.

Mean Values of Residual.	Actual Number.	Probable Number.	$A-P$.
0^{s}	147	137	+ 10
5	221	240	− 19
10	129	166	− 37
15	77	88	− 11
20	38	36	+ 2
25	23	12	+ 11
> 27	49	5	+ 44.

There is, therefore, a large excess of both small and large residuals, which would have been yet more pronounced had the mean error been determined from the sum of the squares of all the residuals.

I find, by several trials, that the residuals which do not exceed 27^{s} can be well represented by the following distribution of precisions and probable errors:

110	observations of	$1:h =$	6^{s}	or probable error	=	$2^{s}.9$,
100	"	" "	10	" "	"	4.8,
400	"	" "	18	" "	"	8.6,
50	"	" "	36	" "	"	17.2.

The comparison of actual and probable numbers of residuals will then be as follows:

s.	Actual.	Probable.	$A-P$.
0	147	143	+ 4
5	221	220	+ 1
10	129	128	+ 1
15	77	76	+ 1
20	38	44	− 6
25	23	23	0
>27	49	26	+ 23.

It must be understood that these four values of $1:h$ and of the consequent probable errors are not four entirely determinate quantities. Really we should consider that the precision has all values between the extreme limits; but it is not at all necessary to consider it as what it really is, a continuously varying quantity. All we have to do is to form an expression which shall represent the relation between the number and magnitude of the residuals; and this we do most conveniently by assuming three, four or more separate values of h, and then finding how many observations we have to assign to each class in order to represent the observed relation. From the above table we may infer that about one-third the observations of transits of Mercury belong to classes which might be called good or very good, the probable error ranging from $\overset{s.}{2\tfrac{1}{2}}$ to $\overset{s.}{6}$; that more than half belong to an average class, of which the probable error may range from 6 to 12 seconds, and that about one-twelfth are made under such unfavorable circumstances that their probable error averages $\overset{s.}{17}$. Even with this large probable error, we see that there is an excess of 23 residuals exceeding 27 seconds, so that we should have increased the number of this imperfect class. But I suspect that many of these arose from errors of a minute in the record, or from other pure blunders.

I am also inclined to think that the comparative excess of very small residuals, indicating that one-fifth of all the observations had a probable error as small as $\overset{s.}{3}$, may be partly due to the fact that many of the residuals are deviations from the mean of a small number of observations, and that no comparison of the separate observations with the final theory founded on the whole mass of observations was made. On the whole, we may suppose that of the actual observations,

$$\begin{aligned} &0.30 \text{ have a precision } h_1 = 1:\overset{s.}{10}, \\ &0.60 \quad \text{"} \quad \text{"} \quad \text{"} \quad h_2 = 1:18, \\ &0.10 \quad \text{"} \quad \text{"} \quad \text{"} \quad h_3 = 1:36, \end{aligned}$$

This will give for the law of probability (*Eq.* 5):

$$\phi(x) = \frac{1}{\sqrt{\pi}}\left\{.030e^{-\left(\frac{x}{10}\right)^2} + .0333 \ldots . e^{-\left(\frac{x}{18}\right)^2} + .00277 \ldots . e^{-\left(\frac{x}{36}\right)^2}\right\}.$$

We have, for the values of w_1, w_2 and w_3, as given by (13),

$$w_1 = .03000e^{-\left(\frac{\xi}{10}\right)^2};\ h_1^2 = 1:\ 100,$$

$$w_2 = .03333e^{-\left(\frac{\xi}{18}\right)^2};\ h_2^2 = 1:\ 324,$$

$$w_3 = .00277e^{-\left(\frac{\xi}{36}\right)^2};\ h_3^2 = 1:1296.$$

In the following table the quantities required are tabulated as a function of the residual ξ of an observation. The w's are multiplied by 1,000, and the W's by 10,000, so as to express them in convenient units:

ξ s.	w_1	w_2	w_3	W
0	30.0	33.3	2.8	6.1
2	28.8	32.9	2.8	6.1
4	25.6	31.7	2.7	5.9
6	20.9	29.8	2.7	5.7
8	15.8	27.4	2.6	5.3
10	11.0	24.5	2.6	4.9
12	7.1	21.4	2.5	4.5
14	4.2	18.2	2.4	4.0
16	2.3	15.1	2.3	3.6
18	1.2	12.3	2.2	3.3
20	0.55	9.70	2.04	3.0
22	0.23	7.49	1.91	2.8
24	0.09	5.63	1.78	2.6
26	0.03	4.14	1.65	2.5
28	0.01	2.96	1.52	2.3
30	0.00	2.07	1.39	2.2

If we could be sure that any one observation belonged to the best class, its weight on the above scale would be 10; were we sure it belonged to the intermediate class, its weight would be 3, and if it belonged to the worst class, it would be 0.77. The value of W for $\xi = 0$, namely, 6.1, falls below 10 in consequence of the probability that an observation of residual zero may belong to one of the poorer classes, and the value of W for an observation of residual

$3^{\mathrm{s}}.0$ is above 0.77 on account of the possibility that such an observation may belong to the intermediate class.

§ 10. *Approximation Expression for the Evil.*

It remains to find an expression for the evil of the best result, as obtained by the preceding method. As already defined and shown, if the probability that the value of the observed quantity is contained between the limits x and $x + dx$ be

$$\theta x dx,$$

then the evil of any assumed value x_0 of the required quantity is given by the equation

$$E = \int_{-\infty}^{+\infty} (x - x_0)^2 \theta x dx.$$

In the case of m observed values of x, $x_1, x_2, \ldots . x_m$, following the general law of error, we have

$$\theta x = a\phi(x_1 - x)\,\phi(x_2 - x) \ldots . \phi(x_m - x),$$

the coefficient a being a constant, determined by the condition

$$\int_{-\infty}^{+\infty} \theta x dx = 1.$$

If we take for x_0 the best value of x, namely, the value which satisfies the condition

$$x_0 = \int_{-\infty}^{+\infty} x\theta x dx,$$

we have for its evil

$$E = \int_{-\infty}^{+\infty} x^2 \theta x dx - x_0^2.$$

It will be noticed that the function θx differs from $\psi(x)$ in (6) only in containing the factor a; that is, we have

$$\theta x = a\psi(x) = a\Sigma P e^{-k^2x^2 + 2bx - c}.$$

In consequence of a we may omit from ψx any constant factor, as $\sqrt{\pi}$. Then from

$$(9) \qquad \psi(\eta) = \Sigma P e^{-k^2\eta^2 + 2b\eta - c}$$

we have

$$\int_{-\infty}^{+\infty} x^2 \psi(x) dx = \Sigma P \frac{k^2 + 2b^2}{2k^5} e^{\frac{b^2}{k^2} - c}.$$

a is determined by the condition

$$a \int_{-\infty}^{+\infty} \psi(x) dx = 1 = a\Sigma w.$$

Comparing with the equations (9) and (11), we find

$$\int_{-\infty}^{+\infty} x^2\theta x dx = \frac{\Sigma w\eta^2}{\Sigma w} + \frac{\frac{1}{2}\Sigma\frac{w}{k^2}}{\Sigma w},$$

the sign Σ extending to the n^m distributions of the measures of precision. We thus have, for the minimum evil,

$$E = \frac{\Sigma w\eta^2}{\Sigma w} - \frac{(\Sigma w\eta)^2}{(\Sigma w)^2} + \frac{\frac{1}{2}\Sigma\frac{w}{k^2}}{\Sigma w} = \frac{\Sigma_{i,j}^2 w_i w_j (\eta_i - \eta_j)^2}{(\Sigma w)^2} + \frac{\Sigma\frac{w}{2k^2}}{\Sigma w}$$

$$(i = 1, 2 \ldots . n^m - 1; j = i + 1, \ldots . n^m) \tag{15}$$

The second term of the right-hand member of this equation is a certain mean among the various values of $\frac{1}{2k^2}$, and coincides with the square of the "mean error" of the usual theory, which, as already shown, coincides with the "evil," as that term is here defined. If there is but one distribution of the h's among the x's, then there will be but one value of η, and the first term of the evil will vanish, so that we shall have no evil left except the "mean error." But when, as here supposed, the weights of the observations are themselves uncertain, then the last equation expresses the logical conclusion that, in order to obtain the total evil, we must add to the result of the mean uncertainty of the observations a quantity depending upon the uncertainty in the weights we should individually or collectively assign to them.

The first term of (15) comprises $\frac{\mu(\mu-1)}{2}$ terms ($\mu = n^m$); its actual computation is therefore out of the question. We may, however, remark that it admits of being expressed in the form

$$\Sigma a_{i,j}(x_i - x_j)^2,$$

$a_{i,j}$ being numerical coefficients. This form contains only $\frac{m(m-1)}{2}$ terms.

To show this, we remark that each value of η may be written in the form

$$\eta = p_1x_1 + p_2x_2 + \ldots . + p_mx_m,$$

where

$$p_1 + p_2 + p_3 + \ldots . + p_m = 1.$$

The difference of any two values of η multiplied by any factor, such as $\frac{\sqrt{w_iw_j}}{(\Sigma w)}$, will therefore be of the form

$$\mu_1 x_1 + \mu_2 x_2 + \dots + \mu_m x_m,$$

where

$$\mu_1 + \mu_2 + \mu_3 + \dots + \mu_m = 0, \tag{16}$$

and where each of the coefficients μ_1, μ_2, etc., takes $\frac{n^m(n^m-1)}{2}$ values, corresponding to the number of differences of the η's. The sum of the squares of these differences will be

$$x_1^2 \Sigma\mu_1^2 + x_2^2 \Sigma\mu_2^2 + \dots + x_m^2 \Sigma\mu_m^2$$
$$+ 2x_1x_2\Sigma\mu_1\mu_2 + 2x_1x_3\Sigma\mu_1\mu_3 + 2x_2x_3\Sigma\mu_2\mu_3 + \text{, etc.}$$

On account of the condition (16) this expression may be transformed into

$$\Sigma^2 A_{ij}(x_i - x_j)^2,$$

where

$$A_{i,j} = -\Sigma\mu_i\mu_j.$$

Here the sign Σ of summation extends to all the $\frac{n^m(n^m-1)}{2}$ products of μ having the same constant suffixes i and j.

Although the value of this expression admits of rigorous algebraic determination, it is doubtful whether there would be any advantage in computing it in any special case. I shall therefore seek only for a rough approximation to its probable value. Returning to the equation (15), we first note that the expression

$$\frac{\Sigma^2 w_i w_j (\eta_i - \eta_j)^2}{(\Sigma w_i)^2} \qquad (i = 1,\ 2, \dots n^m_{-1};\ j = i + 1, \dots n^m)$$

is one-half a weighted mean value of $(\eta_i - \eta_j)^2$, the weight of each being the product $w_i w_j$, and the zero terms $(\eta_i - \eta_i)^2$ being allowed to enter with half weight in taking the mean. Instead of this weighted mean we may take the general mean formed by giving all the differences equal weight.

Now, when the number of observations treated is large, we may consider the amount by which any one value of η differs from the mean of all the values, or x, to be the result of an accumulation of accidental errors; namely, if we put

$$\frac{h_i^2}{k^2} \equiv q_i \tag{17}$$

(where h_i means the precision assigned to x_i), we shall have

$$\eta = q_1 x_1 + q_2 x_2 + \dots + q_m x_m,$$

while we have for x an expression of the form

$$x = q_1' x_1 + q_2' x_2 + \dots + q_m' x_m,$$

q_i' being the weighted mean of all the n^m values of q_i. If we now put, as before, ξ_i for the deviation of x_i from the general mean x, we have, on account of

$$\Sigma q_i = \Sigma q_i' = 1,$$

$$\eta - x = (q_1 - q_1')\,\xi_1 + (q_2 - q_2')\,\xi_2 + \ldots . + (q_m - q_m')\,\xi_m.$$

Now, since k^2 is, in all cases, the sum of *some* m values of h^2, the mean value of q given by (17) is $\frac{1}{m}$. The actual special values can never reach zero as their lower limit, and will seldom exceed $\frac{2}{m}$ as their upper limit. The range of value will, however, depend upon the range of values taken by the precisions h. Unless in extreme cases, the mean deviation of q from $\frac{1}{m}$ cannot exceed $\frac{1}{2m}$; that is, the mean value of $q - q'$ will, in general, be less than $\frac{1}{2m}$. Assigning this mean value, we may regard $\eta - x$ as made up of the probable accumulation of terms

$$\frac{1}{2m}(\pm\,\xi_1 \pm \xi_2 \pm \xi_3 \pm \ldots . \pm \xi_m).$$

If we put Δ^2 for the mean value of ξ^2, then, by the theory of errors, we shall have, for the mean value of $(\eta - x)^2$,

$$\text{mean}\,(\eta - x)^2 = \frac{\Delta^2}{4m}.$$

Hence

$$\text{mean}\,(\eta_i - \eta_j)^2 = \frac{\Delta^2}{2m} = \frac{\Sigma\xi^2}{2m^2}.$$

To compare this with the last term of (15), let us suppose the most probable distribution of precisions with respect to their magnitude to be

m_1 precisions of value h_1;
m_2 " " h_2;
. . . .
. . . .
. . . .
m_n " " h_n.

We shall then have as a close approximation to the mean value of k^2,

$$k^2 = m_1 h_1^2 + m_2 h_2^2 + \ldots . m_n h_n^2,$$

one-half the reciprocal of which may be taken for the last term of (15). Thus we may take for the amount of the evil, in all ordinary cases

$$E = \frac{1}{2}\,\frac{1}{m_1 h_1^2 + \ldots . + m_n h_n^2} + \frac{\Delta^2}{2m}.$$

As already shown, this evil will be the square of the "mean error" to be expected of the usual theory; so that we may take

$$\varepsilon = \pm \sqrt{E}$$

as the mean error to be expected.

I remark, in conclusion, that this theory and method may be extended to the case of several unknown quantities without any other difficulty than that of a resolution of the equations of condition with the *a posteriori* weights. We should first solve the equations if necessary, using equal weights for all, or such system of weights as might be deemed most probable. From the residuals thus obtained we should deduce the law of error, and in practice we should, in order to determine such law, combine with the residuals in question all others that astronomy could furnish pertaining to the same class of observations. Then we should re-solve the equations using the modified weights, which re-solution would give us the definitive result.

Symbolic Finite Solutions and Solutions by Definite Integrals of the Equation $\frac{d^n y}{dx^n} = x^m y$.

By J. C. Fields.

The finite solutions obtained in this paper are analogous to the symbolic solutions of Riccati's equation, and hold in all cases for which $m = \frac{-n(ni+k-1)}{ni+k}$, where k is any integer less than and prime to n, always including unity, and i is any integer positive or negative; for $n=2$, $k=1$, we have the well-known cases for which Riccati's equation is finitely soluble.

The solution by definite integrals of the above equation was first proposed by Lobatto.* Kummer and Spitzer, to whose papers I will refer further on, find the general solution for m any positive integer, and for m a negative integer greater than $2n$, respectively. M. De Tilly treats this equation by a very interesting method.†

Finite Solutions.

I will first find solutions analogous to Riccati's of equations of the third order of the form

$$\frac{d^3 y}{dx^3} = x^m y, \tag{1}$$

and afterwards discuss equations of the n^{th} order having this form.

I here premise that I will throughout consider $\frac{d}{d\Delta}$ and $\left(\frac{d}{d\Delta}\right)^{-1}$ (where Δ is any quantity or symbol treated as a quantity) as commutative, which of course amounts to putting the additive constant introduced by an integration always equal to zero.

Suppose (1) satisfied by the series

$$y = \Sigma a_n x^{n\alpha}, \text{ where } \alpha = m+3.$$

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* Crelle, Vol. XVII.

† Mathesis, Vol. V, supplement.

Substituting this series in (1) and equating coefficients of the same power of x on opposite sides of the equation, we get

(2) $$na\,(na-1)\,(na-2)\,a_n = a_{n-1};$$

$$\therefore\ a_n = \frac{a_{n-1}}{na\,(na-1)\,(na-2)} = \ldots .$$

$$= \frac{a_0}{na\,(na-1)\,(na-2)\,\overline{n-1}a\,(\overline{n-1}\ a-1)\,(\overline{n-1}\ a-2)\ldots .\,a\,(a-1)\,(a-2)}$$

$$= \frac{a^{-3n}}{\lfloor n\,(1+\nu_1)\ldots .(n+\nu_1)(1+\nu_2)\ldots .(n+\nu_2)};$$

putting

$$a_0 = 1,\ -\frac{1}{a} = \nu_1,\ -\frac{2}{a} = \nu_2;$$

$$\therefore\ y = \sum_0^{\infty} \frac{a^{-3n}x^{na}}{\lfloor n\,(1+\nu_1)\ldots .(n+\nu_1)(1+\nu_2)\ldots .(n+\nu_2)},$$

(it is evident from (2) that a_{-1}, a_{-2}, etc., are all zero). Thus

(3) $$y = \Sigma\,\frac{z^n}{\lfloor n R_{\nu_1} R_{\nu_2}},$$

where

$$z = a^{-3}x^a,\ R_{\nu_1} = (1+\nu_1)\ldots .(n+\nu_1),\ R_{\nu_2} = (1+\nu_2)\ldots .(n+\nu_2).$$

Now, $$\frac{z^n}{R_{\nu_2}} = z^{-\nu_2}\left(\frac{d}{dz}\right)^{-n} z^{\nu_2} = z^{-\nu_2}\Delta^n z^{\nu_2}, \text{ where } \left(\frac{d}{dz}\right)^{-1} = \Delta .$$

Substituting in (3) therefore

$$y = \Sigma\,\frac{z^{-\nu_2}\Delta^n z^{\nu_2}}{\lfloor n R_{\nu_1}} = z^{-\nu_2}\Sigma\,\frac{\Delta^n}{\lfloor n R_{\nu_1}}\,.\,z^{\nu_2},$$

and $$\frac{\Delta^n}{R_{\nu_1}} = \Delta^{-\nu_1}\left(\frac{d}{d\Delta}\right)^{-n}\Delta^{\nu_1},$$

(4) $$\therefore\ \ y = z^{-\nu_2}\Delta^{-\nu_1}\Sigma\,\frac{\left(\frac{d}{d\Delta}\right)^{-n}}{\lfloor n}\,.\,\Delta^{\nu_1}\,.\,z^{\nu_2} = z^{-\nu_2}\Delta^{-\nu_1}e^{\left(\frac{d}{d\Delta}\right)^{-1}}\Delta^{\nu_1}\,.\,z^{\nu_2},$$

where the entire functional symbol $\Delta^{\nu_1}e^{\left(\frac{d}{d\Delta}\right)^{-1}}\Delta^{\nu_1}$ is supposed to operate upon z^{ν_2}.

We will now show that if the value of the right-hand side of equation (4) is known and finite for any given values of ν_1, ν_2, it is also known and finite for all values of ν_1, ν_2 differing by integers from these given values. Write for brevity

(5) $$\begin{cases} \phi\,(\Delta) = \Delta^{-\nu_1}e^{\left(\frac{d}{d\Delta}\right)^{-1}}\Delta^{\nu_1}, \\ \phi_i\,(\Delta) = \Delta^{-\nu_1+i}e^{\left(\frac{d}{d\Delta}\right)^{-1}}\Delta^{\nu_1-i}, \end{cases}$$

where i is any integer positive or negative. From the formula

$$\psi(D)\,vu = v\psi(D)\,u + v'\psi'(D)\,u + \frac{v''}{2}\,\psi''(D)\,u + \ldots,$$

where $D = \frac{d}{dz}$, we have, putting $v = z$,

$$\psi(D).zu = z\psi(D)\,u + \psi'(D)\,u.$$

If $\psi(D) = \phi(\Delta)$, we have $\psi'(D) = -\Delta^2\phi'(\Delta)$, since $\Delta = D^{-1}$;

(6) $$\therefore\ \phi(\Delta).zu = z\phi(\Delta)\,u - \Delta^2\phi'(\Delta)\,u.$$

Since $\frac{d}{d\Delta}$ and $e^{\left(\frac{d}{d\Delta}\right)^{-1}}$ are commutative, we have from (5),

$$\phi'(\Delta) = -\nu_1\Delta^{-\nu_1-1}e^{\left(\frac{d}{d\Delta}\right)^{-1}}\Delta^{\nu_1} + \nu_1\Delta^{-\nu_1}e^{\left(\frac{d}{d\Delta}\right)^{-1}}\Delta^{\nu_1-1}$$

$$= -\nu_1\Delta^{-1}\left(\Delta^{-\nu_1}e^{\left(\frac{d}{d\Delta}\right)^{-1}}\Delta^{\nu_1} - \Delta^{-\nu_1+1}e^{\left(\frac{d}{d\Delta}\right)^{-1}}\Delta^{\nu_1-1}\right) = -\nu_1\Delta^{-1}(\phi(\Delta) - \phi_1(\Delta)).$$

Substituting this value of $\phi'(\Delta)$ in (6), we get

$$\phi(\Delta).zu = z\phi(\Delta).u + \nu_1\Delta(\phi(\Delta) - \phi_1(\Delta)).u.$$

Operating on this equation with $\frac{1}{\nu_1}\Delta^{-1}$ and solving for $\phi_1(\Delta)\,u$, we get

$$\phi_1(\Delta)\,u = \frac{1}{\nu_1}\Delta^{-1}\{z\phi(\Delta)\,u - \phi(\Delta).zu\} + \phi(\Delta).u$$

$$= \frac{1}{\nu_1}\frac{d}{dz}\{z\phi(\Delta)\,u - \phi(\Delta)\,zu\} + \phi(\Delta).u$$

$$= \frac{1}{\nu_1}\{z\Delta^{-1}\phi(\Delta)\,u - \nu_2\phi(\Delta)\,u\} + \phi(\Delta)\,u$$

on putting $u = z^{\nu_2}$; thus,

(7) $$\phi_1(\Delta)u = \frac{1}{\nu_1}(z\Delta^{-1}\phi(\Delta)\,u + (\nu_1 - \nu_2)\phi(\Delta))\,u = \frac{1}{\nu_1}(z\Delta^{-1} + \nu_1 - \nu_2)\,\phi(\Delta)\,u.$$

Substituting $\nu_1 - i + 1$ for ν_1 in (7), it becomes

$$\phi_i(\Delta)\,u = \frac{(z\Delta^{-1} + \nu_1 - \nu_2 - i + 1)}{\nu_1 - i + 1}\,\phi_{i-1}(\Delta)\,u,$$

and, by successive substitutions of this kind for $\phi_{i-1}(\Delta)\,u$, etc., we get

(8) $$\phi_i(\Delta)u = \frac{(z\Delta^{-1}+\nu_1-\nu_2-i+1)(z\Delta^{-1}+\nu_1-\nu_2-i+2)\ldots(z\Delta^{-1}+\nu_1-\nu_2)}{\nu_1(\nu_1-1)\ldots(\nu_1-i+1)}\,\phi(\Delta)u.$$

Using the theorem

$$\left(z\frac{d}{dz} + r\right)\left(z\frac{d}{dz} + r - 1\right)\ldots\left(z\frac{d}{dz} + r - n + 1\right)v = z^{n-r}\left(\frac{d}{dz}\right)^n z^r v,$$

(8) becomes $$\phi_i(\Delta)\,u = Cz^{i-\nu_1+\nu_2}\left(\frac{d}{dz}\right)^i z^{\nu_1-\nu_2}\phi(\Delta)\,u,$$

where C is the constant $1 \div \nu_1(\nu_1 - 1) \ldots . (\nu_1 - i + 1)$ and i a positive integer. If in this we replace ν_1 by $(\nu_1 + i)$ and operate on both sides with $z^{-i-\nu_1+\nu_2}\left(\frac{d}{dz}\right)^{-i} z^{\nu_1-\nu_2}$, we get the same formula with $-i$ instead of i, and another constant in place of C.

We have in general, then, where i is any integer, positive or negative,

$$\phi_i(\Delta)\, u = C' z^{i-\nu_1+\nu_2}\left(\frac{d}{dz}\right)^i z^{\nu_1-\nu_2}\phi(\Delta)\, u, \tag{9}$$

where C' is a constant.

Changing ν_2 into $\nu_2 - \kappa$ in (9),

$$\phi_i(\Delta)\, z^{\nu_2-\kappa} = C' z^{i-\nu_1+\nu_2-\kappa}\left(\frac{d}{dz}\right)^i z^{\nu_1-\nu_2+\kappa}\phi(\Delta)\, z^{\nu_2-\kappa}$$

$$= C' z^{i-\nu_1+\nu_2-\kappa}\left(\frac{d}{dz}\right)^i z^{\nu_1-\nu_2+\kappa}\left(\frac{d}{dz}\right)^\kappa \phi(\Delta)\, z^{\nu_2};$$

therefore, $$z^{-\nu_2+\kappa}\phi_i(\Delta)\, z^{\nu_2-\kappa} = C' z^{i-\nu_1}\left(\frac{d}{dz}\right)^i z^{\nu_1-\nu_2+\kappa}\left(\frac{d}{dz}\right)^\kappa \phi(\Delta)\, z^{\nu_2}.$$

We have, then, putting ${}_0\nu_1$, ${}_0\nu_2$ for ν_1, ν_2 respectively,

$$\left\{\begin{aligned} & z^{-{}_0\nu_2+\kappa}\Delta^{-{}_0\nu_1+i}e^{\left(\frac{d}{d\Delta}\right)^{-1}}\Delta^{{}_0\nu_1-i}z^{{}_0\nu_2-\kappa} \\ & = C' z^{i-{}_0\nu_1}\left(\frac{d}{dz}\right)^i z^{{}_0\nu_1-{}_0\nu_2+\kappa}\left(\frac{d}{dz}\right)^\kappa z^{{}_0\nu_2}\left(z^{-{}_0\nu_2}\Delta^{-{}_0\nu_1}e^{\left(\frac{d}{d\Delta}\right)^{-1}}\Delta^{{}_0\nu_1}z^{{}_0\nu_2}\right). \end{aligned}\right. \tag{10}$$

Knowing, therefore, the function $z^{-\nu_2}\Delta^{-\nu_1}e^{\left(\frac{d}{d\Delta}\right)^{-1}}\Delta^{\nu_1}z^{\nu_2}$ for $\nu_1 = {}_0\nu_1$, $\nu_2 = {}_0\nu_2$, we obtain its value for $\nu_1 = {}_0\nu_1 - i$, $\nu_2 = {}_0\nu_2 - \kappa$, by operating on the known function with the operator $C' z^{i-{}_0\nu_1}\left(\frac{d}{dz}\right)^i z^{{}_0\nu_1-{}_0\nu_2+\kappa}\left(\frac{d}{dz}\right)^\kappa z^{{}_0\nu_2}$.

By (4) $y = z^{-\nu_2}\Delta^{-\nu_1}e^{\left(\frac{d}{d\Delta}\right)^{-1}}\Delta^{\nu_1}z^{\nu_2}$ is a solution of equation (1), where $\nu_1 = -\frac{1}{\alpha}$, $\nu_2 = -\frac{2}{\alpha}$, $\alpha = m + 3$; now, for $m = 0$, equation (1) becomes $\frac{d^3y}{dx^3} = y$, and its solution is $C_1e^{-\lambda_1 x} + C_2e^{-\lambda_2 x} + C_3e^{-\lambda_3 x}$, where λ_1, λ_2, λ_3 are the three cube roots of unity; but its symbolic solution (4) is, in this case,

$$y = z^{-{}_0\nu_2}\Delta^{-{}_0\nu_1}e^{\left(\frac{d}{d\Delta}\right)^{-1}}\Delta^{{}_0\nu_1}z^{{}_0\nu_2},$$

where $\alpha_0 = 3$, ${}_0\nu_1 = -\frac{1}{\alpha_0}$, ${}_0\nu_2 = -\frac{2}{\alpha_0}$, or ${}_0\nu_1 = -\frac{2}{\alpha_0}$, ${}_0\nu_2 = -\frac{1}{\alpha_0}$

(since, from the mode of forming the symbolic function in (4), it evidently makes no change in its value to interchange the values of ν_1, ν_2); therefore, for properly chosen values of C_1, C_2, C_3, we have

$$z^{-{}_0\nu_2}\Delta^{-{}_0\nu_1}e^{\left(\frac{d}{d\Delta}\right)^{-1}}\Delta^{{}_0\nu_1}z^{{}_0\nu_2} = C_1e^{-\lambda_1 x} + C_2e^{-\lambda_2 x} + C_3e^{-\lambda_3 x};$$

here $\quad z = a_0^{-3}x^{a_0} = \left(\frac{x}{3}\right)^3; \;\therefore\; x = 3z^{\frac{1}{3}};$ thus,

(11) $$z^{-{}_0\nu_2}\Delta^{-{}_0\nu_1}e^{\left(\frac{d}{d\Delta}\right)^{-1}}\Delta^{{}_0\nu_1}z^{{}_0\nu_2} = C_1e^{-3\lambda_1 z^{\frac{1}{3}}} + C_2e^{-3\lambda_2 z^{\frac{1}{3}}} + C_3e^{-3\lambda_3 z^{\frac{1}{3}}}.$$

Taking, now,

$$\nu_2 = {}_0\nu_2 - \varkappa = -\frac{2}{\alpha}, \quad \nu_1 = {}_0\nu_1 - i = -\frac{1}{\alpha},$$

we have $\quad \varkappa = {}_0\nu_2 - 2{}_0\nu_1 + 2i, \; m + 3 = \alpha = \frac{1}{i - {}_0\nu_1}.$

1st. Assuming

$${}_0\nu_1 = -\frac{1}{a_0} = -\frac{1}{3}, \quad {}_0\nu_2 = -\frac{2}{a_0} = -\frac{2}{3},$$

we get $\quad \varkappa = 2i, \; \alpha = \frac{1}{i + \frac{1}{3}} = \frac{3}{3i+1} = m + 3;$

therefore, $\quad m = \frac{-9i}{3i+1};$

and solution of (1) is by (4), (10) and (11):

$$y = z^{-\nu_2}\Delta^{-\nu_1}e^{\left(\frac{d}{d\Delta}\right)^{-1}}\Delta^{\nu_1}z^{\nu_2} = z^{-{}_0\nu_2+\varkappa}\Delta^{-{}_0\nu_1+i}e^{\left(\frac{d}{d\Delta}\right)^{-1}}\Delta^{{}_0\nu_1-i}z^{{}_0\nu_2-\varkappa}$$

$$= z^{i-{}_0\nu_1}\left(\frac{d}{dz}\right)^i z^{{}_0\nu_1-{}_0\nu_2+\varkappa}\left(\frac{d}{dz}\right)^\varkappa z^{{}_0\nu_2}\left(C_1e^{-3\lambda_1 z^{\frac{1}{3}}} + C_2e^{-3\lambda_2 z^{\frac{1}{3}}} + C_3e^{-3\lambda_3 z^{\frac{1}{3}}}\right)$$

(12) $$= z^{\frac{1}{3}+i}\left(\frac{d}{dz}\right)^i z^{\frac{1}{3}+2i}\left(\frac{d}{dz}\right)^{2i} z^{-\frac{2}{3}}\left(C_1e^{-3\lambda_1 z^{\frac{1}{3}}} + C_2e^{-3\lambda_2 z^{\frac{1}{3}}} + C_3e^{-3\lambda_3 z^{\frac{1}{3}}}\right).$$

2d. Assuming

$${}_0\nu_1 = -\frac{2}{a_0} = -\frac{2}{3}, \quad {}_0\nu_2 = -\frac{1}{a_0} = -\frac{1}{3},$$

we get $\quad \varkappa = 2i + 1, \; \alpha = \frac{1}{i + \frac{2}{3}} = \frac{3}{3i+2} = m + 3;$

therefore, $\quad m = \frac{-3(3i+1)}{3i+2}.$

And solution of (1) is by (4), (10) and (11):

$$y = z^{i-{}_0\nu_1}\left(\frac{d}{dz}\right)^i z^{{}_0\nu_1-{}_0\nu_2+\varkappa}\left(\frac{d}{dz}\right)^\varkappa z^{{}_0\nu_2}\left(C_1e^{-3\lambda_1 z^{\frac{1}{3}}} + C_2e^{-3\lambda_2 z^{\frac{1}{3}}} + C_3e^{-3\lambda_3 z^{\frac{1}{3}}}\right)$$

(12′) $$= z^{\frac{2}{3}+i}\left(\frac{d}{dz}\right)^i z^{\frac{2}{3}+2i}\left(\frac{d}{dz}\right)^{2i+1} z^{-\frac{1}{3}}\left(C_1e^{-3\lambda_1 z^{\frac{1}{3}}} + C_2e^{-3\lambda_2 z^{\frac{1}{3}}} + C_3e^{-3\lambda_3 z^{\frac{1}{3}}}\right).$$

These solutions are evidently of the form

$$\phi_1 e^{-3\lambda_1 z^{\frac{1}{3}}} + \phi_2 e^{-3\lambda_2 z^{\frac{1}{3}}} + \phi_3 e^{-3\lambda_3 z^{\frac{1}{3}}},$$

where the ϕ's are polynomials in $z^{\frac{1}{3}}$, and plainly, on substitution in equation (1), the terms containing each of the three exponentials vanish identically; therefore, the three quantities C_1, C_2, C_3 in the above solutions may be considered as arbitrary constants. We conclude, therefore, from (12) and (12′), that the equation $\frac{d^3y}{dx^3} = x^m y$ is solvable finitely if m be of either of the forms $\frac{-3i}{3i+1}$ or $\frac{-3(3i+1)}{3i+2}$, where i is any integer positive or negative, the general solutions in the two cases being given by (12) and (12′) respectively.

These solutions become, on substituting for z in terms of x from the formula $z = a^{-3}x^a\,(a = m + 3)$,

I. $m = \frac{-9i}{3i+1}$,

$$y = x\left(x^{1-\frac{3}{3i+1}}\frac{d}{dx}\right)^i x^{1+\frac{3i}{3i+1}}\left(x^{1-\frac{3}{3i+1}}\frac{d}{dx}\right)^{2i} x^{-\frac{2}{3i+1}}\Big(C_1 e^{-(3i+1)\lambda_1 x^{\frac{1}{3i+1}}}$$
$$+ C_2 e^{-(3i+1)\lambda_2 x^{\frac{1}{3i+1}}} + C_3 e^{-(3i+1)\lambda_3 x^{\frac{1}{3i+1}}}\Big);$$

II. $m = \frac{-3(3i+1)}{3i+2}$,

$$y = x\left(x^{1-\frac{3}{3i+2}}\frac{d}{dx}\right)^i x^{1+\frac{3i}{3i+2}}\left(x^{1-\frac{3}{3i+2}}\frac{d}{dx}\right)^{2i+1} x^{-\frac{1}{3i+2}}\Big(C_1 e^{-(3i+2)\lambda_1 x^{\frac{1}{3i+2}}}$$
$$+ C_2 e^{-(3i+2)\lambda_2 x^{\frac{1}{3i+2}}} + C_3 e^{-(3i+2)\lambda_3 x^{\frac{1}{3i+2}}}\Big),$$

the C's being arbitrary constants and the λ's the cube roots of unity.

The Equation of the nth Order.

Suppose the equation

$$\frac{d^n y}{dx^n} = x^m y \tag{13}$$

satisfied by the series $y = \Sigma a_\kappa x^{\kappa a}$, where $a = m + n$; on substituting this series in (13) we get for the determination of the coefficients,

$$\kappa a(\kappa a - 1)\ldots\ldots(\kappa a - n + 1)\,a_\kappa = a_{\kappa-1}; \tag{14}$$

therefore,

$$a_\kappa = \frac{a_{\kappa-1}}{\kappa\alpha(\kappa\alpha-1)\dots\dots(\kappa\alpha-n+1)}$$

$$= \frac{a_{\kappa-2}}{\kappa\alpha(\kappa\alpha-1)\dots\dots(\kappa\alpha-n+1).\overline{\kappa-1}\,\alpha(\overline{\kappa-1}\,\alpha-1)\dots\dots(\overline{\kappa-1}\,\alpha-n+1)} = \dots\dots$$

$$= \frac{a_0}{\kappa\alpha.\overline{\kappa-1}\,\alpha\overline{\kappa-2}\,\alpha\dots\alpha.(\kappa\alpha-1)(\overline{\kappa-1}\,\alpha-1)\dots(\alpha-1)\dots(\kappa\alpha-n+1)(\overline{\kappa-1}\,\alpha-n+1)\dots(\alpha-n+1)}$$

$$= \frac{a_0\alpha^{-n\kappa}}{\lfloor\!\underline{\kappa}\,(1+\nu_0)(2+\nu_0)\dots\dots(\kappa+\nu_0)(1+\nu_1)\dots\dots(\kappa+\nu_1)\dots\dots(1+\nu_{n-2})\dots\dots(\kappa+\nu_{n-2})},$$

where

(15) $$\nu_0 = -\frac{1}{\alpha}\ \nu_1 = -\frac{2}{\alpha}\dots\dots\nu_{r-1} = -\frac{r}{\alpha}\ \dots\dots\nu_{n-2} = -\frac{n-1}{\alpha},$$

and therefore,

$$\nu_0:\nu_1:\dots\dots:\nu_{r-1}:\dots\dots:\nu_{n-2}::1:2:\dots\dots:r:\dots\dots:\overline{n-1},$$

$$a_\kappa = \frac{\alpha^{-n\kappa}}{\lfloor\!\underline{\kappa}\,R_{\nu_0}R_{\nu_1}\dots\dots R_{\nu_{n-2}}}\text{; on putting } a_0 = 1,\ R_{\nu_r} \equiv (1+\nu_r)(2+\nu_r)\dots\dots(\kappa+\nu_r),$$

therefore, $$y = \Sigma a_\kappa x^{\kappa\alpha} = \sum_0^\infty{}_\kappa \frac{\alpha^{-n\kappa}x^{\alpha\kappa}}{\lfloor\!\underline{\kappa}\,R_{\nu_0}R_{\nu_1}\dots\dots R_{\nu_{n-2}}}.$$

(It is evident from (14) that the series contains only positive powers of x^α.) Thus

(16) $$y = \Sigma\frac{z^\kappa}{\lfloor\!\underline{\kappa}\,R_{\nu_0}R_{\nu_1}\dots\dots R_{\nu_{n-2}}},\text{ where } z = \alpha^{-n}x^\alpha.$$

Now, if we put

$$\Delta_1 \equiv \left(\frac{d}{dz}\right)^{-1},\ \Delta_2 \equiv \left(\frac{d}{d\Delta_1}\right)^{-1},\dots\dots\Delta_r \equiv \left(\frac{d}{d\Delta_{r-1}}\right)^{-1},\dots\dots\Delta_{n-1} \equiv \left(\frac{d}{d\Delta_{n-2}}\right)^{-1},$$

we have

$$\frac{z^\kappa}{R_{\nu_0}} = \frac{z^\kappa}{(1+\nu_0)\dots\dots(\kappa+\nu_0)} = z^{-\nu_0}\left(\frac{d}{dz}\right)^{-\kappa}z^{\nu_0} = z^{-\nu_0}\Delta_1^\kappa z^{\nu_0},$$

$$\frac{\Delta_1^\kappa}{R_{\nu_1}} = \Delta_1{}^{-\nu_1}\Delta_2^\kappa\Delta_1^{\nu_1},\dots\dots\frac{\Delta_r^\kappa}{R_{\nu_r}} = \Delta_r^{-\nu_r}\Delta_{r+1}^\kappa\Delta_r^{\nu_r}\ \dots\dots;$$

therefore,

$$y = \Sigma\frac{z^\kappa}{\lfloor\!\underline{\kappa}\,R_{\nu_0}R_{\nu_1}\dots\dots R_{\nu_{n-2}}} = z^{-\nu_0}\Sigma\frac{\Delta_1^\kappa}{\lfloor\!\underline{\kappa}\,R_{\nu_1}\dots\dots R_{\nu_{n-2}}}.z^{\nu_0}$$

$$= z^{-\nu_0}\Delta_1^{-\nu_1}\Sigma\frac{\Delta_2^\kappa}{\lfloor\!\underline{\kappa}\,R_{\nu_2}\dots\dots R_{\nu_{n-2}}}.\Delta_1^{\nu_1}.z^{\nu_0}$$

$$= \dots\dots = z^{-\nu_0}\Delta_1^{-\nu_1}\Delta_2^{-\nu_2}\dots\dots\Delta_{n-2}^{-\nu_{n-2}}\Sigma\frac{\Delta_{n-1}^\kappa}{\lfloor\!\underline{\kappa}}\Delta_{n-2}^{\nu_{n-2}}\dots\dots\Delta_2^{\nu_2}\Delta_1^{\nu_1}z^{\nu_0}.$$

Thus

(17) $$y = z^{-\nu_0}\Delta_1^{-\nu_1}\Delta_2^{-\nu_2}\dots\dots\Delta_{n-2}^{-\nu_{n-2}}e^{\Delta_{n-1}}\Delta_{n-2}^{\nu_{n-2}}\dots\dots\Delta_2^{\nu_2}\Delta_1^{\nu_1}z^{\nu_0}.$$

From the mode of formation of this symbolic expression, its meaning is that we operate with $e^{\Delta_{n-1}}$ on $\Delta_{n-2}^{\nu_{n-2}}$, multiply the result by $\Delta_{n-2}^{-\nu_{n-2}}$, and with the complete operator $\Delta_{n-2}^{-\nu_{n-2}}e^{\Delta_{n-1}}\Delta_{n-2}^{\nu_{n-2}}$ in Δ_{n-2} thus obtained operate on $\Delta_{n-3}^{\nu_{n-3}}$, and so on, finally operating on z^{ν_0} with the complete operator in Δ_1 and multiplying by $z^{-\nu_0}$.

For brevity I will adopt the following notation:

$$\phi\,(\nu_0\nu_1\,\ldots\ldots\,\nu_{n-2}) \equiv z^{-\nu_0}\Delta_1^{-\nu_1}\,\ldots\ldots\,\Delta_{n-2}^{-\nu_{n-2}}e^{\Delta_{n-1}}\Delta_{n-2}^{\nu_{n-2}}\,\ldots\ldots\,\Delta_1^{\nu_1}z^{\nu_0},$$

$$\phi_r\,(\nu_r\nu_{r+1}\,\ldots\ldots\,\nu_{n-2}) \equiv \Delta_r^{-\nu_r}\,\ldots\ldots\,\Delta_{n-2}^{-\nu_{n-2}}e^{\Delta_{n-1}}\Delta_{n-2}^{\nu_{n-2}}\,\ldots\ldots\,\Delta_r^{\nu_r},$$

$$_{i_r}\phi_r(\nu_r\nu_{r+1}\,\ldots\ldots\,\nu_{n-2}) \equiv \phi_r\,(\nu_r - i_r,\, \nu_{r+1}\,\ldots\ldots\,\nu_{n-2}),$$

$_p(q) = q\,(q-1)\,\ldots\ldots\,(q-p+1)$, where p is an integer and q any quantity or symbol.

Where there is no fear of ambiguity, I will simply denote the first three of these expressions by ϕ, $\phi_r(\Delta_r)$, $_{i_r}\phi_r(\Delta_r)$, respectively.

We notice that $\phi_r(\Delta_r)$, being a function of Δ_r, is commutative with Δ_r.

I will now proceed to show how from the value of ϕ for certain values of the ν's we can deduce its value for values of the ν's differing from the given ones by integers.

By the formula $\psi\left(\frac{d}{dx}\right).uv = u\psi\left(\frac{d}{dx}\right)v + u'.\psi'\left(\frac{d}{dx}\right).v +$, etc., we have, putting $\phi_r\,(\Delta_r) = \psi\left(\frac{d}{d\Delta_{r-1}}\right)$,

$$\text{(18)}\quad \phi_r\,(\Delta_r).\,\Delta_{r-1}^{\nu_{r-1}+1} = \phi_r\,(\Delta_r).\,\Delta_{r-1}.\,\Delta_{r-1}^{\nu_{r-1}} = \Delta_{r-1}\phi_r\,(\Delta_r)\,\Delta_{r-1}^{\nu_{r-1}} - \Delta_r^2\phi_r'(\Delta_r)\,\Delta_{r-1}^{\nu_{r-1}},$$

since
$$\Delta_r = \left(\frac{d}{d\Delta_{r-1}}\right)^{-1} \text{and } \psi'\left(\frac{d}{d\Delta_{r-1}}\right) = -\,\Delta_r^2\phi_r'(\Delta_r).$$

Now, differentiating $\phi_r\,(\Delta_r) = \Delta_r^{-\nu_r}\phi_{r+1}(\Delta_{r+1})\Delta_r^{\nu_r}$ with respect to Δ_r, we find

$$\text{(19)}\quad \begin{aligned}\phi_r'(\Delta_r) &= -\,\nu_r\Delta_r^{-\nu_r-1}\phi_{r+1}(\Delta_{r+1})\Delta_r^{\nu_r} + \nu_r\Delta_r^{-\nu_r}\phi_{r+1}(\Delta_{r+1})\,\Delta_r^{\nu_r-1}\\ &= \nu_r\Delta_r^{-1}\{{}_1\phi_r(\Delta_r) - \phi\,(\Delta_r)\}.\end{aligned}$$

Substituting this value of $\phi_r'(\Delta_r)$ in (18), we get

$$\phi_r\,(\Delta_r)\,\Delta_{r-1}^{\nu_{r-1}+1} = \Delta_{r-1}\phi_r\,(\Delta_r)\,\Delta_{r-1}^{\nu_{r-1}} - \nu_r\Delta_r\{{}_1\phi_r\,(\Delta_r) - \phi\,(\Delta_r)\}\Delta_{r-1}^{\nu_{r-1}}.$$

Operating on this with $\Delta_r^{-1} = \frac{d}{d\Delta_{r-1}}$, and taking $_1\phi_r(\Delta_r)\,.\,\Delta_{r-1}^{\nu_{r-1}}$ to one side of the equation, we get

$$\text{(20)}\quad {}_1\phi_r\,(\Delta_r)\,\Delta_{r-1}^{\nu_{r-1}} = \frac{1}{\nu_r}\left(\Delta_{r-1}\frac{d}{d\Delta_{r-1}} + \nu_r - \nu_{r-1}\right)\phi_r\,(\Delta_r)\,\Delta_{r-1}^{\nu_{r-1}}.$$

In this equation, substituting $\nu_r - i_r + 1$ for ν_r, ${}_1\phi_r(\Delta_r)$ becomes ${}_{i_r}\phi_r(\Delta_r)$ and $\phi_r(\Delta_r)$ becomes ${}_{i_r-1}\phi_r(\Delta_r)$; $\therefore$ the equation becomes

$$(21) \quad {}_{i_r}\phi_r(\Delta_r)\,\Delta_{r-1}^{\nu_r-1} = \frac{1}{\nu_r - i_r + 1}\left(\Delta_{r-1}\frac{d}{d\Delta_{r-1}} + \nu_r - i_r - \nu_{r-1} + 1\right){}_{i_r-1}\phi_r(\Delta_r)\,\Delta_{r-1}^{\nu_r-1}.$$

Similarly,

$$ {}_{i_r-1}\phi_r(\Delta_r)\,\Delta_{r-1}^{\nu_r-1} = \frac{1}{\nu_r - i_r + 2}\left(\Delta_{r-1}\frac{d}{d\Delta_{r-1}} + \nu_r - i_r - \nu_{r-1} + 2\right){}_{i_r-2}\phi_r(\Delta_r)\,\Delta_{r-1}^{\nu_r-1}$$

. . . .

Substituting successively in (21) from these formulæ, for ${}_{i_r-1}\phi_r(\Delta_r)\,\Delta_{r-1}^{\nu_r-1}$, ${}_{i_r-2}\phi_r(\Delta_r)\,\Delta_{r-1}^{\nu_r-1}$, etc., ${}_1\phi_r(\Delta_r)\,\Delta_{r-1}^{\nu_r-1}$, we get

$$(22) \quad {}_{i_r}\phi_r(\Delta_r)\,\Delta_{r-1}^{\nu_r-1} = \frac{1}{{}_{i_r}(\nu_r)}\cdot {}_{i_r}\!\left(\Delta_{r-1}\frac{d}{d\Delta_{r-1}} + \nu_r - \nu_{r-1}\right).\,\phi_r(\Delta_r)\,\Delta_{r-1}^{\nu_r-1}.$$

By the formula

$$ {}_n\!\left(x\frac{d}{dx} + m\right)u \equiv \left(x\frac{d}{dx} + m\right)\left(x\frac{d}{dx} + m - 1\right)\ldots$$
$$\ldots\left(x\frac{d}{dx} + m - n + 1\right)u = x^{n-m}\left(\frac{d}{dx}\right)^n . x^m u,$$

we have

$$ {}_{i_r}\!\left(\Delta_{r-1}\frac{d}{d\Delta_{r-1}} + \nu_r - \nu_{r-1}\right)u = \Delta_{r-1}^{i_r - \nu_r + \nu_{r-1}}\left(\frac{d}{d\Delta_{r-1}}\right)^{i_r} . \Delta_{r-1}^{\nu_r - \nu_{r-1}};$$

therefore, from (22),

$$ {}_{i_r}\phi_r(\Delta_r)\,\Delta_{r-1}^{\nu_r-1} = \frac{1}{{}_{i_r}(\nu_r)}\cdot \Delta_{r-1}^{i_r - \nu_r + \nu_{r-1}}\left(\frac{d}{d\Delta_{r-1}}\right)^{i_r}\Delta_{r-1}^{\nu_r - \nu_{r-1}}.\,\phi_r(\Delta_r)\,\Delta_{r-1}^{\nu_r-1};$$

whence

$$(23) \quad \Delta_{r-1}^{-\nu_{r-1}}\,{}_{i_r}\phi_r(\Delta_r)\,\Delta_{r-1}^{\nu_r-1} = \frac{1}{{}_{i_r}(\nu_r)}\,\Delta_{r-1}^{i_r - \nu_r}\left(\frac{d}{d\Delta_{r-1}}\right)^{i_r}\Delta_{r-1}^{\nu_r - \nu_{r-1}}\phi_r(\Delta_r)\,\Delta_{r-1}^{\nu_r-1}$$
$$= \frac{1}{{}_{i_r}(\nu_r)}\sum_0^{i_r}{}_{\lambda_{r-1}}\frac{i_r!}{\varkappa_{r-1}!\;\lambda_{r-1}!}\,{}_{\varkappa_{r-1}}(\nu_r - \nu_{r-1})\,{}_{\lambda_{r-1}}(\nu_{r-1})\,{}_{\lambda_{r-1}}\phi_{r-1}(\Delta_{r-1})$$

(where $\varkappa_{r-1}$, λ_{r-1} are integers such that $\varkappa_{r-1} + \lambda_{r-1} = i_r$), on performing the operation indicated by $\left(\frac{d}{d\Delta_{r-1}}\right)^{i_r}$; thus,

$$z^{-\nu_0}\Delta_1^{-\nu_1}\ldots\Delta_{r-1}^{-\nu_{r-1}}\,{}_{i_r}\phi_r(\Delta_r)\,\Delta_{r-1}^{\nu_r-1}\ldots\Delta_1^{\nu_1}z^{\nu_0}$$
$$= \frac{1}{{}_{i_r}(\nu_r)}\sum_0^{i_r}{}_{\lambda_{r-1}}\frac{i_r!}{\varkappa_{r-1}!\;\lambda_{r-1}!}\,{}_{\varkappa_{r-1}}(\nu_r - \nu_{r-1})\,{}_{\lambda_{r-1}}(\nu_{r-1}).\,z^{-\nu_0}\Delta_1^{-\nu_1}\ldots$$
$$\Delta_{r-2}^{-\nu_{r-2}}\,{}_{\lambda_{r-1}}\phi_{r-1}(\Delta_{r-1})\,\Delta_{r-2}^{\nu_r-2}\ldots\Delta_1^{\nu_1}z^{\nu_0}$$
$$= \frac{1}{{}_{i_r}(\nu_r)}\sum_0^{i_r}{}_{\lambda_{r-1}}\frac{i_r!}{\varkappa_{r-1}!\,\lambda_{r-1}!}\,{}_{\varkappa_{r-1}}(\nu_r - \nu_{r-1})\,{}_{\lambda_{r-1}}(\nu_{r-1}).\,\frac{1}{{}_{\lambda_{r-1}}(\nu_{r-1})}\sum_0^{\lambda_{r-1}}{}_{\lambda_{r-2}}\frac{\lambda_{r-1}!}{\varkappa_{r-2}!\;\lambda_{r-2}!}\,{}_{\varkappa_{r-2}}(\nu_{r-1}$$
$$- \nu_{r-2})\,{}_{\lambda_{r-2}}(\nu_{r-2}).\,z^{-\nu_0}\Delta_1^{-\nu_1}\ldots\Delta_{r-3}^{-\nu_{r-3}}\,{}_{\lambda_{r-2}}\phi_{r-2}(\Delta_{r-2})\,\Delta_{r-3}^{\nu_r-3}\ldots\Delta_1^{\nu_1}z^{\nu_0}$$

(where $\varkappa_{r-2}+\lambda_{r-2}=\lambda_{r-1}$), on substituting for $\Delta_{r-2}^{-\nu_{r-2}}{}_{\lambda_{r-1}}\phi_{r-1}(\Delta_{r-1})\Delta_{r-2}^{\nu_{r-2}}$, its equivalent expression similar to that obtained for $\Delta_{r-1}^{-\nu_{r-1}}{}_{i_r}\phi_r(\Delta_r)\Delta_{r-1}^{\nu_{r-1}}$ in (23).

By making such substitutions successively for

$$\Delta_{r-1}^{-\nu_{r-1}}{}_{i_r}\phi_r(\Delta_r)\Delta_{r-1}^{\nu_{r-1}},\ \Delta_{r-2}^{-\nu_{r-2}}{}_{\lambda_{r-1}}\phi_{r-1}(\Delta_{r-1})\Delta_{r-2}^{\nu_{r-2}},\ \Delta_{r-3}^{-\nu_{r-3}}{}_{\lambda_{r-2}}\phi_{r-2}(\Delta_{r-2})\Delta_{r-3}^{\nu_{r-3}},\ \ldots\ \Delta_1^{-\nu_1}{}_{\lambda_2}\phi_2(\Delta_2)\Delta_1^{\nu_1},\ z^{-\nu_0}{}_{\lambda_1}\phi_1(\Delta_1)z^{\nu_0},$$

we ultimately obtain

$$(24)\quad z^{-\nu_0}\Delta_1^{-\nu_1}\ldots\Delta_{r-1}^{-\nu_{r-1}}{}_{i_r}\phi(\Delta_r)\Delta_{r-1}^{\nu_{r-1}}\ldots\Delta_1^{\nu_1}z^{\nu_0}$$

$$=\frac{1}{{}_{i_r}(\nu_r)}\sum_0^{i_r}{}_{\lambda_{r-1}}\frac{i_r!}{\varkappa_{r-1}!\,\lambda_{r-1}!}\,{}_{\varkappa_{r-1}}(\nu_r-\nu_{r-1})_{\lambda_{r-1}}(\nu_{r-1}),$$

$$\frac{1}{{}_{\lambda_{r-1}}(\nu_{r-1})}\sum_0^{\lambda_{r-1}}{}_{\lambda_{r-2}}\frac{\lambda_{r-1}!}{\varkappa_{r-2}!\,\lambda_{r-2}!}\,{}_{\varkappa_{r-2}}(\nu_{r-1}-\nu_{r-2})_{\lambda_{r-2}}(\nu_{r-2})\,\frac{1}{{}_{\lambda_{r-2}}(\nu_{r-2})}\sum_0^{\lambda_{r-2}}{}_{\lambda_{r-3}}\ldots,$$

$$\frac{1}{{}_{\lambda_2}(\nu_2)}\sum_0^{\lambda_2}{}_{\lambda_1}\frac{\lambda_2!}{\varkappa_1!\,\lambda_1!}\,{}_{\varkappa_1}(\nu_2-\nu_1)_{\lambda_1}(\nu_4)\cdot\frac{1}{{}_{\lambda_1}(\nu_1)}\sum_0^{\lambda_1}{}_{\lambda_0}\frac{\lambda_1!}{\varkappa_0!\,\lambda_0!}\,{}_{\varkappa_0}(\nu_1-\nu_0)_{\lambda_0}(\nu_0)_{\lambda_0}\phi_0(z).$$

$$=\frac{1}{{}_{i_r}(\nu_r)}\sum_0^{i_r}{}_{\lambda_{r-1}}\frac{i_r!}{\varkappa_{r-1}!\,\lambda_{r-1}!}\,{}_{\varkappa_{r-1}}(\nu_r-\nu_{r-1}),$$

$$\sum_0^{\lambda_{r-1}}{}_{\lambda_{r-2}}\frac{\lambda_{r-1}}{\varkappa_{r-2}!\,\lambda_{r-2}!}\,{}_{\varkappa_{r-2}}(\nu_{r-1}-\nu_{r-2})\sum_0^{\lambda_{r-2}}{}_{\lambda_{r-3}}\ldots,$$

$$\sum_0^{\lambda_2}{}_{\lambda_1}\frac{\lambda_2!}{\varkappa_1!\,\lambda_1!}\,{}_{\varkappa_1}(\nu_2-\nu_1)\cdot\sum_0^{\lambda_1}{}_{\lambda_0}\frac{\lambda_1!}{\varkappa_0!\,\lambda_0!}\,{}_{\varkappa_0}(\nu_1-\nu_0)_{\lambda_0}(\nu_0)z^{-\nu_0+\lambda_0}\phi_1(\Delta_1)z^{\nu_0-\lambda_0}.$$

Now,

$$\sum_0^{\lambda_1}{}_{\lambda_0}\frac{\lambda_1!}{\varkappa_0!\,\lambda_0!}\,{}_{\varkappa_0}(\nu_1-\nu_0)_{\lambda_0}(\nu_0)z^{-\nu_0+\lambda_0}\phi_1(\Delta_1)z^{\nu_0-\lambda_0}$$

$$=\sum_0^{\lambda_1}{}_{\lambda_0}\frac{\lambda_1}{\varkappa_0!\,\lambda_0!}\,{}_{\varkappa_0}(\nu_1-\nu_0)\,z^{-\nu_0+\lambda_0}\left(\frac{d}{dz}\right)^{\lambda_0}\phi_1(\Delta_1)z^{\nu_0}$$

$$=z^{-\nu_1+\lambda_1}\left(\frac{d}{dz}\right)^{\lambda_1}z^{\nu_1-\nu_0}\phi_1(\Delta_1)z^{\nu_0};$$

thus,

$$\sum_0^{\lambda_2}{}_{\lambda_1}\frac{\lambda_2}{\varkappa_1!\,\lambda_1!}\,{}_{\varkappa_1}(\nu_2-\nu_1)\sum_0^{\lambda_1}{}_{\lambda_0}\frac{\lambda_1!}{\varkappa_0!\,\lambda_0!}\,{}_{\varkappa_0}(\nu_1-\nu_0)_{\lambda_0}(\nu_0)z^{-\nu_0+\lambda_0}\phi_1(\Delta_1)z^{\nu_0-\lambda_0}$$

$$=\sum_0^{\lambda_2}{}_{\lambda}\frac{\lambda_2!}{\varkappa_1!\,\lambda_1!}\,{}_{\varkappa_1}(\nu_2-\nu_1)\,z^{-\nu_1+\lambda_1}\left(\frac{d}{dz}\right)^{\lambda_1}.z^{\nu_1-\nu_0}\phi_1(\Delta_1)z^{\nu_0}$$

$$=z^{-\nu_2+\lambda_2}\left(\frac{d}{dz}\right)^{\lambda_2}z^{\nu_2-\nu_1}.\,z^{\nu_1-\nu_0}\phi_1(\Delta_1)z^{\nu_0};$$

and proceeding in this manner, we find that (24) becomes

(25) $$z^{-\nu_0}\Delta_1^{-\nu_1} \ldots \Delta_{r-1}^{-\nu_{r-1}}\, {}_{i_r}\phi_r(\Delta_r)\, \Delta_{r-1}^{\nu_{r-1}} \ldots \Delta_1^{\nu_1} z^{\nu_0}$$
$$= \frac{1}{{}_{i_r}(\nu_r)} z^{-\nu_r + i_r}\left(\frac{d}{dz}\right)^{i_r} z^{\nu_r - \nu_{r-1}} . z^{\nu_{r-1} - \nu_{r-2}} \ldots z^{\nu_1 - \nu_0} . \phi_1(\Delta_1) z^{\nu_0}$$
$$= \frac{1}{{}_{i_r}(\nu_r)} z^{-\nu_r + i_r}\left(\frac{d}{dz}\right)^{i_r} z^{\nu_r - \nu_0} \phi_1(\Delta_1) z^{\nu_0}.$$

Operating on both sides of equation (25) with ${}_{i_r}(\nu_r)\, z^{-\nu_r}\left(\frac{d}{dz}\right)^{-i_r} z^{\nu_r - i_r}$, and substituting $(\nu_r + i_r)$ for ν_r, we obtain

(26) $$z^{-\nu_0}\Delta_1^{-\nu_1} \ldots \Delta_{r-1}^{-\nu_{r-1}}\, {}_{-i_r}\phi_r(\Delta_r)\Delta_{r-1}^{\nu_{r-1}} \ldots \Delta_1^{\nu_1} z^{\nu_0} = {}_{i_r}(\nu_r + i_r) z^{-\nu_r - i_r}\left(\frac{d}{dz}\right)^{-i_r} z^{\nu_r - \nu_0}\phi_1(\Delta_1) z^{\nu_0}.$$

Thus, from (25) and (26), we get

(27) $$z^{-\nu_0}\Delta_1^{-\nu_1} \ldots \Delta_{r-1}^{-\nu_{r-1}}\, {}_{i_r}\phi_r(\Delta_r)\Delta_{r-1}^{\nu_{r-1}} \ldots \Delta_1^{\nu_1} z^{\nu_0} = C_r z^{-\nu_r + i_r}\left(\frac{d}{dz}\right)^{i_r} z^{\nu_r} . z^{-\nu_0}\phi_1(\Delta_1) z^{\nu_0}$$
$$= C_r z^{-\nu_r + i_r}\left(\frac{d}{dz}\right)^{i_r} z_{\nu_r} . \phi,$$

where i_r is any integer positive or negative and C_r is a constant.

Now, since (27) holds for all values of $\nu_0, \nu_1 \ldots \nu_r \ldots$, we have

(28) $$\phi(\nu_0 - i_0,\, \nu_1 - i_1 \ldots \nu_{r-1} - i_{r-1},\, \nu_r - i_r \ldots \nu_{n-2} - i_{n-2})$$
$$= \omega_r . (\nu_0 - i_0,\, \nu_1 - i_1, \ldots \nu_{r-1} - i_{r-1},\, \nu_r, \ldots, \nu_{n-2} - i_{n-2}),$$

where ω_r denotes the operator $C_r z^{-\nu_r + i_r}\left(\frac{d}{dz}\right)^{i_r} z^{\nu_r}$ and the i's are any integers positive or negative. By successive applications of the formula (28) for all values of r from 0 up to $(n-2)$ we obtain

(29) $$\phi(\nu_0 - i_0,\, \nu_1 - i_1, \ldots, \nu_r - i_r \ldots, \nu_{n-2} - i_{n-2})$$
$$= \Pi(\omega_r) . \phi(\nu_0 . \nu_1, \ldots, \nu_r \ldots \nu_{n-2}),$$

where $\Pi(\omega_r)$ designates the symbolic operator $\omega_0 . \omega_1 \ldots \omega_{n-2}$.

Substituting now for $\nu_0, \nu_1 \ldots \nu_r \ldots$ in (29) ${}_0\nu_0, {}_0\nu_1, \ldots {}_0\nu_r \ldots$ respectively, and putting ${}_0\nu_0 - i_0 = \nu_0,\ {}_0\nu_1 - i_1 = \nu_1 \ldots {}_0\nu_r - i_r = \nu_r \ldots$ (29) becomes

(30) $$\phi(\nu_0, \nu_1, \ldots \nu_r, \ldots \nu_{n-2}) = \Pi({}_0\omega_r)\, \phi({}_0\nu_0, {}_0\nu_1, \ldots {}_0\nu_r \ldots {}_0\nu_{n-2}),$$

where ${}_0\omega_r = C_r z^{-{}_0\nu_r + i_r}\left(\frac{d}{dz}\right)^{i_r} z^{{}_0\nu_r}$ and $\nu_r = {}_0\nu_r - i_r$.

If therefore we are given the solution $\phi({}_0\nu_0, {}_0\nu_1, \ldots {}_0\nu_r, \ldots {}_0\nu_{n-2})$ of an equation of the form (13), we can, by (30), find the solutions $\phi(\nu_0, \nu_1 \ldots \nu_r \ldots \nu_{n-2})$ of other equations of this form such that m is changed subject to the conditions that $\nu_0, \nu_1 \dots \nu_{n-2}$ differ by integers from the given quantities ${}_0\nu_0, {}_0\nu_1, \dots {}_0\nu_{n-2}$, and also to the condition (15), viz., $\nu_0 : \nu_1 \dots : \nu_{n-2} :: 1 : 2 : \dots : \overline{n-1}$, the value of m being determined from the equation $\nu_0 = -\frac{1}{a} = -\frac{1}{m+n}$. While the order of the ν's in the proportion just mentioned is perfectly arbitrary, I will in general consider them in this order except in the case of the initial values ${}_0\nu_0, {}_0\nu_1 \dots, {}_0\nu_{n-2}$, whose arrangement I will not limit, viz.,

(31) $${}_0\nu_0 : {}_0\nu_1 : \dots : {}_0\nu_{n-2} :: \varkappa_1 : \varkappa_2 : \dots : \varkappa_{n-1},$$

where the $\varkappa$'s are all different from each other and each equal to one of the integers $1, 2, \dots \overline{n-1}$.

While these different ways of distributing their values to ${}_0\nu_0, {}_0\nu_1 \dots : {}_0\nu_{n-2}$ does not affect the value of $\phi\,({}_0\nu_0, {}_0\nu_1 \dots {}_0\nu_{n-2})$, it will be seen that it does affect the derived quantities $\nu_0, \nu_1, \dots, \nu_{n-2}$, giving rise to different sets of values for these, and consequently also giving different values of m for which equation (13) is soluble.

Now, when $m = 0$ we know the general solution of (13) to be

(32) $$C_1 e^{-\mu_1 x} + C_2 e^{-\mu_2 x} + \dots + C_n e^{-\mu_n x},$$

where the μ's are the nth roots of unity and the C's are arbitrary constants.

Taking this as our initial case, we have

$$a_0 = m + n = n;\ {}_0\nu_0, {}_0\nu_1, \dots {}_0\nu_{n-2} \text{ equal } \frac{-\varkappa_1}{n}, \frac{-\varkappa_2}{n}, \dots \frac{-\varkappa_{n-1}}{n}$$

respectively, by (15) and (31); and by (16) $z = n^{-n}x^n$; therefore $x = nz^{\frac{1}{n}}$.

Substituting now for x in terms of z, (32) becomes

(33) $$C_1 e^{-\mu_1 n z^{\frac{1}{n}}} + C_2 e^{-\mu_2 n z^{\frac{1}{n}}} + \dots + C_n e^{-\mu_n n z^{\frac{1}{n}}},$$

which is the general solution of (13), and therefore includes $\phi\,({}_0\nu_0, {}_0\nu_1, \dots, {}_0\nu_{n-2})$, which is a particular solution; we have then

(34) $$\phi\,({}_0\nu_0, {}_0\nu_1 \dots {}_0\nu_{n-2}) = C_1 e^{-\mu_1 n z^{\frac{1}{n}}} + C_2 e^{-\mu_2 n z^{\frac{1}{n}}} + \dots + C_n e^{-\mu_n n z^{\frac{1}{n}}},$$

the constants C being properly chosen; and therefore from (30)

(35) $$\phi\,(\nu_0, \nu_1 \dots \nu_{n-2}) = \Pi\,({}_0\omega_r)\left(C_1 e^{-\mu_1 n z^{\frac{1}{n}}} + C_2 e^{-\mu_2 n z^{\frac{1}{n}}} + \dots + C_n e^{-\mu_n n z^{\frac{1}{n}}}\right).$$

Now, as stated before, the conditions to which the ν's are subject are

$$\nu_{r-1} = r\nu_0 \text{ for all integer values of } r \text{ from 1 up to } \overline{n-1},$$

$$\text{and } \nu_{r-1} = {}_0\nu_{r-1} - i_{r-1} \text{ for all integer values of } r \text{ from 1 up to } \overline{n-1},$$

where i_{r-1} is any integer positive or negative; thus,

(36) $${}_0\nu_{r-1} - i_{r-1} = r\,({}_0\nu_0 - i_0);$$

and since ${}_0\nu_{r-1} = -\frac{x_r}{n}$, ${}_0\nu_0 = -\frac{x_1}{n}$, this becomes

(37) $$\frac{rx_1 - x_r}{n} = i_{r-1} - ri_0.$$

Now, the x's must all be positive integers different from each other and less than n. Choosing then any one of these integers as the value of x_1, it is evident that we can for any chosen value of r find an integer value of x_r less than n such that $\frac{rx_1 - x_r}{n} = .I_r$, where I_r is zero or an integer positive or negative, and there is plainly only one such value of x_r, and the value of i_{r-1} obtained from the equation $i_{r-1} - ri_0 = I_r$ is of course an integer.

We know then that for each value of x_1 there is one, and only one, value of x_r satisfying equation (37) for each value of r, and it only remains to find in what cases the values of x_r are all different for the different values of r.

Suppose $x_r = x_{r'}$; then $\frac{rx_1 - x_r}{n} - \frac{r'x_1 - x_{r'}}{n} =$ integer; therefore $\frac{(r - r')\,x_1}{n}$ = integer, and since $r - r' < n_1 n$ and x_1 must have a factor in common; and conversely, if n, x_1 have a common factor, we can choose $\overline{r - r'}$, so as to contain n's remaining factor, and therefore $\frac{(r - r')\,x_1}{n}$ = integer; consequently $x_r = x_{r'}$, and the x's are not all different.

Thus, in order that the x's should be all distinct, it is necessary and sufficient that x_1 be prime to n.

Taking, then, ${}_0\nu_0 = -\frac{x_1}{n}$, we have $\nu_0 = {}_0\nu_0 - i_0 = -\left(\frac{x_1}{n} + i_0\right)$, where x_1 is less than n and prime to it and i_0 is any integer; and for this value of ν_0 with the values of ν_{r-1} derived from the formula $\nu_{r-1} = r\nu_0$, for all integer values of r from 2 up to $\overline{n-1}$, we have $\phi\,(\nu_0, \nu_1 \ldots \nu_{n-2})$ expressible finitely and given by (35). Now, from

$$-\left(\frac{x_1}{n} + i_0\right) = \nu_0 = -\frac{1}{a} = -\frac{1}{m+n}$$

we obtain $m = \frac{-n\,\{ni_0 + x_1 - 1\}}{ni_0 + x}$, for all which values of m equation (13) is finitely integrable; or, putting i, x for i_0, x_1 respectively, equation (13) is finitely integrable for all values of m given by the equation

(38) $$m = \frac{-n\,\{ni + x - 1\}}{ni + x},$$

where κ is any integer less than and prime to n, and i is any integer whatever, positive or negative, the integral being given by equation (35), its value being found by substituting in the operator $\Pi\,({}_0\omega_r)$ the values of the ν's derived from the equations $\nu_0 = -\frac{\kappa}{n}$, $\nu_{r-1} = r\nu_0$, and the values of the i's derived from the equations $i_0 = i$, $\frac{r\kappa - \kappa_r}{n} = i_{r-1} - ri$. It is easily seen that all the C's in (35) are different from zero, and therefore that on the substitution of the right-hand member of (35) for y in (13) the resulting identity contains n terms of the form $\psi_r . e^{-\mu_r n z^{\frac{1}{n}}}$ (ψ_r being a polynomial in $z^{\frac{1}{n}}$), due respectively to each of the n exponentials occurring in (35); and since, evidently, each of these n terms must identically vanish, equation (13) is satisfied by

$$(39) \qquad \Pi\,({}_0\omega_r)\,(C_1 e^{-\mu_1 n z^{\frac{1}{n}}} + C_2 e^{-\mu_2 n z^{\frac{1}{n}}} + \ldots + C_n e^{-\mu_n n z^{\frac{1}{n}}}),$$

where the C's are all *arbitrary* constants.

We may write (39) in the form

$$(40) \qquad \Omega_{n-2}\Omega_{n-3}\ldots\Omega_0 . z^{-\frac{\kappa}{n}}(C_1 e^{-\mu_1 n z^{\frac{1}{n}}} + C_2 e^{-\mu_2 n z^{\frac{1}{n}}} + \ldots + C_n e^{-\mu_n n z^{\frac{1}{n}}}),$$

where $\Omega_r = z^{i_r + 1 - \frac{ni+\kappa}{n}}\left(\frac{d}{dz}\right)^{i_r}$, the integers i_r being determined from the equation (37) $i_r = (r+1)\,i + \frac{(r+1)\,\kappa - \kappa_{r+1}}{n}$, κ_{r+1} being equal to zero or any integer less than n, no two of κ's being equal, and r taking the values $0, 1, 2, \ldots, (n-1)$.

Substituting in (40) by the formulae, $z = \alpha^{-n}x^{\alpha}$, $\alpha = \frac{n}{ni+\kappa}$, we have for the general solution of equation (13), for values of m given by (38),

$$(41) \qquad y = \Omega_{n-2}\Omega_{n-3}\ldots\Omega_0 . x^{-\frac{\kappa}{ni+\kappa}}(C_1 e^{-\mu_1 (ni+\kappa) x^{\frac{1}{ni+\kappa}}} + C_2 e^{-\mu_2 (ni+\kappa) x^{\frac{1}{ni+\kappa}}} + \ldots + C_n e^{-\mu_n (ni+\kappa) x^{\frac{1}{ni+\kappa}}}),$$

where the C's are arbitrary constants and

$$\Omega_r = x^{\frac{ni_r+1}{ni+\kappa} - 1}\left(x^{-\frac{n}{ni+\kappa}+1}\frac{d}{dx}\right)^{i_r}.$$

For $n = 2$, $\kappa = 1$, (38) gives the cases $m = \frac{-4i}{2i+1}$, for which Riccati's equation is finitely integrable. We might consider $m = -n$ as the limiting case of (38) for which $i = \infty$.

For $m = -2n$ (38) gives $x = n - 1$, $i = -1$; we find $i_r = -1$, $\Omega_r = z^{-1+\frac{1}{n}}\left(\frac{d}{dz}\right)^{-1}$ for all values of r, and we may write (40) in the form

(42) $$z^{-1+\frac{1}{n}}\Omega^{n-1}\left(C_1 e^{-\mu_1 n z^{\frac{1}{n}}} + \ldots . + C_n e^{-\mu_n z^{\frac{1}{n}}}\right),$$

where $$\Omega . u = \left(\frac{d}{dz}\right)^{-1} z^{-1+\frac{1}{n}} . u = n\left(\frac{d}{dz^{\frac{1}{n}}}\right)^{-1} . u .$$

Evidently $\Omega^{n-1} e^{-\mu n z^{\frac{1}{n}}} = \left(-\frac{1}{\mu}\right)^{n-1} e^{-\mu n z^{\frac{1}{n}}}$; therefore, (42) becomes

$$z^{-1+\frac{1}{n}}\left(C_1 e^{-\mu_1 n z^{\frac{1}{n}}} + \ldots . + C_n e^{-\mu_n n z^{\frac{1}{n}}}\right),$$

and this, on substituting $z = (-nx)^{-n}$, takes the form

$$x^{n-1}\left(C_1 e^{\frac{\mu_1}{x}} + \ldots . + C_n e^{\frac{\mu_n}{x}}\right),$$

which is the well-known solution of $\frac{d^n y}{dx^n} = x^{-2n} y$.

Riccati's Equation.

The preceding method, applied to the equation of the second order, suggests also the following mode of solving Riccati's equation:

(43) $$\frac{d^2 y}{dx^2} = x^m y .$$

Putting $z = a^2 x^{-\frac{1}{a}}$, $a = -\frac{1}{m+2}$, (43) becomes

(44) $$z\frac{d^2 y}{dz^2} + (a+1)\frac{dy}{dz} - y = 0 .$$

Differentiate (44) with respect to z, and put $\frac{dy}{dz} = u$, when we get

$$z\frac{d^2 u}{dz^2} + (a+2)\frac{du}{dz} - u = 0 ,$$

which is the same in form as (44), a being increased by unity. Evidently, then, the solution of (44) being given for any value a_0 of a, that for $a = a_0 + i$ (where i is any integer) is the i^{th} differential of the given solution.

Now, for $m = 0$, $a = -\frac{1}{2}$, the solution of (43) is $Ae^{-x} + Be^{x}$, and, therefore,

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Note.—I might here say that formula (25) can be more briefly obtained by a consideration of the general term of the series in (16); but as I first obtained it from (17), and the treatment of the symbolic form there given seems to me somewhat interesting as an application of symbolic methods, I have let it stand.

that of (44) is $Ae^{2\sqrt{z}}+Be^{-2\sqrt{z}}$ (since $x=-2\sqrt{z}$); therefore, for $a=i-\frac{1}{2}$, the solution of (44) is

(45) $$y=\frac{d^i}{dz^i}\left(Ae^{2\sqrt{z}}+Be^{-2\sqrt{z}}\right),$$

where $z=a^2x^{-\frac{1}{a}}=\left(\frac{2i-1}{2}\right)^2x^{-\frac{2}{2i-1}}$.

It may easily be verified that (45) is identical with

(46) $$y=z^{-i+i}\frac{d^{-i}}{dz^{-i}}z^{-i}\left(Ae^{2\sqrt{z}}+Be^{-2\sqrt{z}}\right);$$

consequently, whether i be positive or negative, the solution need only involve the direct operation of differentiation by using (45) when i is positive and (46) when i is negative.

We might notice that if in (44) we put $y=z^{-a}v$ and operate on the resultant equation in v with $e^{-\Delta}\left(\Delta\equiv\left(\frac{d}{dz}\right)^{-1}\right)$, this equation at once reduces to $z\frac{d}{dz}\left(e^{-\Delta}\frac{dv}{dz}\right)+(1-a)e^{-\Delta}\frac{dv}{dz}=0$, an equation of the first order in $e^{-\Delta}\frac{dv}{dz}$, from which $v=e^{\Delta}z^a$, and, therefore, $y=z^{-a}v=z^{-a}e^{\Delta}z^a$, the form (17) thus obtained directly from the equation, without first considering the solution in the form of an infinite series.

From the solutions of Riccati's equation we may also derive those of the equation $x\frac{du}{dx}-au+bu^2=cx^n$; for, in this equation, putting

$$u=\frac{a-1}{2b}+\frac{x}{bv}\cdot\frac{dv}{dx},$$

it becomes

(47) $$\frac{d^2v}{dx^2}=(c_1x^{n-2}+c_2x^{-2})\,v,\text{ where } c_1=bc,\ c_2=\frac{a^2-1}{4}.$$

Substituting $z=c_1n^{-2}x^n$, $v=z^ry$, (47) becomes

(48) $$z\frac{d^2y}{dz^2}+\left(\frac{n-1}{n}+2r\right)\frac{dy}{dz}=\left(c_2n^{-2}-r(r-1)-\frac{n-1}{n}.r\right)z^{-1}y,$$

and this reduces to equation (44) when we choose r, so that

$$c_2n^{-2}-r(r-1)-\frac{n-1}{n}r=0;$$

therefore, for (47) finitely integrable $2r-\frac{1}{n}=a=i-\frac{1}{2}$, eliminating r

between this equation and the quadratic in r just given, and putting $c_2=\frac{a^2-1}{4}$, we obtain, as condition for finite integrability, $\frac{n\pm 2a}{2n}=i$, and when this condition holds, the solution of $x\frac{du}{dx}-au+bu^2=cx^n$ is easily seen to be

$$u=\frac{a-1}{2b}+\frac{n}{b}z^{1-r}\frac{d}{dz}\cdot z^ry, \tag{49}$$

where $r=\frac{1\pm a}{2n}$ and y is given by equations (45) and (46).

Solutions by Definite Integrals.

Kummer has shown* that if $\psi(x)$ be the general solution of equation $\frac{d^{n+1}z}{dx^{n+1}}=x^{m-1}z$, the general solution of $\frac{d^ny}{dx^n}=x^my$ may be expressed by the integral $\int_0^\infty u^{m-1}e^{-\frac{u^{m+n}}{m+n}}\psi(xu)\,du$, there being a certain relation among the $(n+1)$ constants involved in $\psi(x)$; and by successive applications of this formula has obtained the solution in definite integral form of the equation $\frac{d^ny}{dx^n}=x^my$ for all positive integral values of m; Spitzer, by a modification of Kummer's method, has shown† that if $\psi(x)$ be the general solution of $x^{m+1}\frac{d^{n+1}z}{dx^{n+1}}=\varepsilon z$, the general solution of $x^m\frac{d^ny}{dx^n}=-\varepsilon y$ may be expressed by $\int_0^\infty u^{m-1}e^{-\frac{u^{m-n}}{m-n}}\psi\left(\frac{x}{u}\right)du$, a certain relation holding among the $(n+1)$ constants of $\psi(x)$, and has thus found in definite integral form the general solution of $\frac{d^ny}{dx^n}=x^my$ for all negative integer values of m numerically greater than $2n$.

We might express both Kummer's and Spitzer's definite integrals under one form; thus, if $\psi(x)$ be the general solution of $\frac{d^{n+1}z}{dx^{n+1}}=bx^{m-1}z$, the general solution of $\frac{d^ny}{dx^n}=ax^my$ may be expressed by

$$y=\int_0^\infty u^{m-1}e^{-\frac{b}{a}\frac{u^{m+n}}{m+n}}\psi(xu)\,du, \tag{50}$$

* Crelle, Vol. 19.

† Crelle, Vol. 57.

a certain relation holding among the $(n+1)$ arbitrary constants of $\psi(x)$, the conditions being involved that $m+n$ and m are of the same sign, and $\frac{b}{a}$ positive or negative according as this sign is plus or minus. This may be easily verified by differentiating $\frac{d^n y}{dx^n} - ax^m y = 0$ and substituting for $\frac{d^{n+1}y}{dx^{n+1}}$, y' and y from (50). Kummer's case is that of m and $m+n$ positive, Spitzer's m and $m+n$ negative. Both Kummer and Spitzer always suppose n positive. There is nothing, however, in the verification of (50) to require this, and (50) holds equally well whether n be positive or negative, providing that m and $m+n$ fulfil the requisite conditions; thus, from the solution of $\frac{d^{-n+1}z}{dx^{-n+1}} = x^{m-1}z$ we may derive that of $\frac{d^{-n}y}{dx^{-n}} = x^m y$. In the former of these equations, putting $x^{m-1}z = v$, it becomes $x^{-m+1}v = \frac{d^{n-1}v}{dx^{n-1}}$ and the latter similarly becomes $x^{-m}u = \frac{d^n u}{dx^n}$; thus, from the solution of $\frac{d^{n-1}v}{dx^{n-1}} = x^{-m+1}v$ we derive a solution of $\frac{d^n u}{dx^n} = x^{-m}u$; in $\frac{d^{-n}y}{dx^{-n}}$ we will suppose the additive constants due to the integrations always equal to zero, so that the solutions of $y = \frac{d^n}{dx^n}(x^m y)$ are also solutions of $\frac{d^{-n}y}{dx^{-n}} = x^m y$.

Now, we know the solution of $\frac{d^n y}{dx^n} = x^{-2n}y$ to be $y = x^{n-1}\Sigma C_r e^{-\frac{\mu_r}{x}}$, where the μ's are the nth roots of unity and the C's arbitrary constants, and the solution of $\frac{d^{-n}z}{dx^{-n}} = x^{2n}z$ is $z = x^{-2n}y = x^{-n-1}\Sigma C_r e^{-\frac{\mu_r}{x}}$.

Starting out now from the equation

$$\frac{d^{-n}z}{dx^{-n}} = x^{2n}z, \tag{51}$$

by (50) we have, as the solution of equation $\frac{d^{-n-1}y}{dx^{-n-1}} = x^{2n+1}y$,

$$y = x^{-n-1}\int_0^\infty u^{n-1} e^{-\frac{u^n}{n}} \Sigma C_r e^{-\frac{\mu_r}{xu}} . du,$$

and by successive applications of formula (50) we obtain as the solution of equation $\frac{d^{-n-i}y}{dx^{-n-i}} = x^{2n+i}y$,

$$y = x^{-n-1}\int_0^\infty \int_0^\infty \dots \int_0^\infty \prod_1^i{}_\kappa\, u_\kappa^{n+\kappa-2} . e^{-\frac{1}{n}\sum_1^i{}_\kappa u_\kappa^n} \Sigma C_r e^{-\mu_r (xu_1 \dots u_i)^{-1}} . du_1, du_2 \dots du_i.$$

Substituting n for $(n+i)$, we have, as solution of equation $\frac{d^{-n} y}{dx^{-n}} = x^{2n-i} y$,

(52) $y = x^{-n+i-1} \int_0^\infty \int_0^\infty \cdots \int_0^\infty \prod_1^i {}_\kappa u_\kappa^{n-i+\kappa-2} . e^{-\frac{1}{n-i} \sum_1^i {}_\kappa u_\kappa^{n-i}} \Sigma C_r e^{-\mu r (x u_1 \ldots u_i)^{-1}} du_1 \ldots du_i$,

where by $\prod_1^i {}_\kappa u_\kappa^{n-i+\kappa-2}$ I mean the product $u_1^{n-i-1} . u_2^{n-i} \ldots u_i^{n-2}$; evidently i is always a positive integer less than n; the μ's are here the $(n-i)$th roots of unity.

In the equation $\frac{d^n u}{dx^n} = x^{-2n+i} u$ we have $u = x^{2n-i} y$; therefore, the solution of the equation

(53) $$\frac{d^n y}{dx^n} = x^{-2n+i} y \text{ is}$$

(54) $y = x^{n-1} \int_0^\infty \int_0^\infty \cdots \int_0^\infty \prod_1^i {}_\kappa u_\kappa^{n-i+\kappa-2} . e^{-\frac{1}{n-i} \sum_1^i {}_\kappa . u_\kappa^{n-i}} \Sigma C_r e^{-\mu r (x u_1 \ldots u_i)^{-1}} du_1 \ldots du_i$.

By (54) we have then a definite integral solution of the equation $\frac{d^n y}{dx^n} = x^m y$ for all negative integer values of m between n and $2n$; this solution is not, indeed, the most general one, but contains $-(m+n) = n - i$ arbitrary constants.

We can further find a particular solution of this equation for any value of m.

If we write $\frac{d^{-n} z}{dx^{-n}} = x^m z$, $\frac{d^{-n-1} z_1}{dx^{-n-1}} = x^{m+1} z_1$, ... $\frac{d^{-n-i} z_i}{dx^{-n-i}} = x^{m+i} z_i$, solutions of these equations being respectively $\psi(x)$, $\psi_1(x) \ldots \psi_i(x)$, and m a positive quantity greater than n, m and $m-n$ are both positive, and we have, by (50),

$$\psi_1(x) = \int_0^\infty u_1^m e^{-\frac{u_1^{m-n}}{m-n}} \psi(x u_1) du_1,$$

$$\psi_2(x) = \int_0^\infty u_2^{m+1} e^{-\frac{u_2^{m-n}}{m-n}} \psi_1(x u_2) du_2,$$

. . . .

$$\psi_i(x) = \int_0^\infty u_i^{m+i-1} e^{-\frac{u_i^{m-n}}{m-n}} \psi_{i-1}(x u_i) du_i;$$

therefore, the solution of $\frac{d^{-n-i} z_i}{dx^{-n-i}} = x^{m+i} z_i$ is

$$z_i = \psi_i(x) = \int_0^\infty \int_0^\infty \cdots \int_0^\infty u_1^m u_2^{m+1} \ldots u_i^{m+i-1} e^{-\frac{1}{m-n}(u_1^{m-n} + \ldots + u_i^{m-n})} \psi(x u_1 u_2 \ldots u_i) du_1 du_2 \ldots du_i.$$

Now, putting $n=1$, m is any quantity greater than unity, and $\psi(x)$ is the solution of equation $\frac{d^{-1}z}{dx^{-1}}=x^m z$; therefore, $\psi(x)=x^{-m}e^{\frac{x^{1-m}}{1-m}}$.

Now, substituting for $\psi(xu_1 \dots u_i)$ in $\psi_i(x)$, and putting m, n for $m+i$, $n+i\,(=1+i)$ respectively, we have the solution of $\frac{d^{-n}z_i}{dx^{-n}}=x^m z_i$,

$$z_i=x^{-m+n-1}\int_0^\infty\int_0^\infty\dots\int_0^\infty u_2u_3^2\dots u_{n-1}^{n-2}e^{\frac{1}{n-m}\left(u_1^{m-n}+\dots+u_{n-1}^{m-n}+(xu_1\dots u_{n-1})^{n-m}\right)}du_1\dots du_{n-1};$$

replacing z_i by $x^{-m}y$, we find the equation

$$(55)\qquad \frac{d^n y}{dx^n}=x^{-m}y$$

to have a solution:

$$(56)\qquad y=x^{n-1}\int_0^\infty\int_0^\infty\dots\int_0^\infty u_2u_3^2\dots u_{n-1}^{n-2}e^{\frac{1}{n-m}\left(u_1^{m-n}+\dots+u_{n-1}^{m-n}+(xu_1\dots u_{n-1})^{n-m}\right)}du_1\dots du_{n-1};$$

where m is any positive quantity greater than n. We can find a similar solution for any other value of m by starting out from equation $\frac{d^{-1}y}{dx^{-1}}=x^m y$, where $m-1$ is any negative quantity, by successive use of formula (50) with $\frac{b}{a}=-1$.

Now, referring back to formula (30); if in this we substitute y_0 for $\phi({}_0\nu_0\dots{}_0\nu_{n-2})$, where y_0 is the general solution of the equation of which $\phi({}_0\nu_0\dots{}_0\nu_{n-2})$ is a particular solution, the equation

$$(57)\qquad y=\Pi({}_0\omega_r)\cdot y_0$$

gives y the general solution of the equation of which $\phi(\nu_0\dots\nu_{n-2})$ is a particular solution, as is easily verified.

We can now with the help of (57) derive from (56) a solution of $\frac{d^n y}{dx^n}=x^{m_1}y$ for any real value of m_1.

We know that (57) holds for

$$(58)\qquad \nu_0={}_0\nu_0-i,$$

where i is any integer, the values of the other ν's being derived from equations

$$\nu_r={}_0\nu_r-i_r=(r+1)\nu_0.$$

Now, being given any value of ν_0, we can always satisfy (58) by a positive value of ${}_0\nu_0$, by properly choosing the integer i, which we may do in an infinite number of ways.

Now, putting $\nu_0 = -\frac{1}{m_1 + n}$, ${}_0\nu_0 = -\frac{1}{{}_0m_1 + n}$, we have, when ${}_0\nu_0$ is positive, ${}_0m_1$, a negative quantity greater than n; therefore, ${}_0m_1 = -m$ where $m > n$, and, therefore, (56) gives a solution of (55), that is, of $\frac{d^n y}{dx^n} = x^{{}_0m_1} y$. Being, therefore, given any equation,

$$\text{(59)} \qquad \frac{d^n y}{dx^n} = x^{m_1} y,$$

find ν_0, satisfy (58) by a positive value of ${}_0\nu_0$, find ${}_0m_1$ from equation ${}_0\nu_0 = -\frac{1}{{}_0m_1 + n}$; this will give us ${}_0m_1 = -m$, where m is a value for which (56) holds as the solution of (55); but the general solution of $\frac{d^n y}{dx^n} = x^{{}_0m_1} y$ is y_0, and, therefore, (56) is contained as a particular solution under y_0.

Now, y_0 in (57) is supposed expressed as a function of z, where z is connected with x in $\frac{d^n y}{dx^n} = x^{-m} y$ by the equation $z = (n-m)^{-n} x^{n-m}$ (see (16)); transforming (56) from x to z by this transformation, and substituting (56) thus transformed for y_0 in (57), we get, as a solution of (59), expressed in terms of z,

$$\text{(60)} \quad y = \Pi({}_0\omega_r)\, z^{\frac{n-1}{n-m}}$$
$$\int_0^\infty \int_0^\infty \dots \int_0^\infty u_2 u_3^2 \dots u_{n-1}^{n-2} e^{\frac{1}{n-m}\left(u_1^{n-m} + \dots + u_{n-1}^{n-m}\right) + (n-m)^{n-1}(u_1 \dots u_{n-1})^{n-m} \cdot z} du_1 \dots du_{n-1},$$

for all values of m_1, m being obtained from the positive value of ${}_0\nu_0$ satisfying (58).

The x in (59) is connected with z in (60) by the relation $z = \alpha^{-n} x^{\alpha} = (m_1 + n)^{-n} x^{m_1 + n}$ (see (16)), by which equation, transforming (60) to terms of x, putting ${}_0\nu_r = (r+1)\,{}_0\nu_0$, $i_r = (r+1)\,i$, ${}_0\nu_0 = \frac{1}{m-n}$, and instead of ${}_0\omega_r = z^{-{}_0\nu_r + i_r}\left(\frac{d}{dz}\right)^{i_r} z^{{}_0\nu_r}$ writing

$$\text{(61)} \qquad \omega_r = x^{r+1}\left(x^{-(m_1+n)+1}\frac{d}{dx}\right)^{(r+1)i} x^{(r+1)\{(m_1+n)i-1\}},$$

which differs only by a constant multiplier from ${}_0\omega_r$, (60) becomes

$$\text{(62)} \quad y = \Pi(\omega_r)\, x^{-(n-1)\{(m_1+n)i-1\}}$$
$$\int_0^\infty \int_0^\infty \dots \int_0^\infty u_2 u_3^2 \dots u_{n-1}^{n-2} e^{\frac{1}{n-m}\left\{\left(u_1^{n-m} + \dots + u_{n-1}^{n-m}\right) + \left(\frac{n-m}{n+m_1}\right)^n (u_1 \dots u_{n-1})^{n-m} x^{m_1+n}\right\}} du_1 \dots du_{n-1};$$

here the product $\Pi(\omega_r)$ is taken for all integer values of r from 0 to $(n-1)$ inclusive.

Solutions, then, of equation (59) are given by (62) for all values of m_1, the values of m and i involved in these solutions being any that will satisfy equation (58); *i. e.* $\frac{1}{m_1+n}+\frac{1}{m-n}=i$, i being any integer and $m-n$ any positive quantity, and for the same value of m_1 (62) assumes different forms, according to the values we may choose for m and i.

If we put $i=0$, we have $m_1=-m$, $\Pi(\omega_r)=1$, and (62) reduces back to (56).

We might also obtain a more general form, including that of (62), by taking ${}_0\nu_0=\frac{x}{m-n}$ instead of $\frac{1}{m-n}$, where x is chosen subject to the condition (37), with $n-m$ substituted for m, the x's being positive integers different from each other and less than n.

By combinations of formulae (50) and (57) we may obtain solutions of equations of form (59) in a variety of forms on starting out from some equation of this form whose solution is known, either finitely or as a definite integral; (50) connects the solutions of all equations (59) for which m_1+n remains constant, and (57) those of equations (59) for which the values of $(m_1+n)^{-1}$ only differ by integers.

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